Wave Geometry No. 43

Cosmology in Terms of Wave Geometry (VIII) Observation Systems in Cosmology.

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(Received January 11, 1941.)

§ 1. Introduction.

In cosmology in terms of wave geometry,[1] we have adopted the idea that a nebula has two aspects, one being that of a particle in detecting the universe, the other being of probability-existence; that is to say

(1) the path of each nebula, as a particle, is a geodesic line in the space time continuum.

(2) as a probability-existence, the momentum density of nebulae is a function of ψ, the solution of the fundamental equation.

On the basis of these considerations, with the condition that the vector $u^l=\psi^\dagger A\gamma^l\psi$ considered as expressing the momentum density vector of nebulae smeared out, always generates a geodesic line in the space-time continuum, we have established our theory.

In this paper, first we shall show that our theory of cosmology is characterized by a *homogeneous property for observations* in the universe. Next, we shall obtain the relations between two observation-systems in the universe and using these relations we shall show that the Hubble's velocity distance relation in terms of wave geometry is also deducible.

§ 2. Homogeneous property of the universe.

In cosmology in terms of wave geometry,[1] we have obtained the fundamental equation for ψ:

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=\frac{k}{2}\gamma_i\psi\,,\qquad (i=1,\ldots,4)\tag{2.1}$$

and, as condition for complete integrability of (2.1),

$$K_{ij}{}^{kl}=2k^2\delta^k_{[j}\delta^l_{i]}\,,\qquad (i,j,k\,l=1,\ldots,4)\tag{2.2}$$

(1) Y. Mimura and T. Iwatsuki, this Journal **8** (1938), 193, (W. G. No. 28).
T. Sibata, this Journal **8** (1938), 199, (W. G. No. 29).
H. Takeno, this Journal **8** (1938), 223, (W. G. No. 30).
K. Itimaru, this Journal **8** (1938), 239, (W. G. No. 31).

which gives the line element of the form

$$ds^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+(1-k^2r^2)dt^2\,, \tag{2.3}$$

putting $x^1=r$, $x^2=\theta$, $x^3=\varphi$, $x^4=t$ (r, θ, φ and t mean polar coordinates and time coordinate respectively).

In the procedure above we find that the origin of the coordinates is taken on a nebula arbitrary chosen, and for each coordinate system r, θ, φ, t, the cosmological laws are described in the same form. This means that the universe has the property to be called as *homogeneous* for set of observers resting on nebulae. In the following we shall investigate this homogeneous property in detail.

The homogeneous property of the universe can, at least, be regarded that for each observer (resting on a nebula), who assigns coordinates by the same method, the interval ds^2 must have the same form. Namely, if (r, θ, φ, t) and $(r', \theta', \varphi', t')$ are the coordinate systems for two different observers the interval ds^2 corresponding to $(r', \theta', \varphi', t')$ must have the same form as (2.3), i. e.

$$ds^2=-\frac{dr'^2}{1-k^2r'^2}-r'^2d\theta'^2-r'^2\sin^2\theta' d\varphi'^2+(1-k^2r'^2)dt'^2\,.$$

So that (r, θ, φ, t) and $(r', \theta', \varphi', t')$ are connected in the relations

$$\left.\begin{aligned} &r\to r'=R(r,\theta,\varphi,t)\,, && \theta\to\theta'=\Theta(r,\theta,\varphi,t)\,,\\ &\varphi\to\varphi'=\Phi(r,\theta,\varphi,t)\,, && t\to t'=T(r,\theta,\varphi,t)\,, \end{aligned}\right\} \tag{2.4}$$

which make (2.3) invariant. The actual forms of (2.4) are given[(1)], in the form of infinitesimal transformations, as follows:

$$\left.\begin{aligned}
&U_1\equiv\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\,, \qquad U_3\equiv\frac{\partial}{\partial\varphi}\,,\\
&U_2\equiv\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\,, \qquad U_4\equiv\frac{\partial}{\partial t}\,,\\
&U_5\equiv e^{kt}\left[\sqrt{1-k^2r^2}\cos\theta\frac{\partial}{\partial r}-\frac{\sqrt{1-k^2r^2}}{r}\sin\theta\frac{\partial}{\partial\theta}+\frac{kr}{\sqrt{1-k^2r^2}}\cos\theta\frac{\partial}{\partial t}\right]\\
&U_6\equiv e^{-kt}\left[\sqrt{1-k^2r^2}\cos\theta\frac{\partial}{\partial r}-\frac{\sqrt{1-k^2r^2}}{r}\sin\theta\frac{\partial}{\partial\theta}-\frac{kr}{\sqrt{1-k^2r^2}}\cos\theta\frac{\partial}{\partial t}\right]\\
&U_7\equiv e^{kt}\left[\sqrt{1-k^2r^2}\cos\varphi\sin\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\cos\varphi\cos\theta\frac{\partial}{\partial\theta}\right.\\
&\qquad\left.-\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\sin\varphi\frac{\partial}{\partial\varphi}+\frac{kr}{\sqrt{1-k^2r^2}}\cos\varphi\sin\theta\frac{\partial}{\partial t}\right]
\end{aligned}\right|$$

(1) Note I, p. 26.

$$\left.\begin{aligned}
U_8 &\equiv e^{-kt}\Big[\sqrt{1-k^2r^2}\cos\varphi\sin\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\cos\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&\qquad -\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\sin\varphi\frac{\partial}{\partial\varphi}-\frac{kr}{\sqrt{1-k^2r^2}}\cos\varphi\sin\theta\frac{\partial}{\partial t}\Big]\\
U_9 &\equiv e^{kt}\Big[\sqrt{1-k^2r^2}\sin\varphi\sin\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\sin\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&\qquad +\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\cos\varphi\frac{\partial}{\partial\varphi}+\frac{kr}{\sqrt{1-k^2r^2}}\sin\varphi\sin\theta\frac{\partial}{\partial t}\Big]\\
U_{10} &\equiv e^{-kt}\Big[\sqrt{1-k^2r^2}\sin\varphi\sin\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\sin\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&\qquad +\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\cos\varphi\frac{\partial}{\partial\varphi}-\frac{kr}{\sqrt{1-k^2r^2}}\sin\varphi\sin\theta\frac{\partial}{\partial t}\Big].
\end{aligned}\right\} \quad (2.5)$$

On the other hand we can show that *the fundamental equation for* ψ (2.1), *the cosmological law in our cosmology, is invariant by* (2.5),(1) *and conversely the most general fundamental equations for* ψ:

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=\Sigma_i\psi$$

which are invariant by (2.5) *are reducible to the form* (2.1).(2) Therefore, our cosmology itself is consistent with the homogeneous property of the universe.

§ 3. Classification of the transformations (2.5).

The transformations (2.5) can be classified in the following two groups G_4 and G_6:

$$G_4: \quad U_1,\ U_2,\ U_3,\ U_4,$$

$$G_6: \quad U_5,\ U_6,\ \ldots,\ U_{10}.$$

G_4 is a transformation group which makes $r=0$ invariant and represents rotations in 3-dimensional space and translation in the axis of time; that is to say G_4 is a transformation group between two observation-systems on the same nebula. G_6 is the transformation group which transforms the origin $r=0$ to the other; that is to say G_6 is a transformation group between two observation-systems each resting on different nebulae. Now we shall study G_6 in its fimite form.

Rewriting G_6 in the rectangular coordinates $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$, we have

(1) Note II and III, p. 33 and 36.

(2) Note III, p. 36.

$$\left.\begin{aligned}
U_5&=e^{kt}\left[\sqrt{1-k^2r^2}\,\frac{\partial}{\partial z}+\frac{kz}{\sqrt{1-k^2r^2}}\,\frac{\partial}{\partial t}\right],\\
U_6&=e^{-kt}\left[\qquad ,, \qquad - \qquad ,, \qquad\right],\\
U_7&=e^{kt}\left[\sqrt{1-k^2r^2}\,\frac{\partial}{\partial x}+\frac{kx}{\sqrt{1-k^2r^2}}\,\frac{\partial}{\partial t}\right],\\
U_8&=e^{-kt}\left[\qquad ,, \qquad - \qquad ,, \qquad\right],\\
U_9&=e^{kt}\left[\sqrt{1-k^2r^2}\,\frac{\partial}{\partial y}+\frac{ky}{\sqrt{1-k^2r^2}}\,\frac{\partial}{\partial t}\right],\\
U_{10}&=e^{-kt}\left[\qquad ,, \qquad - \qquad ,, \qquad\right].
\end{aligned}\right\}\tag{3.1}$$

The finite forms of the equations of the transformations generated by the operators above are,(1) τ, τ' being parameters,

$$\left.\begin{aligned}
&z'=z+e^{kt}\sqrt{1-k^2r^2}\,\tau, \qquad x'=x, \qquad y'=y,\\
&e^{kt'}=e^{kt}[1-k^2e^{2kt}\tau^2-2k^2ze^{kt}\tau/\sqrt{1-k^2r^2}\,]^{-\frac{1}{2}}
\end{aligned}\right\}\tag{3.2}$$

$$\left.\begin{aligned}
&z'=z+e^{-kt}\sqrt{1-k^2r^2}\,\tau', \qquad x'=x, \qquad y'=y,\\
&e^{-kt'}=e^{-kt}[1-k^2e^{-2kt}\tau^2-2k^2ze^{-kt}\tau/\sqrt{1-k^2r^2}\,]^{-\frac{1}{2}}
\end{aligned}\right\}\tag{3.3}$$

and the equations obtained by cyclic interchange of x, y, z in the equations above. Combining (3.2) and (3.3) in a single form, we have(2)

$$\left.\begin{aligned}
&x'=x, \qquad y'=y\\
&z'=z+\sqrt{1-k^2r^2}\,\{e^{kt}(1-k^2\tau\tau')\tau+e^{-kt}\tau'\}-2k^2z\tau\tau'\\
&e^{kt'}=e^{kt}\left[\frac{(1-k^2\tau\tau')^2-k^2e^{-2kt}\tau'^2-2k^2ze^{-kt}(1-k^2\tau\tau')\tau'/\sqrt{1-k^2r^2}}{1-k^2e^{2kt}\tau^2-2k^2ze^{kt}\tau/\sqrt{1-k^2r^2}}\right]^{\frac{1}{2}}
\end{aligned}\right\}\tag{3.4}$$

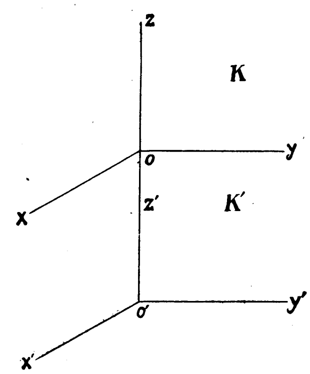
So we have the result: *the relations between the coordinates of the set of observers which are moving to each other* (G_6) *are decomposed into the following 3 kind:* (3.4) *and the equations obtained by cyclic interchange of* x, y, z *in* (3.4).

§ 4. Velocity of an observer which are moving to another observer.

Let us consider the case when the coordinates x, y, z, t of K system and x', y', z', t' of K' system are related by the equations (3.4), and obtain

(1) Note V, p. 40.
(2) Note V, p. 41.

the velocity v of the K system with reference to the K' system. This velocity v is obtained by putting $x=y=z=0$ and $\left[\frac{dz}{dt}\right]_{x=y=z=0}=0$ in the expression $\frac{dz'}{dt'}$, i. e.



$$v=\left[\frac{dz'}{dt'}\right]_{\substack{x=y=z=0\\ \left[\frac{dz}{dt}\right]_{x=y=z=0}=0}} . \tag{4.1}$$

Since

$$\frac{dz'}{dt'}=\frac{\frac{\partial z'}{\partial t'}+\frac{\partial z'}{\partial z}\frac{dz}{dt}}{\frac{\partial t'}{\partial t}+\frac{\partial t'}{\partial z}\frac{dz}{dt}},$$

we have, from (4.1),

$$\left.\begin{aligned} v&=\left[\frac{\partial z'}{\partial t}\Big/\frac{\partial t'}{\partial t}\right]_{x=y=z=0}\\ &=\frac{k\{e^{kt}(1-k^2\tau\tau')\tau-e^{-kt}\tau'\}\{1-2k^2\tau\tau'+k^4\tau^2\tau'^2-k^2e^{-2kt}\tau'^2\}}{(1-2k^2\tau\tau')\{1+k^2e^{2kt}\tau^2/(1-k^2e^{2kt}\tau^2)\}} .\end{aligned}\right\} \tag{4.2}$$

Expressing the right-hand side in terms of z' and t', we have[1]

$$v=\frac{-\tau'(1-k^2\tau\tau')e^{-kt'}+\tau e^{kt'}}{\tau'(1-k^2\tau\tau')e^{-kt'}+\tau e^{kt'}}\cdot kz'(1-k^2z'^2) . \tag{4.3}$$

This is the required form of velocity, and can be regarded as the velocity of the origin of K with respect to K', z' and t' being the coordinates of the origin of K-system in reference to K'-system. So we can say that the origin of K is in a motion along the axis of z' with the velocity v given by (4.3); and the same can be said for the velocity in the direction of the axes of x' and y'. Therefore, we have the result: *In reference to a system of coordinates* (K'), *the origin of another system of coordinates* (K) *connected by* G_6 *is in a motion in a radial direction, with velocity of the form:*

$$v=\frac{-pe^{-kt'}+qe^{kt'}}{pe^{-kt'}+qe^{kt'}}kr'(1-k^2r^2) \tag{4.4}$$

This shows that origins of all the coordinate-systems which are transformable by G_6 moves with the velocity (4.4) to each other, namely, *nebulae which are constituents of the universe, moves with the velocity* (4.4) *in the direction joining nebulae to each other.* This result coincide with that obtained by using the momentum density vector $u^l=\psi^\dagger A\gamma^l\psi$ of nebula in

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(1) Note VI, p. 42.

our cosmology.[1] So we see that Hubble's velocity distance relation in the wave geometrical cosmology is also deducible from the theory of group based on G_6.

§ 5. The fundamental equation for ψ invariant by G_6.

In § 3 and 4, we have seen that G_6 give the transformations between the observation-systems resting on the different nebulae which are in the relative motion satisfying the Hubble's velocity distance relation in the wave geometrical cosmology. Here we have a question: What are the phenomena observed as homogeneous in all the observation-systems connected by G_6? or in wave geometrical terms: What is the fundamental equation for ψ which is invariant by G_6?

By actual calculation[2] we see that the tensor invariant by G_6 is $\delta_i^j C$ (C=constant), so that the invariant equation for ψ is

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=(\alpha\gamma_i+\beta\gamma_i\gamma_5)\psi\,,$$

which can be reduced to the form

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi'=\frac{k}{2}\gamma_i\psi' \qquad \left(\alpha^2+\beta^2=\frac{k^2}{4}\right)$$

by a gauge transformation:

$$\psi=\left\{\beta+\left(\alpha-\frac{k}{2}\gamma_5\right)\right\}\psi'\,.$$

So we have as result: *The fundamental equation for ψ invariant by G_6 is reducible to the form* (2.1). That is to say *our cosmology is characterized by the homogeneous property by the observation-systems connected by G_6.*

Notes.

I. The group of motions admitted by the space whose line element is de Sitter form.

We shall obtain the continuous group G which preserves the line element of de Sitter form:

$$ds^2=-\frac{1}{1-k^2r^2}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+(1-k^2r^2)dt^2\,. \qquad (1)$$

Denote r, θ, φ, t by x^1, x^2, x^3, x^4 respectively, and let the infinitesimal transformations of the continuous group be

(1) K. Itimaru, ibid., 245.

(2) Note VII, p. 42.

$$x'^i = x^i + \xi^i(x)\delta\tau . \qquad (i=1, 2, 3, 4) \tag{2}$$

That the form (1) remains invariant for the group generated by (2) means that for the two systems of coordinates, x^i and x'^i, the corresponding fundamental tensors g_{ij} and g'_{ij} are the same functions of x^i and x'^i respectively, i. e.,

$$g'_{ij} = g_{ij}(x+\xi\delta\tau) , \qquad (i, j=1, 2, 3, 4) \tag{3}$$

or

$$g_{lm}(x) = g_{ij}(x+\xi\delta\tau)\frac{\partial x'^i}{\partial x^l}\frac{\partial x'^j}{\partial x^m} , \qquad (l, m=1, 2, 3, 4) . \tag{4}$$

From this, since the coefficient of $\delta\tau$ in the expansion of the right-hand side of (4) into power series of $\delta\tau$ must vanish identically, it follows that

$$\xi^h\frac{\partial g_{lm}}{\partial x^h} + \frac{\partial \xi^i}{\partial x^l}g_{im} + \frac{\partial \xi^j}{\partial x^m}g_{lj} = 0 . \tag{5}$$

By use of the covariant derivative, it is shown that (5) is equivalent to[1]

$$\nabla_i\xi_j + \nabla_j\xi_i = 0 , \qquad (\xi_i \equiv g_{ih}\xi^h) ,$$

or

$$\frac{\partial \xi_j}{\partial x^i} + \frac{\partial \xi_i}{\partial x^j} - 2\left\{ {h \atop ij} \right\}\xi_h = 0 , \tag{6}$$

where $\left\{ {h \atop ij} \right\}$ are Christoffel symbols:

$$\left\{ {h \atop ij} \right\} = \frac{1}{2}g^{hk}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right) . \tag{7}$$

Now we shall solve equation (6) and find the general solutions of ξ^i. By (1), the fundamental tensor g_{ij} has the form

$$\left.\begin{aligned} & g_{11} = -\frac{1}{1-k^2r^2} , \quad g_{22} = -r^2 , \quad g_{33} = -r^2\sin^2\theta , \quad g_{44} = -(1-k^2r^2) \\ & g_{ij} = 0 \quad \text{if} \quad i \neq j \end{aligned}\right\} \tag{8}$$

Hence, by (7), the Christoffel symbols $\left\{ {h \atop ij} \right\}$ can be calculated[2]; the results are

$$\left.\begin{aligned} & \left\{ {1 \atop 11} \right\} = -\frac{1}{2}\left(\log(1-k^2r^2)\right)' , \quad \left\{ {2 \atop 12} \right\} = \frac{1}{r} , \quad \left\{ {3 \atop 13} \right\} = \frac{1}{r} , \quad \left\{ {4 \atop 14} \right\} = \frac{1}{2}\left(\log(1-k^2r^2)\right)' \\ & \left\{ {1 \atop 22} \right\} = -r(1-k^2r^2) , \quad \left\{ {3 \atop 23} \right\} = \cot\theta , \quad \left\{ {1 \atop 33} \right\} = -r\sin^2\theta(1-k^2r^2) , \quad \left\{ {2 \atop 33} \right\} = -\sin\theta\cos\theta \\ & \left\{ {1 \atop 44} \right\} = \frac{1}{2}(1-k^2r^2)^2(1-k^2r^2)' , \qquad \text{the remaining} \quad \left\{ {i \atop jk} \right\} = 0 \end{aligned}\right\} \tag{9}$$

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(1) Eisenhart; Reimannian Geometry (1926), 234.

(2) The actual calculations are given in Eddington's The Mathematical Theory of Relativity (1930), 84.

where the accent denotes differentiation with respect to r.

Substituting (9) into (6) for each value of $i, j=1, 2, 3, 4$, we have the following ten equations:—

$$\frac{\partial\xi_1}{\partial r}+\frac{1}{2}\left(\log(1-k^2r^2)\right)'\xi_1=0, \quad (6.11) \qquad \frac{\partial\xi_2}{\partial\theta}+r(1-k^2r^2)\xi_1=0, \quad (6.22)$$

$$\frac{\partial\xi_3}{\partial\varphi}+r\sin^2\theta(1-k^2r^2)\xi_1+\sin\theta\cos\theta\xi_2=0, \quad (6.33) \qquad \frac{\partial\xi_4}{\partial t}+k^2r(1-k^2r^2)^2\xi_1=0, \quad (6.44)$$

$$\frac{\partial\xi_1}{\partial\theta}+\frac{\partial\xi_2}{\partial r}-2\frac{1}{r}\xi_2=0, \quad (6.12) \qquad \frac{\partial\xi_1}{\partial\varphi}+\frac{\partial\xi_3}{\partial r}-2\frac{1}{r}\xi_3=0, \quad (6.13)$$

$$\frac{\partial\xi_1}{\partial t}+\frac{\partial\xi_4}{\partial r}-\left(\log(1-k^2r^2)\right)'\xi_4=0, \quad (6.14) \qquad \frac{\partial\xi_2}{\partial\varphi}+\frac{\partial\xi_3}{\partial\theta}-2\cot\theta\xi_3=0, \quad (6.23)$$

$$\frac{\partial\xi_2}{\partial t}+\frac{\partial\xi_4}{\partial\theta}=0, \quad (6.24) \qquad \frac{\partial\xi_3}{\partial t}+\frac{\partial\xi_4}{\partial\varphi}=0. \quad (6.34)$$

From (6.11), we have

$$\xi_1=C(\theta, \varphi, t)\frac{1}{\sqrt{1-k^2r^2}} \qquad (10)$$

where $C(\theta, \varphi, t)$ does not contain r. Substituting (10) into (6.22) and (6.12), we have

$$\frac{\partial\xi_2}{\partial\theta}+r\sqrt{1-k^2r^2}\cdot C=0 \qquad (6.22)'$$

and

$$\frac{\partial\xi_2}{\partial r}+\frac{\partial C}{\partial\theta}\frac{1}{\sqrt{1-k^2r^2}}-\frac{2}{r}\xi_2=0 \qquad (6.12)'$$

Eliminating ξ_2 from (6.22)′ and (6.12)′, we have

$$\frac{d}{dr}(r\sqrt{1-k^2r^2})C-\frac{\partial^2C}{\partial\theta^2}\frac{1}{\sqrt{1-k^2r^2}}-\frac{2}{r}\cdot r\sqrt{1-k^2r^2}\,C=0,$$

or

$$\frac{\partial^2C}{\partial\theta^2}+C=0,$$

hence

$$C=A(\varphi, t)\cos\theta+B(\varphi, t)\sin\theta, \qquad (11)$$

where A and B do not contain r and θ. Substituting (11) into (6.22)′ and (6.12)′, and solving for ξ_2, we have

$$\xi_2=\{-A(\varphi, t)\sin\theta+B(\varphi, t)\cos\theta\}r\sqrt{1-k^2r^2}+r^2D(\varphi, t) \qquad (12)$$

where $D(\varphi, t)$ does not contain r and θ. Then (6.33) and (6.44) become, respectively,

$$\frac{\partial \xi_3}{\partial \varphi}+B(\varphi, t)\sin\theta r\sqrt{1-k^2r^2}+D(\varphi, t)r^2\sin\theta\cos\theta=0 \qquad (6.33)'$$

and

$$\frac{\partial \xi_4}{\partial t}+k^2r\{A(\varphi, \mathrm{t})\cos\theta+B(\varphi, t)\sin\theta\}\sqrt{1-k^2r^2}=0\,. \qquad (6.44)'$$

Differentiating (6.13) by φ and using (6.33)′, we have

$$\frac{\partial^2 A}{\partial \varphi^2}\cos\theta+\left\{\frac{\partial^2 B(\varphi, t)}{\partial \varphi^2}+B(\varphi, t)\right\}\sin\theta=0\,,$$

hence it follows that

$$\frac{\partial^2 A(\varphi, t)}{\partial \varphi^2}=0\,, \qquad \frac{\partial^2 B(\varphi, t)}{\partial \varphi^2}+B(\varphi, t)=0\,,$$

therefore,

$$A=T_1(t)\varphi+T_2(t)\,, \qquad B=T_3(t)\cos\varphi+T_4(t)\sin\varphi\,, \qquad (13)$$

where T_1, T_2, T_3, T_4 are functions of t alone. Similarly, differentiating (6.14) by t, and using (6.44)′, we have

$$\left(\frac{\partial^2 A}{\partial t^2}-k^2A\right)\cos\theta+\left(\frac{\partial^2 B}{\partial t^2}-k^2B\right)\sin\theta=0\,,$$

hence

$$\frac{\partial^2 A}{\partial t^2}-k^2A=0\,, \qquad \frac{\partial^2 B}{\partial t^2}-k^2B=0\,.$$

Therefore from (13), we have

$$T_i=a_ie^{kt}+b_ie^{-kt}\,, \qquad (i=1,\ldots\ldots, 4) \qquad (14)$$

where a_i and b_i are any constants. Also, from (6.23), differentiating (6.23) by φ, and using (6.33)′ and (13), we have

$$\frac{\partial^2 D}{\partial \varphi^2}+D=0\,,$$

from which

$$D=p(t)\cos\varphi+q(t)\sin\varphi\,, \qquad (15)$$

where $p(t)$ and $q(t)$ are functions of t alone.

Then, from (6.33)′, we have

$$\left.\begin{aligned}\xi_3=&\{-T_3\sin\varphi+T_4\cos\varphi\}r\sin\theta\sqrt{1-k^2r^2}\\&+r^2\sin\theta\cos\theta\{-p(t)\sin\varphi+q(t)\cos\varphi\}+P_3(r, \theta, t)\,,\end{aligned}\right\} \qquad (16)$$

where P_3 does not contain φ. And, from (6.44)′, we have

$$\left.\begin{aligned}\xi_4=&-r\sqrt{1-k^2r^2}\left[\left(\frac{dT_1}{dt}\varphi+\frac{dT_2}{dt}\right)\cdot\cos\theta\right.\\&\left.+\left(\frac{dT_3}{dt}\cos\varphi+\frac{dT_4}{dt}\sin\varphi\right)\sin\theta\right]+P_4(r, \theta, \varphi)\end{aligned}\right\} \qquad (17)$$

where P_4 does not contain t. Substituting ξ_1, ξ_2, ξ_3 and ξ_4, defined by (10), (12), (16) and (17), into the equations (6.13), (6.14), (6.23), (6.24), (6.34), respectively, we have

$$\frac{1}{\sqrt{1-k^2r^2}}T_1\cos\theta+\left(\frac{\partial P_3}{\partial r}-\frac{2}{r}P_3\right)=0\,, \tag{6.13$'$}$$

$$\frac{\partial P_4}{\partial r}-\left(\log(1-k^2r^2)\right)'P_4=0 \tag{6.14$'$}$$

$$-T_1 r\sqrt{1-k^2r^2}\sin\theta+\frac{\partial P_3}{\partial\theta}-2\cot\theta P_3=0 \tag{6.23$'$}$$

$$r^2\frac{\partial D}{\partial t}+\frac{\partial P_4}{\partial\theta}=0 \tag{6.24$'$}$$

$$\left.\begin{aligned}&\frac{\partial P_3}{\partial t}+r^2\sin\theta\cos\theta\{-p'(t)\sin\varphi+q'(t)\cos\varphi\}\\&\quad+\frac{\partial P_4}{\partial\varphi}-r\sqrt{1-k^2r^2}\cos\theta\frac{dT_1}{dt}=0\end{aligned}\right\} \tag{6.34$'$}$$

From (6.14)$'$, we have

$$P_4=(1-k^2r^2)Q_4(\theta,\varphi) \tag{18}$$

where $Q_4(\theta,\varphi)$ does not contain r and t. Substituting (18) into (6.24)$'$, it follows that

$$\frac{\partial Q_4}{\partial\theta}=0\,,\qquad \frac{\partial D}{\partial t}=0\,,$$

i. e.

$$Q_4=\Phi(\varphi)\,,\qquad p,q \text{ in (15)}=\text{constant.} \tag{19}$$

Therefore, (6.34)$'$ becomes

$$\frac{\partial P_3}{\partial t}+(1-k^2r^2)\Phi'(\varphi)-r\sqrt{1-k^2r^2}\cos\theta\frac{dT_1}{dt}=0\,,$$

from which we have

$$\Phi(\varphi)=c\varphi+d\,,\quad \text{(because of } \Phi''(\varphi)=0 \text{ from the equation above)}, \tag{20}$$

accordingly

$$P_3=-c(1-k^2r^2)t+r\sqrt{1-k^2r^2}\cos\theta T_1+N(r,\theta)\,. \tag{21}$$

Substituting (21) into (6.13)$'$, we have

$$\frac{\partial N}{\partial r}+\frac{2}{r}Ct-\frac{2}{r}N=0\,,$$

from which we have

$$c=0\,,\qquad \text{(since } N \text{ does not contain } t),$$

accordingly

$$N=r^2\Theta(\theta)\,,$$

hence $$P_3 = r\sqrt{1-k^2r^2}\cos\theta T_1 + r^2\Theta(\theta) = 0\,. \tag{22}$$

also (18) becomes $$P_4 = d\cdot(1-k^2r^2)\,. \tag{23}$$

Lastly, substituting (22) into (6.23)′, we have

$$-\frac{2}{\sin\theta} r\sqrt{1-k^2r^2}\,T_1 + r^2\Theta'(\theta) - 2\cot\theta\cdot r^2\Theta(\theta) = 0\,,$$

from which it follows that

$$a_1 = b_1 = 0\,, \qquad \text{(i. e., } T_1 = 0)$$
$$\Theta = e\sin^2\theta$$

where e is any constant. Hence

$$P_3 = er^2\sin^2\theta\,. \tag{24}$$

Putting together the results obtained above, we have

$$\left.\begin{aligned}
\xi_1 &= \frac{1}{\sqrt{1-k^2r^2}}[T_2\cos\theta + (T_3\cos\varphi + T_4\sin\varphi)\sin\theta] \\
\xi_2 &= r\sqrt{1-k^2r^2}[-T_2\sin\theta + (T_3\cos\varphi + T_4\sin\varphi)\cos\theta] \\
&\quad + r^2(p\cos\varphi + q\sin\varphi) \\
\xi_3 &= r\sqrt{1-k^2r^2}\sin\theta[-T_3\sin\varphi + T_4\cos\varphi] \\
&\quad + r^2\sin\theta\cos\theta(-p\sin\varphi + q\cos\varphi) + er^2\sin^2\theta \\
\xi_4 &= -kr\sqrt{1-k^2r^2}[(a_2e^{kt} - b_2e^{-kt})\cos\theta + \{a_3e^{kt} - b_3e^{-kt})\cos\varphi \\
&\quad + (a_4e^{kt} - b_4e^{-kt})\sin\varphi\}\sin\theta] + d\cdot(1-k^2r^2)\,,
\end{aligned}\right\}\tag{25}$$

where $$T_i = a_ie^{kt} + b_ie^{-kt} \qquad (i=2, 3, 4)\,,$$

and $a_i, b_i, p, q, d,$ and e are arbitrary constants. (25) is the general solution of (6). From (25), $\xi^l \equiv g^{lm}\xi_m$ are calculated as follows:

$$\left.\begin{aligned}
\xi^1 &= -\sqrt{1-k^2r^2}\,[T_2\cos\theta + (T_3\cos\varphi + T_4\sin\varphi)\sin\theta] \\
\xi^2 &= -\frac{\sqrt{1-k^2r^2}}{r}[-T_2\sin\theta + (T_3\cos\varphi + T_4\sin\varphi)\cos\theta] \\
&\quad -(p\cos\varphi + q\sin\varphi) \\
\xi^3 &= -\frac{\sqrt{1-k^2r^2}}{r\sin\theta}(-T_3\sin\varphi + T_4\cos\varphi) - \cot\theta(-p\sin\varphi + q\cos\varphi) - e \\
\xi^4 &= -\frac{kr}{\sqrt{1-k^2r^2}}[(a_2e^{kt} - b_2e^{-kt})\cos\theta + \{(a_3e^{kt} - b_3e^{-kt})\cos\varphi \\
&\quad + (a_4e^{kt} - b_4e^{-kt})\sin\varphi\}\sin\theta] + d
\end{aligned}\right\}\tag{26}$$

If we express the group G by the operator $\xi^l\frac{\partial}{\partial x^l}$, since (26) contains ten arbitrary constants (a_i, b_i ($i=2, 3, 4$), p, q, d, e), we have the following ten operators:

$$\left.\begin{aligned}
&(\text{coefficients of } p) && -\cos\varphi\frac{\partial}{\partial\theta}+\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\\
&(\quad ,, \quad q) && -\sin\varphi\frac{\partial}{\partial\theta}-\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\\
&(\quad ,, \quad e) && \frac{\partial}{\partial\varphi}\\
&(\quad ,, \quad d) && \frac{\partial}{\partial t}\\
&(\quad ,, \quad a_2) && e^{kt}\Big[-\sqrt{1-k^2r^2}\cos\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\sin\theta\frac{\partial}{\partial\theta}\\
&&& \quad -\frac{kr}{\sqrt{1-k^2r^2}}\cos\theta\frac{\partial}{\partial t}\Big]\\
&(\quad ,, \quad b_2) && e^{-kt}\Big[-\sqrt{1-k^2r^2}\cos\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\sin\theta\frac{\partial}{\partial\theta}\\
&&& \quad +\frac{kr}{\sqrt{1-k^2r^2}}\cos\theta\frac{\partial}{\partial t}\Big]\\
&(\quad ,, \quad a_3) && e^{kt}\Big[-\sqrt{1-k^2r^2}\cos\varphi\sin\theta\frac{\partial}{\partial r}-\frac{\sqrt{1-k^2r^2}}{r}\cos\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&&& \quad +\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\sin\varphi\frac{\partial}{\partial\varphi}-\frac{kr}{\sqrt{1-k^2r^2}}\cos\varphi\cdot\sin\theta\frac{\partial}{\partial t}\Big]\\
&(\quad ,, \quad b_3) && e^{-kt}\Big[-\sqrt{1-k^2r^2}\cos\varphi\sin\theta\frac{\partial}{\partial r}-\frac{\sqrt{1-k^2r^2}}{r}\cos\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&&& \quad +\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\sin\varphi\frac{\partial}{\partial\varphi}+\frac{kr}{\sqrt{1-k^2r^2}}\cos\varphi\cdot\sin\theta\frac{\partial}{\partial t}\Big]\\
&(\quad ,, \quad a_4) && e^{kt}\Big[\sqrt{1-k^2r^2}\sin\varphi\sin\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\sin\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&&& \quad +\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\cos\varphi\frac{\partial}{\partial\varphi}+\frac{kr}{\sqrt{1-k^2r^2}}\sin\varphi\cdot\sin\theta\cdot\frac{\partial}{\partial t}\Big]\\
&(\quad ,, \quad b_4) && e^{-kt}\Big[\sqrt{1-k^2r^2}\sin\varphi\sin\theta\frac{\partial}{\partial r}+\frac{\sqrt{1-k^2r^2}}{r}\sin\varphi\cos\theta\frac{\partial}{\partial\theta}\\
&&& \quad +\frac{\sqrt{1-k^2r^2}}{r\sin\theta}\cos\varphi\frac{\partial}{\partial\varphi}-\frac{kr}{\sqrt{1-k^2r^2}}\sin\varphi\cdot\sin\theta\cdot\frac{\partial}{\partial t}\Big],
\end{aligned}\right\} \tag{27}$$

which generate the continuous group G_{10}.

II. On the fundamental differential equations for ψ remaining invariant by transformations of coordinates.

We shall consider the properties of ψ when the fundamental differential equation for ψ remains invariant by transformations of coordinates. Let us consider the transformations of coordinates:

$$x'^i = f^i(x)\,, \tag{1}$$

which make the fundamental tensor g_{ij} invariant, i. e., $g'_{ij}=g_{ij}(x')$. Taking the 4–4 matrices γ_i, γ_5 satisfying the relations:

$$g_{ij}=\gamma_{(i}\gamma_{j)}\,, \qquad \gamma_{(i}\gamma_{5)}=0\,, \qquad \gamma_5\gamma_5=-1\,,$$

consider the fundamental differential equation for ψ:

$$\nabla_i\psi=\sum\nolimits_i\psi\,, \tag{2}$$

where

$$\nabla_i\equiv\frac{\partial}{\partial x^i}-\Gamma_i\,,$$

Γ_i being defined by

$$\frac{\partial\gamma_h}{\partial x^i}=\{{}^{\,j}_{hi}\}\gamma_j+\Gamma_i\gamma_h-\gamma_h\Gamma_i\,,$$

and $\sum_i$ being any 4–4 matrices expressed by sedenion as follows:

$$\sum\nolimits_i=A_i^{pq}\gamma_p\gamma_q+A_i+A_i^{\cdot 5}\gamma_5+A_i^{\cdot p}\gamma_p+A_i^{\cdot p5}\gamma_p\gamma_5 \qquad (A_i^{\cdot pq}=A_i^{\cdot qp})\,. \tag{3}$$

We say that the fundamental equations for ψ (2) are invariant by transformation (1) when the tensors $A_i^{\cdot pq}$, $A_i^{\cdot p}$, $A_i^{\cdot p5}$, vectors $A_i^{\cdot 5}$ and scalar A_i in (3) are invariant by (1); i. e., for the two systems of coordinates, x^i and x'^i, the corresponding quantities $A_i^{\cdot pq}, A_i^{\cdot p}\dots\dots A_i$ and $A_i'^{\cdot pq}, A_i'^{\cdot p}\dots\dots A'_i$ are the same functions of x^i and x'^i respectively ($A_i'^{\cdot pq}=A_i^{\cdot pq}(x')$ etc.). In such circumstances we shall examine the properties of (2).

By transformation (1), g_{ij}, $A_i^{\cdot pq}$, γ_i, Γ_i and ψ are transformed as

$$g_{ii}\rightarrow g'_{ij}=\frac{\partial x^l}{\partial' x^i}\frac{\partial x^m}{\partial x'^j}g_{lm}\,, \qquad A_i^{\cdot pq}\rightarrow A_i'^{\cdot pq}=\frac{\partial x^l}{\partial x'^i}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^q}{\partial x^n}A_l^{\cdot mn}\,, \qquad \text{etc.}$$

$$\gamma_i\rightarrow\gamma'_i=\frac{\partial x^l}{\partial x'^i}\gamma_l\,, \qquad \Gamma_i\rightarrow\Gamma'_i=\frac{\partial x^l}{\partial' x^i}\Gamma_l\,,$$

$$\gamma_5\rightarrow\gamma'_5=\gamma_5\,; \qquad \psi\rightarrow\psi'=\psi(x(x'))\,,$$

accordingly

$$\nabla_i\psi\equiv\Big(\frac{\partial}{\partial x^i}-\Gamma_i\Big)\psi\rightarrow\nabla'_i\psi'\equiv\Big(\frac{\partial}{\partial x'^i}-\Gamma'_i\Big)\psi'=\frac{\partial x^l}{\partial x'^i}\nabla_l\psi\,.$$

$$\sum\nolimits_i\rightarrow\Sigma'_i\equiv A_i'^{\cdot pq}\gamma'_p\gamma'_q+A'_i+A_i'^{\cdot 5}\gamma'_5+A_i'^{\cdot p}\gamma'_p+A_i'^{\cdot p5}\gamma'_p\gamma'_5=\frac{\partial x^l}{\partial x'^i}\sum\nolimits_l$$

and (1): $$\nabla_i\psi=\Sigma_i\psi\rightarrow\nabla'_i\psi'=\Sigma'_i\psi'.$$

But (1) makes g_{ij} invariant, i. e., $g'_{ij}=g_{ij}(x')$; hence

$$\gamma'_i\gamma'_j+\gamma'_j\gamma'_i=\gamma_i(x')\gamma_j(x')+\gamma_j(x')\gamma_i(x'),$$

therefore γ'_i and $\gamma_i(x')$ are related by the equations

$$\gamma'_i=S\gamma_i(x')S^{-1}, \tag{4}$$

where S is a certain 4–4 matrix which shall be determined according to (1). (Note that in general γ'_i are not equal to $\gamma_i(x')$).

Next we shall find that the relations between Γ'_i and $\Gamma_i(x')$. Γ'_i and $\Gamma_i(x')$ are determined from the equations:

$$\frac{\partial\gamma'_j}{\partial x'^i}=\{{}^{\,k}_{ij}\}'\gamma'_k+\Gamma'_i\gamma'_j-\gamma'_j\Gamma'_i$$

and

$$\frac{\partial\gamma_j(x')}{\partial x'^i}=\{{}^{\,k}_{ij}\}(x')\gamma_k(x')+\Gamma_i(x')\gamma_j(x')-\gamma_j(x')\Gamma_i(x'),$$

where $\{{}^{\,k}_{ij}\}(x')$ express the functions in which x^i in $\{{}^{\,k}_{ij}\}$ are replaced by x'^i. From the equations above, since

$$\{{}^{\,k}_{ij}\}'=\{{}^{\,k}_{ij}\}(x') \qquad \text{(because of } g'_{ij}=g_{ij}(x'))$$

and

$$\gamma'_j=S\gamma_j(x')S^{-1},$$

we have

$$\Gamma'_i=S\Gamma_i(x')S^{-1}+\frac{\partial S}{\partial x'^i}S^{-1}. \tag{5}$$

Therefore we have

$$\nabla'_i\psi'=S\Big(\frac{\partial}{\partial x'^i}-\Gamma_i(x')\Big)S^{-1}\psi'.$$

Hence, when (2) is invariant by (1), since

$$\Sigma'_i\psi'=S\Sigma_i(x')\cdot S^{-1}\psi',$$

the transformed equation of (2) becomes

$$\Big(\frac{\partial}{\partial x'^i}-\Gamma_i(x')\Big)S^{-1}\psi'=\Sigma_i(x')\cdot S^{-1}\psi'.$$

Comparing these equations with (2) we see that when ψ is the solution of (2), $S^{-1}\psi'$ is also the solution of (2) in which x^i are replaced by x'^i. Hence if we denote the solution ψ of (2) by $\psi(x, c)$, where $c\,(c^1, c^2, c^3, c^4)$ are integration constants, we have

$$S^{-1}\psi'=\psi(x';\ c')$$

or

$$\psi'\equiv\psi\big(x(x');\ c\big)=S\psi(x';\ c'); \tag{6}$$

where c' are in general not equal to c.

So we have the result: *That the differential equation for* ψ:

$$\left\{\frac{\partial}{\partial x^i}-\Gamma_i(x)\right\}\psi=\Sigma_i(x)\psi\,, \tag{2}$$

is invariant by transformations of coordinates $x'^i=f^i(x)$ *which make* g_{ij} *invariant* $\big($*i. e.* $g'_{ij}=g_{ij}(x')\big)$, *means that for the two systems of coordinates,* x^i *and* x'^i, *there exists a matrix* S *such that*

$$S^{-1}\gamma'_i S=\gamma_i(x')$$

and the corresponding $S^{-1}\psi'$ *and* $\psi(x')$ *satisfy the differential equations of the same form, i. e.*

$$\left\{\frac{\partial}{\partial x'^i}-\Gamma_i(x')\right\}S^{-1}\psi'=\Sigma_i(x')S^{-1}\psi'\,;$$

accordingly, in this case, the solution $\psi(x.c)$, $c(c_1,c_2,c_3,c_4)$ *being constants of integration, is related to* $\psi'\equiv\psi\big(x(x');c\big)$ *by the relation*

$$\psi'=S\psi(x;c)\,,$$

where c's are in general not equal to c.

From the result above we can prove: *When the fundamental equations for* ψ (2) *are invariant by* (1) *which makes* g_{ij} *invariant, for the two systems of coordinates,* x^i *and* x'^i, *the following scalars, vectors, and tensor, etc.:*

$$\left.\begin{aligned}&M\equiv\psi^\dagger A\psi\,,\quad N\equiv\psi^\dagger A\gamma_5\psi\,,\quad u^i\equiv\psi^\dagger A\gamma^i\psi\,,\\&u^i_{\cdot 5}\equiv\psi^\dagger A\gamma^i\gamma_5\psi\,,\quad u^{ij}\equiv\psi^\dagger A\gamma^{[i}\gamma^{j]}\psi\end{aligned}\right\} \tag{7}$$

(*A being a hermite matrix which makes* $A\gamma_i$ *hermite*) *satisfy the following relations:*

$$\left.\begin{aligned}&M'\equiv M\big(x(x');c\big)=M(x';c')\,,\qquad N'\equiv N\big(x(x');c\big)=N(x';c')\\&u'^i\equiv\frac{\partial x'^i}{\partial x^l}u^l(x;c)=u^i(x';c')\,;\qquad u'^i_{\cdot 5}=u'^i_{\cdot 5}(x',c')\\&u'^{ij}=u^{ij}(x';c')\,.\end{aligned}\right\} \tag{8}$$

Proof. By (6) and (4), we have

$$\left.\begin{aligned}&M'\equiv\psi'^\dagger A\psi'=\psi^\dagger(x',c')S^\dagger AS\psi(x';c')\\&u'^i\equiv\psi'^\dagger A\gamma'^i\psi'=\psi^\dagger(x';c')S^\dagger AS\gamma^i(x')\cdot\psi(x';c')\,,\\&\text{etc.}\end{aligned}\right\} \tag{9}$$

But, since $\frac{\partial x'^i}{\partial x^j}$ are real, A makes $A\gamma'_i \equiv \frac{\partial x^l}{\partial' x^i} A\gamma_l$ also hermite, i. e., by (4),

$$AS\gamma_i(x')S^{-1} = \left(AS\dot{\gamma}_i(x')S^{-1}\right)^\dagger ,$$

or

$$S^\dagger AS \cdot \gamma_i(x') = \gamma_i^\dagger(x) \cdot S^\dagger AS$$

which shows that $S^\dagger AS$ makes $S^\dagger AS \cdot \gamma_i(x')$ hermite, so that

$$S^\dagger AS = A(x')$$

Substituting this into (9), we have

$$M' = M(x'\,;\,c')\,, \qquad u'^i = u^i(x'\,;\,c') \qquad \text{etc.} \qquad \text{Q. E. D.}$$

[N. B.] These results are also easily shown by construction of the differential equations for the quantities defined by (7) from the differential equations for ψ (2). By the construction of the differential equations for the quantities defined by (7) we can easily see:

When the fundamental equations for ψ are invariant by the transformations of coordinates $x'^i = f^i(x)$, for the two systems of coordinates, x^i and x'^i, the corresponding differential equations[(1)] for u^l, $u^l_{\cdot 5}$, M, N, etc., and u'^l, u'^l_5, M', N', etc.,.... have the same form in x and x' respectively.

III. Fundamental differential equations for ψ invariant by G_{10}.

In cosmology in terms of wave geometry[(2)] we have obtained de Sitter type universe in which the line element is of de Sitter form and the fundamental equation for ψ is

$$\nabla_i \psi = (A_i + A_i^{\cdot 5}\gamma_5 + \alpha\gamma_i + \beta\gamma_i\gamma_5)\psi\,, \tag{1}$$

which is reduced to

$$\nabla_i \psi = \frac{k}{2}\gamma_i \psi\,. \tag{2}$$

We see that (2) is invariant by all the transformations of the continuous group G_{10} which make the line element of de Sitter form invariant. For $\frac{k}{2}$ is invariant for the group G_{10}. Conversely, now we shall examine the most general fundamental equation for ψ:

$$\nabla_i \psi = \textstyle\sum_i \psi \tag{3}$$

which is invariant for G_{10}. For this purpose, expanding $\sum_i$ by sedenion as (II, 3), we shall determine $A_i^{\cdot pq}$ etc. such that they are invariant by G_{10}.

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(1) Such equations are given in Spinor calculus II, T. Sibata, this Journal **9** (1939), 184, (W. G. No. 34).

(2) T. Sibata, this Journal **8** (1938), 206, (W. G. No. 29).

By actual calculation we see that[1] such $A_i^{;pq}$ etc. are given by

$$A_i^{;pq}=0\,, \qquad A_i^{;p}=\delta_i^{;p}\alpha$$

$$A_i=0\,, \qquad A_i^{;5}=0\,, \qquad A_i^{;p5}=\delta_i^{;p}\beta\,.$$

So that (2) becomes

$$\nabla_i\psi=(\alpha\gamma_i+\beta\gamma_i\gamma_5)\psi\,. \qquad (\alpha,\ \beta \text{ are constants}) \tag{4}$$

So we have the result: *The most general fundamental equation for* ψ *invariant by* G_{10} *is given by* (4).

By if we put

$$\psi=\{\beta+(\alpha-f)\gamma_5\}\overline{\psi}\,, \qquad (f^2=\alpha^2+\beta^2) \tag{5}$$

(3) is reduced to

$$\nabla_i\overline{\psi}=f\gamma_i\overline{\psi}\,, \qquad \left(f=\frac{k}{2}\right)$$

identical with (2). So we can say that *the invariant equations for* G_{10} *are reduced to the form* (2).

The property above is interpreted physically as follows: *The two observation-systems defined in* § 2 *are connected by transformations of* G_{10}, *therefore the property that the equation* (2) *is invariant by* G_{10} *means that for two observation-systems the fundamental equation for* ψ*—cosmological law—is expressed in the same form, i. e. our cosmology has the invariant property for observations.* So we can say that our cosmology defined by (2) is characterized by the invariant property for observations.

IV. Invariant vector and tensors by G_4 or G_{10}.

In order to obtain the differential equation for ψ invariant by G_4 or G_{10} in this section we shall obtain the vector and tensors which are invariant by G_4 or G_{10}. The group G_{10} is generated by infinitesimal transformations (I, 27), and G_4, the sub-group of G_{10} making $r=0$ invariant, is generated by the following operators:

$$\left.\begin{aligned} U_1&\equiv-\cos\varphi\frac{\partial}{\partial\theta}+\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\,,\\ U_2&\equiv-\sin\varphi\frac{\partial}{\partial\theta}-\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\,,\\ U_3&\equiv\frac{\partial}{\partial\varphi}\,,\\ U_4&\equiv\frac{\partial}{\partial t}\,. \end{aligned}\right\} \tag{1}$$

(1) Note IV, p. 37.

Writing the operators above in the form

$$U=\xi^l\frac{\partial}{\partial x^l}, \tag{2}$$

the infinitesimal transformations generated by U are written as

$$x'^l=x^l+\xi^l\delta\tau. \tag{3}$$

The vector A_i and tensors $A_i^{\cdot j}$, A_{ijk} which are invariant by (3) must satisfy the relations:

$$\left.\begin{aligned}&A_l\frac{\partial x^l}{\partial x'^i}\equiv A_i'=A_i(x'),\\&A_l^{\cdot m}\frac{\partial x^l}{\partial x'^i}\frac{\partial x'^j}{\partial x^m}\equiv A_i'^{\cdot j}=A_i^{\cdot j}(x'),\\&A_{lmn}\frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^j}\frac{\partial x^n}{\partial x'^k}\equiv A'_{ijk}=A_{ijk}(x'),\end{aligned}\right\} \tag{4}$$

respectively. Expanding these relations into power series of $\delta\tau$, we have

$$\left.\begin{aligned}&\xi^l\frac{\partial A_i}{\partial x^l}+\frac{\partial\xi^l}{\partial x^i}A_l=0, &(5.1)\\&\xi^l\frac{\partial A_i^{\cdot j}}{\partial x^l}+\frac{\partial\xi^l}{\partial x^i}A_l^{\cdot j}-\frac{\partial\xi^j}{\partial x^l}A_i^{\cdot l}=0, &(5.2)\\&\xi^l\frac{\partial A_{ijk}}{\partial x^l}+\frac{\partial\xi^l}{\partial x^i}A_{ljk}+\frac{\partial\xi^l}{\partial x^j}A_{ilk}+\frac{\partial\xi^l}{\partial x^k}A_{ijl}=0 &(5.3)\end{aligned}\right\} \tag{5}$$

respectively.

First we shall solve (5.1): For U_1, ξ^l have the forms:

$$\xi^1=0,\qquad \xi^2=-\cos\varphi,\qquad \xi^3=\cot\theta\sin\varphi,\qquad \xi^4=0,$$

hence (5.1) becomes

$$\left.\begin{aligned}&\cos\varphi\frac{\partial A_i}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial A_i}{\partial\varphi}+\delta_i^2\frac{1}{\sin^2\theta}\sin\varphi A_3\\&\qquad-\delta_i^3(\sin\varphi A_2+\cot\theta\cos\varphi A_3)=0.\end{aligned}\right\} \tag{5.1$'$}$$

Similarly, for U_2 and U_3,

$$\left.\begin{aligned}&\sin\varphi\frac{\partial A_i}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial A_i}{\partial\varphi}-\delta_i^2\frac{1}{\sin^2\theta}\cos\varphi A_3\\&\qquad+\delta_i^3(\cos\varphi A_3-\cot\theta\sin\varphi A_3)=0,\end{aligned}\right\} \tag{5.1$''$}$$

$$\frac{\partial A_i}{\partial\varphi}=0.$$

From (5.1)′ and (5.1)″, we have

$$\frac{\partial A_i}{\partial\theta}-\delta_i^2\cot\theta\, A_3=0\,,$$

$$\delta_i^2\frac{1}{\sin^2\theta}A_3-\delta_i^3A_2=0\,,$$

from which we have

$$A_2=A_3=0\,,$$

and A_1, A_4 are arbitrary functions of r and t. But from invariancy for U_4, A_1 and A_4 do not contain t, so that for (1) the solutions of (5.1) are given by

$$A_1(r)\,,\qquad A_2=A_3=0\,,\qquad A_4(r)\,. \tag{A}$$

Next, we shall solve (5.2). For U_1, U_2, and U_3, (5.2) becomes

$$\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)A_{hi}+\delta_h^2\frac{1}{\sin^2\theta}\sin\varphi A_{3i}-\delta_h^3(\sin\varphi A_{2i}+\cot\theta\cos\varphi A_{3i})$$

$$+\delta_i^2\frac{1}{\sin^2\theta}\sin\varphi A_{h3}-\delta_i^3(\sin\varphi A_{h2}+\cot\theta\cos\varphi A_{h3})=0\,,$$

$$\left(\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)A_{hi}-\delta_h^2\frac{1}{\sin^2\theta}\cos\varphi A_{3i}+\delta_h^3(\cos\varphi A_{2i}-\cot\theta\sin\varphi A_{3i})$$

$$-\delta_i^2\frac{1}{\sin^2\theta}\cos\varphi A_{h3}+\delta_i^3(\cos\varphi A_{h2}-\cot\theta\sin\varphi A_{h3})=0\,,$$

$$\frac{\partial}{\partial\varphi}A_{hi}=0\,.$$

From these equations we have

$$\frac{\partial}{\partial\theta}A_{hi}-\cot\theta(\delta_h^3A_{3i}-\delta_i^3A_{h3})=0\,, \tag{5.2$'$}$$

$$\frac{1}{\sin^2\theta}(\delta_h^2A_{3i}+\delta_i^2A_{h3})-(\delta_h^3A_{2i}+\delta_i^3A_{h2})=0\,. \tag{5.2$''$}$$

From (5.2)″ we have

$$A_{32}+A_{23}=0\,,\qquad \frac{1}{\sin^2\theta}A_{33}-A_{22}=0\,;$$

$$A_{31}=A_{34}=A_{13}=A_{12}=A_{21}=A_{24}=A_{43}=A_{42}=0\,,$$

and from (5.2)′ we have

$$A_{23}=\sin\theta B_{23}(r,t)\,,$$

$$A_{33}=\sin^2\theta B_{33}(r,t)\,,$$

and A_{11}, A_{14}, A_{22}, B_{23}, B_{33}, A_{41}, A_{44} do not contain θ. So that the surviving A_{ij} are the following:

$$\left.\begin{array}{lll} A_{11}(r)\,, & A_2^{\cdot 2}=A_3^{\cdot 3}(r)\,, & A_{41}(r)\,, \\ A_{14}(r)\,, & A_{23}=A_{32}=\sin\theta B_{23}(r)\,, & A_{44}(r)\,; \\ \text{the other} & A_{ij}=0\,. & \end{array}\right\} \quad \text{(B)}$$

By the same method, we have the solutions A_{ijk} of (5.3) as

$$\left.\begin{array}{lll} A_{114}(r)\,, & A_2^{\cdot 21}=A_3^{\cdot 31}(r)\,, & A_4^{\cdot 14}(r)\,, \\ A_{123}=\sin\theta B_{123}(r)\,, & A_2^{\cdot 24}=A_3^{\cdot 34}(r)\,; & A_{423}=\sin\theta R_{423}(r)\,, \\ & A_{231}=-A_{321}=\sin\theta R_{231}(r)\,, & \\ & A_{234}=-A_{324}=\sin\theta R_{234}(r)\,; & \\ \text{the other} & A_{ijk}=0\,. & \end{array}\right\} \quad \text{(C)}$$

The vector and tensors given by (A), (B), and (C) are the general solution invariant by group G_4. Further, for these vector and tensors to be also invariant by G_{10}, we can easily see that

$$A_i=0\,, \qquad A_i^{\cdot j}=a\delta_i^{\cdot j} \quad (a=\text{constant})\,, \qquad A_{ijk}=0\,,$$

the calculation being omitted.

V. Finite form of the group G_6 transforming the origin $r=0$.

The finite forms of the equations of the transformations generated by the operators:

$$U_5=\sqrt{1-k^2r^2}\,e^{kt}\frac{\partial}{\partial z}+\frac{kz}{\sqrt{1+k^2r^2}}e^{kt}\frac{\partial}{\partial t}\,, \tag{1}$$

are obtained by solving the equations

$$\left.\begin{array}{lll} \dfrac{dz'}{d\tau}=\sqrt{1-k^2r'^2}\,e^{kt'}\,, & \dfrac{dx'}{d\tau}=0\,, & \dfrac{dy'}{d\tau}=0\,, \\ \dfrac{dt'}{d\tau}=\dfrac{kz'}{\sqrt{1-k^2r'^2}}e^{kt'}\,, & (r'^2\equiv x'^2+y'^2+z'^2) & \end{array}\right\} \tag{2}$$

where τ is a parameter and chosen such that when $\tau=0$ x', y', z', t', coincide with x, y, z, t respectively. From (2), we have

$$x'=x\,, \qquad y'=y \tag{3}$$

and

$$\frac{dz'}{dt'}=\frac{1-k^2r'^2}{kz'} \qquad \text{or} \qquad \frac{kz'dz'}{1-k^2r'^2}=dt'\,,$$

from which

$$\sqrt{1-k^2r'^2}\cdot e^{kt'}=\text{constant},$$

i. e.

$$\sqrt{1-k^2r'^2}\,e^{kt'}=\sqrt{1-k^2r^2}\,e^{kt}\,.$$

Hence

$$\frac{dz'}{d\tau}=\sqrt{1-k^2r^2}\,e^{kt},$$

therefore

$$z'=z+\sqrt{1-k^2r^2}\,e^{kt}\tau. \qquad (4)$$

So that the transformations generated by (1) are given by (3) and (4), τ being a parameter. Similarly, from U_6, we have (3.3) (in p. 24).

Next we shall obtain the equations of transformations produced by combining (3.2) and (3.3). Writing (3.3) in the form:

$$\left.\begin{aligned} z''&=z'+\sqrt{1-k^2r'^2}\,e^{-kt'}\tau' \\ e^{-kt''}&=e^{-kt'}\left[1-k^2e^{-2kt'}\tau'^2-\frac{2k^2z'}{\sqrt{1-k^2r'^2}}\,e^{-kt'}\tau'\right]^{-\frac{1}{2}} \end{aligned}\right\} \qquad (5)$$

and substituting (3.2) into it, we have

$$\left.\begin{aligned} z''&=z+\sqrt{1-k^2r^2}\,e^{kt}\tau+\sqrt{1-k^2r^2}\,e^{kt-2kt'}\tau' \\ &=z+\sqrt{1-k^2r^2}\,\{e^{kt}\tau(1-k^2\tau\tau')+e^{-kt}\tau'\}-2k^2z\tau\tau' \\ &\qquad\left(\text{Because of } e^{kt-2kt'}=e^{-kt}\left(1-k^2e^{2kt}\tau^2-2z\frac{k^2}{\sqrt{1-k^2r^2}}\,e^{kt}\tau\right)\right) \\ e^{-kt''}&=\frac{1}{\sqrt{1-k^2r''^2}}\sqrt{1-k^2r'^2}\,e^{-kt'} \\ &=\frac{1}{\sqrt{1-k^2r''^2}}\sqrt{1-k^2r^2}\,e^{-kt}\left(1-k^2e^{2kt}\tau^2-2z\frac{k^2}{\sqrt{1-k^2r^2}}\,e^{kt}\tau\right). \end{aligned}\right\} \qquad (6)$$

But, since

$$\begin{aligned} 1-k^2r''^2&=1-k^2[r'^2+e^{-2kt'}(1-k^2r'^2)\tau'^2+2z'\sqrt{1-k^2r'^2}\,e^{-kt'}\tau'] \\ &=(1-k^2r^2)\left[1-k^2e^{2kt}\tau^2-\frac{2k^2z}{\sqrt{1-k^2r^2}}\,e^{kt}\tau-k^2e^{2kt-4kt'}\tau'^2\right. \\ &\qquad\left.-\frac{2k^2z'}{\sqrt{1-k^2r^2}}\,e^{kt-2kt'}\tau'\right], \end{aligned}$$

we have

$$\begin{aligned} e^{-kt''}&=\left[\left(1-k^2e^{2kt}\tau^2-\frac{2k^2z}{\sqrt{1-k^2r^2}}\,e^{kt}\tau\right)e^{4kt'-2kt}-k^2\tau'^2\right. \\ &\qquad\left.-2k^2\left(\frac{z}{\sqrt{1-k^2r^2}}+e^{kt}\tau\right)e^{2kt'-kt}\tau'\right]^{-\frac{1}{2}} \\ &=e^{-kt}\left[\frac{1}{1-k^2e^{2kt}\tau^2-2k^2ze^{kt}\tau/\sqrt{1-k^2r^2}}-k^2e^{-2kt}\tau'^2\right. \\ &\qquad\left.-2k^2\left(\frac{z}{\sqrt{1-k^2r^2}}+e^{kt}\tau\right)\frac{e^{-kt}\tau'}{1-k^2e^{2kt}\tau^2-2k^2ze^{kt}\tau/\sqrt{1-k^2r^2}}\right]^{-\frac{1}{2}} \end{aligned}$$

so that,

$$e^{kt''}=e^{kt}\left[\frac{1-2k^2\tau\tau'-2k^2ze^{-kt}\tau'/\sqrt{1-k^2r^2}}{1-k^2e^{2kt}\tau^2-2k^2ze^{kt}\tau/\sqrt{1-k^2r^2}}-k^2e^{-2kt}\tau'^2\right]^{-\frac{1}{2}}. \quad (7)$$

Therefore (6) and (7) give the required equations.

VI.

When $x=y=z=0$, (3.4) (in p. 24) becomes

$$z'=e^{kt}(\tau-k^2\tau^2\tau')+e^{-kt}\tau'$$

$$e^{kt'}=e^{kt}\left[\frac{1-2k^2\tau\tau'}{1-k^2e^{2kt}\tau^2}-k^2e^{-2kt}\tau'^2\right]^{\frac{1}{2}}$$

and hence

$$1-k^2z'^2=1-k^2\{e^{2kt}(\tau-k^2\tau^2\tau')^2+e^{-2kt}\tau'^2+2\tau\tau'(1-k^2\tau\tau')\}$$
$$=(1-k^2e^{2kt}\tau^2)(1-2k^2\tau\tau'+k^4\tau^2\tau'^2-k^2e^{-2kt}\tau'^2).$$

Using the above, (4.2) (in p. 25) is expressed as

$$v=\frac{e^{kt}(\tau-k^2\tau^2\tau')-e^{-kt}\tau'}{1-2k^2\tau\tau'}k(1-k^2z'^2). \quad (※)$$

Further, since

$$\frac{e^{kt}(\tau-k^2\tau^2\tau')-e^{-kt}\tau'}{1-2k^2\tau\tau'}=\frac{-\tau'(1-k^2\tau\tau')e^{kt'}+\tau e^{kt'}}{\tau'(1-k^2\tau\tau')e^{-kt'}+\tau e^{kt'}}\cdot z'$$

(※) becomes

$$v=\frac{-\tau'(1-k^2\tau\tau')e^{-kt'}+\tau e^{kt'}}{\tau'(1-k^2\tau\tau')e^{-kt'}+\tau e^{kt'}}\cdot kz'(1-k^2z'^2).$$

VII. Invariant tensors by Transformations G_6.

If we write (3.1) in the form $\xi^l\frac{\partial}{\partial x^l}$, then the invariant vector A^i and tensors $A_i^{\cdot j}$, $A_i^{\cdot jk}$ are defined by equations (IV, 5). (IV, 5.3) is expressed in the form:

$$\xi^l\frac{\partial A_i^{\cdot jk}}{\partial x^l}+A_l^{\cdot jk}\frac{\partial \xi^l}{\partial x^i}-A_i^{\cdot lk}\frac{\partial \xi^j}{\partial x^l}-A_i^{\cdot jl}\frac{\partial \xi^k}{\partial x^l}=0. \quad (1)$$

First we shall solve equation (1). For the operator U_6, since ξ^l has the forms

$$\xi^1=\xi^2=0, \qquad \xi^3=e^{kt}\sqrt{1-k^2r^2}, \qquad \xi^4=e^{kt}\frac{kz}{\sqrt{1-k^2r^2}},$$

(1) becomes

$$e^{kt}\left(\sqrt{1-k^2r^2}\,\frac{\partial}{\partial z}+\frac{kz}{\sqrt{1-k^2r^2}}\,\frac{\partial}{\partial t}\right)A_i^{\cdot jk}$$

$$+A_i^{\cdot jk}\left[\delta_i^a\left(\delta_3^l\frac{\partial\xi^3}{\partial x^a}+\delta_4^l\frac{\partial\xi^4}{\partial x^a}\right)+\delta_i^4k(\delta_3^l\xi^3+\delta_4^l\xi^4)\right]$$

$$-A_i^{\cdot ak}\left(\delta_3^j\frac{\partial\xi^3}{\partial x^a}+\delta_4^j\frac{\partial\xi^4}{\partial x^a}\right)-A^{\cdot 4k}k(\delta_3^j\xi^3+\delta_4^j\xi^4)$$

$$-A_i^{\cdot ja}\left(\delta_3^k\frac{\partial\xi^3}{\partial x^a}+\delta_4^k\frac{\partial\xi^4}{\partial x^a}\right)-A_i^{\cdot j4}k(\delta_3^k\xi^3+\delta_4^k\xi^4)=0\,,\quad(a=1,2,3)\,.$$

Similarly, from U_7 we get the equations by changing k to $-k$ above. Then, combining these twe equations, we have

$$\left.\begin{aligned}&e^{kt}\sqrt{1-k^2r^2}\,\frac{\partial A_i^{\cdot jk}}{\partial z}+\delta_i^aA_3^{\cdot jk}\frac{\partial\xi^3}{\partial x^a}+\delta_i^4A_4^{\cdot jk}k\xi^4-\delta_3^jA_i^{\cdot ak}\frac{\partial\xi^3}{\partial x^a}\\&\quad-\delta_4^jA_i^{\cdot 4k}k\xi^4-\delta_3^kA_i^{\cdot ja}\frac{\partial\xi^3}{\partial x^a}-\delta_4^kA_i^{\cdot j4}k\xi^4=0\,,\end{aligned}\right\}\qquad(2)$$

$$\left.\begin{aligned}&e^{kt}\frac{kz}{\sqrt{1-k^2r^2}}\,\frac{\partial A_i^{\cdot jk}}{\partial t}+\delta_i^aA_4^{\cdot jk}\frac{\partial\xi^4}{\partial x^a}+\delta_i^4kA_3^{\cdot jk}\xi^3-\delta_4^jA_i^{\cdot ak}\frac{\partial\xi^4}{\partial x^a}\\&\quad-\delta_3^jA_i^{\cdot 4k}k\xi^3-\delta_4^kA_i^{\cdot ja}\frac{\partial\xi^4}{\partial x^a}-\delta_3^kA_i^{\cdot j4}k\xi^3=0\end{aligned}\right\}\qquad(3)$$

By putting $i=4$, $j=b$ $(b=1,2)$, $k=c$ $(c=1,2)$, these equations become

$$e^{kt}\sqrt{1-k^2r^2}\,\frac{\partial A_4^{\cdot bc}}{\partial z}+A_4^{\cdot bc}k\xi^4=0\,,\qquad(4)$$

$$e^{kt}\frac{kz}{\sqrt{1-k^2r^2}}\,\frac{\partial A_4^{\cdot bc}}{\partial t}+A_3^{\cdot bc}k\xi^3=0\,.\qquad(5)$$

Similarly, by putting $i=d$ $(d=1,2,3)$, we have

$$e^{kt}\sqrt{1-k^2r^2}\,\frac{\partial A_d^{\cdot bc}}{\partial z}+A_3^{\cdot bc}\frac{\partial\xi^3}{\partial x^d}=0\,,\qquad(6)$$

$$e^{kt}\frac{kz}{\sqrt{1-k^2r^2}}\,\frac{\partial A_d^{\cdot bc}}{\partial t}+A_4^{\cdot bc}\frac{\partial\xi^4}{\partial x^d}=0\,.\qquad(7)$$

From (4) and (6) (for $d=3$), we have

$$\left.\begin{aligned}&A_4^{\cdot bc}=\sqrt{1-k^2r^2}\,B_4^{\cdot bc}(x,y,t)\\&A_3^{\cdot bc}=\frac{1}{\sqrt{1-k^2r^2}}B_3^{\cdot bc}(x,y,t)\,,\end{aligned}\right.\qquad(b,c=1,2)$$

where $B_4^{\cdot bc}$ and $B_3^{\cdot bc}$ do not contain z. Substituting these values into (5), we have

$$\frac{z}{(1-k^2r^2)}\frac{\partial B_4^{\cdot bc}}{\partial t}+B_3^{\cdot bc}=0,$$

from which it must follow that

$$\frac{\partial B_4^{\cdot bc}}{\partial t}=0, \qquad B_3^{\cdot bc}=0;$$

hence, from (7),

$$A_4^{\cdot bc}=0.$$

But, since from $U_8, \ldots, U_{10}$, by changing x, y, and z cyclically, the same relations would be obtained, it must follow that

$$A_a^{\cdot bc}=0, \qquad (a \neq b, c;\ a, b, c=1, 2, 3),$$

$$A_4^{\cdot bc}=0.$$

Further, by putting $i=4$, $j=b$ $(b=1, 2)$, $k=4$ in (2) and (3), we have

$$e^{kt}\sqrt{1-k^2r^2}\frac{\partial A_4^{\cdot b4}}{\partial z}=0, \tag{8}$$

$$e^{kt}\frac{kz}{\sqrt{1-k^2r^2}}\frac{\partial A_4^{\cdot b4}}{\partial t}+A_3^{\cdot b4}k\xi^3=0; \tag{9}$$

and, by putting $i=c$, $j=b$ $(b=1, 2)$, $k=4$,

$$e^{kt}\sqrt{1-k^2r^2}\frac{\partial A_c^{\cdot b4}}{\partial z}+A_3^{\cdot b4}\frac{\partial \xi^3}{\partial x^c}-A_c^{\cdot b4}k\xi^4=0, \tag{10}$$

$$e^{kt}\frac{kz}{\sqrt{1-k^2r^2}}\frac{\partial A_c^{\cdot b4}}{\partial t}+A_4^{\cdot b4}\frac{\partial \xi^4}{\partial x^c}-A_c^{\cdot 3c}\frac{\partial \xi^4}{\partial x^c}=0, \tag{11}$$

From (10) (for $c=3$), we have

$$\sqrt{1-k^2r^2}\frac{\partial A_3^{\cdot b4}}{\partial z}-2A_3^{\cdot b4}\frac{k^2z}{\sqrt{1-k^2r^2}}=0,$$

from which

$$A_3^{\cdot b4}=(1-k^2r^2)^{-1}B_3^{\cdot b4}(x, y, t).$$

Substituting this into (9), we have

$$z\frac{\partial A_4^{\cdot b4}}{\partial t}+B_3^{\cdot b4}=0.$$

But, by (8), $A_4^{\cdot b4}$ does not contain z; hence it must be true that

$$\frac{\partial A_4^{\cdot b4}}{\partial t}=0, \qquad B_3^{\cdot b4}=0.$$

Therefore

$$A_3^{\cdot b4}=0,$$

by which, from (11), we have

$$A_3^{\cdot b3} = A_4^{\cdot b4}.$$

Since from $U_8, \ldots, U_{10}$ similar relations would be obtained, we have

$$A_a^{\cdot b4} = 0 \qquad (a \neq b,\ a, b = 1, 2, 3)$$

$$A_a^{\cdot ba} = A_4^{\cdot b4} \qquad \text{(a not summed)}$$

$$\frac{\partial A_4^{\cdot a4}}{\partial t} = 0$$

Lastly, by putting $i=4$, $j=3$, $k=4$ in (2) and (3), we have

$$e^{kt}\sqrt{1-k^2r^2}\frac{\partial A_4^{\cdot 34}}{\partial z} - A_4^{\cdot a4}\frac{\partial \xi^3}{\partial x^a} = 0, \qquad (12)$$

and

$$A_3^{\cdot 34} = 0 \qquad \left(\text{because of } \frac{\partial A_3^{\cdot 34}}{\partial t} = 0\right)$$

But $A_4^{\cdot 34}$ does not contain x and y; hence, from (12), we have

$$A_4^{\cdot 14} = A_4^{\cdot 24} = 0.$$

So that, combining the results obtained above, we have all

$$A_i^{\cdot jk} = 0.$$

So we have the result: the invariant tensor $A_i^{\cdot jk}$ by G_6 must vanish. Similarly, we see that the tensor $A_i^{\cdot j}$ and vector A_i which are invariant by G_6 are given by

$$A_i^{\cdot j} = \delta_i^{\cdot j} c \qquad (c = \text{constant}),$$

$$A_i = 0.$$

This problem was discussed at a special Seminar of Geometry and Theoretical Physics in the Hirosima University, and carried on under the Scientific Research Fund of the Monbusyo, the Department of Education.

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