INDEPENDENCE OF QUADRATIC QUANTITIES IN A NORMAL SYSTEM

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In the study of independence of quadratic quantities in a normal system Craig-Sakamoto's lemma is a fundamental one [1]. We give here its simple proof, the method of which will also enable us in a short way to characterize quadratic quantities distributed in χ^2-distributions (Theorem 1) and to obtain an analogous result for Wishart's distributions (Theorem 14). In Part I we investigate the independence of quadratic quantities in a normal system to obtain results related to Cochran's theorem, and in Part II we extend results so obtained to the case of Wishart's distributions.

Part I. χ^2-distributions and Cochran's theorem

§ 1. Craig-Sakamoto's lemma.

In the following, unless otherwise stated, $A, B, C, \ldots$ stand for real symmetric matrices of order $k \cdot k$.

Lemma 1. The following two conditions are equivalent:

(i) $$AB = 0$$

(ii) $$|E-sA|\,|E-tB| = |E-sA-tB|$$

for arbitrary real scalars s, t.

Proof. Since evidently (i) implies (ii), so we shall show the converse. Let $K=(E-sA)^{-1} = E+sA+s^2A^2+\ldots$, then we may write (ii) as

$$|E-tB| = |E-tKB|$$

or

$$\sum_{n=1}^{\infty} \frac{t^n}{n} trB^n = \sum_{n=1}^{\infty} \frac{t^n}{n} tr\,(KB)^n .$$

Comparing the coefficients of s^2t^2 on both sides of this equation, we have

$$tr\,(ABAB+2A^2B^2) = 0 .$$

(1) A. T. Craig [1]. H. Sakamoto [1]. Numbers in brackets refer to the list of references at the end of this paper.

Therefore we obtain

$$tr\,(AB+BA)^2+2\,tr\,(AB)(AB)' = 2\,tr\,(ABAB+2A^2B^2) = 0.$$

Since $(AB+BA)^2$, $(AB)(AB)'$ are semi-definite positive, it follows that $AB=0$, which completes the proof.

§ 2. χ^2-distribution.

Let $x_1, x_2, \dots, x_k$ be normally correlated variables with moment matrix M. Since M is semi-definite positive, there exists a uniquely determined semi-definite positive matrix N such that $N^2=M$, which we denote by $M^{\frac{1}{2}}$. Let $\mathfrak{x}$ stand for a column vector, $(x_1, x_2, \dots, x_k)$ and $\mathfrak{a}$ the mean vector of $\mathfrak{x}$, where i-th component of $\mathfrak{a}$ is the mean of x_i. We consider a quadratic quantity $\mathfrak{x}'A\mathfrak{x}$, the m. g. f. $\varphi(\theta)$ of which is given by

$$(1) \qquad |E-2\theta A_1|^{-\frac{1}{2}} \exp \mathfrak{a}'\left\{A+2\theta^2 AM^{\frac{1}{2}}(E-2\theta A_1)^{-1}M^{\frac{1}{2}}A\right\}\mathfrak{a},$$

where $A_1=M^{\frac{1}{2}}AM^{\frac{1}{2}}$. Then $\mathfrak{x}'A\mathfrak{x}$ is distributed in a χ^2-distribution with r degrees of freedom if and only if $\varphi(\theta)=(1-2\theta)^{-\frac{r}{2}}$, that is,

$$(2) \qquad |E-2\theta A_1| = (1-2\theta)^r$$

$$(3) \qquad \mathfrak{a}'\left\{A+2\theta^2 AM^{\frac{1}{2}}(E-2\theta A_1)^{-1}M^{\frac{1}{2}}A\right\}\mathfrak{a} = 0$$

hold. And (2) is written as

$$\sum_1^\infty \frac{(2\theta)^n}{n} trA_1^n = \sum_1^\infty \frac{(2\theta)^n}{n} r,$$

or

$$(4) \qquad trA_1^n = r, \quad n = 1, 2, \dots.$$

By making use of (4) we get $tr\,(A_1-A_1^2)^2=r-2r+r=0$, so we obtain

$$(5) \qquad M^{\frac{1}{2}}AM^{\frac{1}{2}} \quad \text{is a projection.}$$

And (5) reduces (3) to

$$(6) \qquad \mathfrak{a}'A\mathfrak{a} = 0, \quad \mathfrak{a}'AMA\mathfrak{a} = 0 \quad (\text{or } MA\mathfrak{a} = 0).$$

Since $\mathfrak{a}'A\mathfrak{x}$ has the mean $\mathfrak{a}'A\mathfrak{a}$ and the variance $\mathfrak{a}'AMA\mathfrak{a}$, so (6) is equivalent to

$$(7) \qquad \mathfrak{x}'A\mathfrak{x} = (\mathfrak{x}'-\mathfrak{a}')\,A\,(\mathfrak{x}-\mathfrak{a}) \quad \text{holds with a probability 1.}$$

If $|M|\neq 0$, that is, for non-singular case (5), (6), (7) are equivalent to respectively

$$(8) \qquad AMA = A$$

(9) $$A\mathfrak{a} = 0$$

(10) $$\mathfrak{x}'A\mathfrak{x} = (\mathfrak{x}'-\mathfrak{a}')A(\mathfrak{x}-\mathfrak{a}).$$

Thus we obtain the following

THEOREM 1. A necessary and sufficient condition that $\mathfrak{x}'A\mathfrak{x}$ is distributed in a χ^2-distribution with r degrees of freedoms is that $M^{\frac{1}{2}}AM^{\frac{1}{2}}$ is a projection (or $MAMAM=MAM$) and $\mathfrak{x}'A\mathfrak{x}=(\mathfrak{x}'-\mathfrak{a}')A(\mathfrak{x}-\mathfrak{a})$ holds with a probability 1 (or $\mathfrak{a}'A\mathfrak{a}=0$, $\mathfrak{a}'AMA\mathfrak{a}=0$) and $r=tr(AM)$, or $r=rank\ of\ MAM$.

For non-singular case [1] the condition is reduced to that $A=AMA$, $A\mathfrak{a}=0$, and $r=rank\ of\ A$.

Example [2]. Let A be a rectangular matrix of rank r, and of order $k\cdot r$, where $r\leq k$. And assume that $|M|\neq 0$, and $\mathfrak{a}=0$. Then $\mathfrak{x}'A(A'MA)^{-1}A'\mathfrak{x}$ is distributed in a χ^2-distribution with r degrees of freedoms. For $\{A(A'MA)^{-1}A'\}M\{A(A'MA)^{-1}A'\}=A(A'MA)^{-1}A'$, and $tr(MA(A'MA)^{-1}A')$ $=tr(A'MA(A'MA)^{-1})=r$, since $(A'MA)(A'MA)^{-1}$ is a unit matrix of order $r\cdot r$.

Next consider a quantity $\mathfrak{x}'A\mathfrak{x}+2\mathfrak{q}'\mathfrak{x}+c$, where $\mathfrak{q}$ is a constant column vector and c is a constant. Such a quantity is regarded as quadratic when we introduce a new variable $x_{k+1}=1$, so Theorem I shows

THEOREM 2. A necessary and sufficient condition that $\mathfrak{x}'A\mathfrak{x}+2\mathfrak{q}'\mathfrak{x}+c$ is distributed in a χ^2-distribution with r degree of freedoms is that $M^{\frac{1}{2}}AM^{\frac{1}{2}}$ is a projection, $\mathfrak{x}'A\mathfrak{x}+2\mathfrak{q}'\mathfrak{x}+c=(\mathfrak{x}'-\mathfrak{a}')A(\mathfrak{x}-\mathfrak{a})$ holds with a probability 1, and $r=tr(AM)$, or $r=rank\ of\ MAM$.

For non-singular case the condition is reduced to that $AMA=A$, $\mathfrak{x}'A\mathfrak{x}+2\mathfrak{q}'A+c=(\mathfrak{x}'-\mathfrak{a}')A(\mathfrak{x}-\mathfrak{a})$, and $r=rank\ of\ A$.

§ 3. Independence of quadratic quantities.

We consider quadratic quantities $Q=\mathfrak{x}'A\mathfrak{x}$, $R=\mathfrak{x}'B\mathfrak{x}$, and let $\varphi_1(\theta_1)$, $\varphi_2(\theta_2)$, $\varphi(\theta_1,\theta_2)$ be m. g. f. of $Q, R, (Q, R)$ respectively. Their expressions are similar to § 2 (1). Then Q, R are independent if and only if

$$\varphi_1(\theta_1)\,\varphi_2(\theta_2) = \varphi(\theta_1,\theta_2),$$

or

(1) $$|E-2\theta_1A_1|\,|E-2\theta_2B_1| = |E-2\theta_1A_1-2\theta_2B_1|$$

(2) $$\mathfrak{a}'\Big\{\theta_1^2AM^{\frac{1}{2}}(E-2\theta_1A_1)^{-1}M^{\frac{1}{2}}A+\theta_2^2BM^{\frac{1}{2}}(E-2\theta_2B_1)^{-1}M^{\frac{1}{2}}B\Big\}\mathfrak{a}$$
$$=\mathfrak{a}'\Big\{(\theta_1A+\theta_2B)\,M^{\frac{1}{2}}(E-2\theta_1A_1-2\theta_2B_1)^{-1}M^{\frac{1}{2}}(\theta_1A+\theta_2B)\Big\}\mathfrak{a}$$

(1) H. Sakamoto [2]. Theorem II.
(2) H. Cramér [1]. 432.

hold, where

$$A_1 = M^{\frac{1}{2}}AM^{\frac{1}{2}}, \quad B_1 = M^{\frac{1}{2}}BM^{\frac{1}{2}}.$$

By making use of Lemma 1 it is not difficult to see that (1), (2) are equivalent to the following conditions:

(3) $$MAMBM = 0$$

(4) $$MAMB\mathfrak{a} = 0 \quad MBMA\mathfrak{a} = 0$$

(5) $$\mathfrak{a}'AMB\mathfrak{a} = 0.$$

Furthermore if $|M| \neq 0$, then these conditions are reduced to $AMB=0$. Thus we obtain

THEOREM 3. It is necessary and sufficient for Q, R to be independent that (3), (4), (5) hold. And for non-singular case they are independent if and only if $AMB=0$ (1).

If Q, R are distributed in χ^2-distributions, then $Q=(\mathfrak{x}'-\mathfrak{a}')A(\mathfrak{x}-\mathfrak{a})$, $R=(\mathfrak{x}'-\mathfrak{a}')B(\mathfrak{x}-\mathfrak{a})$ hold with a probability 1. Therefore Theorem 3 implies

THEOREM 4. If Q, R are distributed in χ^2-distributions, then it is necessary and sufficient for Q, R to be independent that $MAMBM=0$ holds.

If we introduce a new variable $x_{k+1}=1$, then Theorem 3 shows

THEOREM 5. It is necessary and sufficient for $\mathfrak{x}'A\mathfrak{x}+2\mathfrak{q}'\mathfrak{x}+c$, $\mathfrak{x}'B\mathfrak{x}+2\mathfrak{r}'\mathfrak{x}+d$ to be independent that for case $\mathfrak{a}=0$, conditions $MAMBM=0$, $MAM\mathfrak{r}=0$, $MBM\mathfrak{q}=0$, and $\mathfrak{q}'M\mathfrak{r}=0$ hold, and for case $|M| \neq 0$, conditions $AMB=0$, $BM\mathfrak{q}=0$, $AM\mathfrak{r}=0$ and $\mathfrak{q}'M\mathfrak{r}=0$ hold.

As a consequence of this Theorem we get

THEOREM 6. Assume that $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'B\mathfrak{x}$ are independent and we are in non singular case. If B, C are semi-definite positive, and $B \geqq C$, then $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'C\mathfrak{x}$ are independent.

Proof. Theorem 5 shows that $\mathfrak{x}'A\mathfrak{x}$ is independent to $\mathfrak{y}$, where $\mathfrak{y}=B\mathfrak{x}$. Since $\mathfrak{x}'C\mathfrak{x}$ is expressed as a quadratic form of $\mathfrak{y}$, it follows that $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'C\mathfrak{x}$ are independent.

§ 4. Cochran's Theorem.

To carry out the proofs of the following theorems we need the following

Lemma 2. Let $C=A+B$, where C is a projection. Then

(1) A, B are projections, if and only if $AB=0$.

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(1) H. Sakamoto [2]. Theorem I.

(2) If A is a projection and B is semi-definite positive, then B is a projection.

(3) If *rank of* A + *rank of* $B \leq$ *rank of* C, then A, B are projections. We first show

THEOREM 7. Let $C=A+B$. Assume that $\mathfrak{x}'C\mathfrak{x}$, $\mathfrak{x}'A\mathfrak{x}$ are distributed in χ^2-distributions and that B is semi-definite positive. Then $\mathfrak{x}'B\mathfrak{x}$ is distributed in a χ^2-distribution and $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'B\mathfrak{x}$ are independent.

Proof. We put $A_1=M^{\frac{1}{2}}AM^{\frac{1}{2}}$, $B_1=M^{\frac{1}{2}}BM^{\frac{1}{2}}$, and $C_1=M^{\frac{1}{2}}CM^{\frac{1}{2}}$, then we have $C_1=A_1+B_1$. From the assumption B_1 becomes semi-definite positive. It follows from Theorem 1 that A_1, C_1 are projections, whence by Lemma 2 B_1 is also a projection and $A_1B_1=0$. Since by Theorem 1 $\mathfrak{x}'C\mathfrak{x}=(\mathfrak{x}'-\mathfrak{a}')C(\mathfrak{x}-\mathfrak{a})$, $\mathfrak{x}'A\mathfrak{x}=(\mathfrak{x}'-\mathfrak{a}')A(\mathfrak{x}-\mathfrak{a})$ hold with a probability 1, so also for $\mathfrak{x}'B\mathfrak{x}$, whence $\mathfrak{x}'B\mathfrak{x}$ is distributed in a χ^2-distribution. Theorem 4 shows that $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'B\mathfrak{x}$ are independent.

COROLLARY. Let $A=A_1+\dots+A_n$. Assume that A, A_i $i=1, 2, \dots, n-1$ are distributed in χ^2-distributions and that A_n is semi-definite positive. Then $\mathfrak{x}'A_n\mathfrak{x}$ is distributed in a χ^2-distribution, and $\mathfrak{x}'A_i\mathfrak{x}$ $i=1, 2, \dots, n$ are independent.

The proof is similar to that of the theorem.

As a consequence of this corollary we have

THEOREM 8. Let $A=A_1+\dots+A_n$. If $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'A_i\mathfrak{x}$ $i=1, 2, \dots, n$ are distributed in χ^2-distributions, then $\mathfrak{x}'A_i\mathfrak{x}$ are independent.

On account of the addition theorem(1) of χ^2-distributions $\mathfrak{x}'A\mathfrak{x}+\mathfrak{x}'B\mathfrak{x}$ is distributed in a χ^2-distribution if $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'B\mathfrak{x}$ are independent and distributed in χ^2-distributions. Conversely

THEOREM 9. Let $C=A+B$. Assume that C is distributed in a χ^2-distribution, and that $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'B\mathfrak{x}$ are independent. Then $\mathfrak{x}'A\mathfrak{x}$, $\mathfrak{x}'B\mathfrak{x}$ are distributed in χ^2-distributions if and only if $\mathfrak{a}'A\mathfrak{a}=\mathfrak{a}'B\mathfrak{a}=0$. The last conditions are trivial when A, B are semi-definite positive or $\mathfrak{a}=0$ or when we are in non-singular case.

Proof. Put $A_1=M^{\frac{1}{2}}AM^{\frac{1}{2}}$, $B_1=M^{\frac{1}{2}}BM^{\frac{1}{2}}$, $C_1=M^{\frac{1}{2}}CM^{\frac{1}{2}}$. Theorem 4 and Lemma 2 show that A_1, B_1 are projections and $\mathfrak{a}'AMB\mathfrak{a}=0$, whence $\mathfrak{a}'AMA\mathfrak{a}+\mathfrak{a}'BMB\mathfrak{a}=\mathfrak{a}'(A+B)M(A+B)\mathfrak{a}=\mathfrak{a}'CMC\mathfrak{a}=0$, so that we have $\mathfrak{a}'AMA\mathfrak{a}=0$, $\mathfrak{a}'BMB\mathfrak{a}=0$. Then by Theorem 1 we have the first part of the theorem. If A, B are semi-definite positive, then $\mathfrak{a}'A\mathfrak{a}+\mathfrak{a}'B\mathfrak{a}=\mathfrak{a}'C\mathfrak{a}=0$, whence $\mathfrak{a}'A\mathfrak{a}$

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(1) H. Cramér [1], 234. S. S. Wilks [1], 105.

$=0$, $\mathfrak{a}'B\mathfrak{a}=0$. For non-singular case $\mathfrak{a}'AMA\mathfrak{a}=0$, $\mathfrak{a}'BMB\mathfrak{a}=0$ imply $A\mathfrak{a}=B\mathfrak{a}=0$.

Corollary. Let $A=A_1+\dots+A_n$. Assume that A is distributed in a χ^2-distribution and that A_i, $i=1,\dots,n$ are independent. Then $\mathfrak{x}'A_i\mathfrak{x}$ $i=1,\dots,n$ are distributed in χ^2-distributions if and only if $\mathfrak{a}'A_i\mathfrak{a}=0$ $i=1,\dots,n$.

The last condition is trivial when A_i $i=1,\dots,n$ are semi-definite positive or $\mathfrak{a}=0$ or when we are in non-singular case.

The proof is similar to that of the theorem.

Next we turn to Cochran's theorem

THEOREM 10. Let $A=A_1+\dots+A_n$. Assume that $\mathfrak{x}'A\mathfrak{x}$ is distributed in a χ^2-distribution and that $\sum(\textit{rank of } A_i)\leq \textit{rank of } A$. If we are in non-singular case, then $\mathfrak{x}'A_i\mathfrak{x}$, $i=1,\dots,n$ are distributed in χ^2-distributions and are independent.

Proof. Put $P=M^{\frac{1}{2}}AM^{\frac{1}{2}}$, $P_i=M^{\frac{1}{2}}A_iM^{\frac{1}{2}}$. Then $P=P_1+\dots+P_n$. Theorem 1, 3 and Lemma 2 show the conclusion.

Similarly we have

THEOREM 11. Let $A=A_1+\dots+A_n$. Assume that $\mathfrak{x}'A\mathfrak{x}$ is distributed in a χ^2-distribution and $\mathfrak{a}=0$, and that $\sum(\textit{rank of } MA_iM)\leq \textit{rank of } MAM$. Then $\mathfrak{x}'A_i\mathfrak{x}$ $i=1,\dots,n$ are distributed in χ^2-distributions and are independent.

§ 5. Some consequences of the preceding §§

In this § let M be a diagonal matrix. We may assume $M=E$ (without loss of generality). Then $\mathfrak{x}'A\mathfrak{x}$ is distributed in a χ^2-distribution if and only if A is a projection and when this is the case, then

degree of freedom of $\mathfrak{x}'A\mathfrak{x}$ $=$ $tr\,A$ $=$ *rank of* A=*dimension of range* A.

THEOREM 12. Let $A=B+C$. If A is distributed in a χ^2-distribution and $\mathfrak{x}'A\mathfrak{x}=\mathfrak{x}'B^2\mathfrak{x}+\mathfrak{x}'C^2\mathfrak{x}$, then $\mathfrak{x}'B\mathfrak{x}$, $\mathfrak{x}'C\mathfrak{x}$ are distributed in χ^2-distributions, and are independent.

Proof. On account of Theorem 3 it suffices to show that $BC=0$. Since $\mathfrak{x}'(E-A)\mathfrak{x}$, $\mathfrak{x}'A\mathfrak{x}$ are independent, it follows from Theorem 7 that $(E-A)B^2=B^2(E-A)=0$ whence, $(E-A)B=B(E-A)=0$, therefore $AB=BA$, which implies $CB=BC$. Since $(B+C)^2=A^2=A=B+C$, that is, $BC+CB=0$, whence we have $BC=0$.

In the next theorem we shall not assume that A_i are symmetric.

(1) S. S. Wilks [1], 107.

THEOREM 13. Let $A=\sum_{i=1}^{n} A_i$. If $\mathfrak{x}'A\mathfrak{x}$ is distributed in a X^2-distribution and $\mathfrak{x}'A\mathfrak{x}=\sum_{i=1}^{n}\mathfrak{x}'A_i'A_i\mathfrak{x}$ holds, then the following conditions are equivalent:

(i) $\mathfrak{x}'A_i'A_i\mathfrak{x}$ $i=1,\dots,n$ are distributed in X^2-distributions,

(ii) $\mathfrak{x}'A_i'A_i\mathfrak{x}$ $i=1,\dots,n$ are independent,

(iii) $A_i\mathfrak{x}$ $i=1,\dots,n$ are independent,

(iv) $A_iA_j'=0$ for $i\neq j$,

(v) $AA_i=A_i$, $A_i'A_j=0$ for $i\neq j$,

(vi) A_i $i=1,\dots,n$ are projections,

(vii) $AA_i=A_i$, $A_iA_j=0$ for $i\neq j$.

Proof. Owing to Theorem 3 $\mathfrak{x}'(E-A)\mathfrak{x}$, $\mathfrak{x}'A\mathfrak{x}$ are independent. Therefore, by Theorem 7, we have $A_i'A_i(E-A)=(E-A)A_i'A_j=0$, that is $A_i=A_iA$, and $A_i'=AA_i'$. Then

(i) $\rightarrow$ (ii) is immediate by Theorem 8.

(ii) $\rightarrow$ (iv) is immediate by Theorem 5.

(iv) $\rightleftarrows$ (iii) is immediate by Theorem 5.

(iv) $\rightarrow$ (vi): $A_i=A_iA=A_iA'=A_i\sum A_j'=A_iA_i'$, whence A_i are projections.

(vi) $\rightarrow$ (v) follows from Lemma 2.

(v) $\rightarrow$ (vii): $A_i=AA_i=(\sum A_j')A_i=A_i'A_i$, whence A_i are symmetric, so that $A_iA_j=0$ $(i\neq j)$, $AA_i=A_i$.

(vii) $\rightarrow$ (i): $A_i=AA_i=(\sum A_j)A_i=A_i^2$, whence $A_i=AA_i=(\sum A_j'A_j)A_i=A_i'A_i$, so that A_i and also $A_i'A_i$ are projections. Theorem 8, shows that (vii) implies (i).

Part II. Wishart's Distributions

§ 6. Wishart's distributions

Let $\mathfrak{x}_\alpha=(x_{1\alpha}, x_{2\alpha},\dots,x_{k\alpha})$, $\alpha=1,\dots,n$ stand for independent column vectors, where $x_{1\alpha},\dots,x_{k\alpha}$ are normally correlated variables with the same non-singular moment matrix M. Let $\mathfrak{a}_\alpha$ be the mean vector of $\mathfrak{x}_\alpha$. Put $l_{ij}=\sum_{\alpha\beta} c_{\alpha\beta}x_{i\alpha}x_{j\beta}$, where C is a real symmetric matrix of order $n\cdot n$. We say that $(l_{11},\dots,l_{kk})$ is distributed in a Wishart's distribution $W_{r,n}(M)$ when m. g. f. $\varphi(\theta_{11},\theta_{12},\dots,\theta_{kk})$, or briefly $\varphi(\Theta)$, of $(l_{11}\dots l_{kk})$ is equal to [1]

$$|\underset{k}{E}-2\Theta M|^{-\frac{r}{2}} \quad \text{or} \quad |\underset{k}{E}-2M^{\frac{1}{2}}\Theta M^{\frac{1}{2}}|^{-\frac{r}{2}},$$

(1) S. S. Wilks [1], 232.

where $\underset{k}{E}$ stand for a unit matrix of order $k\cdot k$ and $\Theta=(\theta_{ij})$, $\theta_{ij}=\theta_{ji}$.

When we regard $x_{11}x_{21}\dots x_{k1}$, $x_{12}x_{22}\dots x_{k2}, \dots, x_{1n}x_{2n}\dots x_{kn}$, which we denote by a column vector $\mathfrak{z}$, as normally correlated variables, they have the moment matrix $\underset{n}{E}\times M$, (Kronecker product), and $\mathfrak{z}$ has the mean $\mathfrak{a}=(\mathfrak{a}_1\dots\mathfrak{a}_n)$. Since $\sum\theta_{ij}l_{ij}=\mathfrak{z}'C\times\Theta\mathfrak{z}$, so m. g. f. of $(l_{11}\dots l_{kk})$ is given by

$$\varphi(\Theta)=|\underset{n}{E}\times\underset{k}{E}-2C\times M^{\frac{1}{2}}\Theta M^{\frac{1}{2}}|^{-\frac{1}{2}}\exp\mathfrak{a}'\Big\{C\times\Theta M^{\frac{1}{2}}\cdot(\underset{n}{E}\times\underset{k}{E}-2C\times M^{\frac{1}{2}}\Theta M^{\frac{1}{2}})^{-1}\cdot M^{\frac{1}{2}}\Theta\times C\Big\}\mathfrak{a}\,.$$

THEOREM 14. $(l_{11},\dots,l_{kk})$ is distributed in $W_{rk}(M)$ if and only if C is a projection, and $C\mathfrak{a}_\alpha=0$ $(\alpha=1,2,\dots,n)$, and $r=tr\,C$.

Proof. It is necessary and sufficient for $(l_{11},\dots,l_{kk})$ to be distributed in $W_{rn}(M)$ that

$$|E-2M^{\frac{1}{2}}\Theta M^{\frac{1}{2}}|^{-\frac{r}{2}}=|\underset{n}{E}\times\underset{k}{E}-2C\times M^{\frac{1}{2}}\Theta M^{\frac{1}{2}}|^{-\frac{1}{2}}\times$$
$$\exp\mathfrak{a}'\Big\{C\Theta M^{\frac{1}{2}}(\underset{n}{E}\times\underset{k}{E}-2C\times M^{\frac{1}{2}}\Theta M^{\frac{1}{2}})^{-1}M^{\frac{1}{2}}\Theta\times C\Big\}\mathfrak{a}\,,$$

namely

(1) $$|E-2M^{\frac{1}{2}}\Theta M^{\frac{1}{2}}|^{r}=|\underset{n}{E}\times\underset{k}{E}-2C\times M^{\frac{1}{2}}\Theta M^{\frac{1}{2}}|,$$

(2) $$\mathfrak{a}'\Big\{C\times\Theta M^{\frac{1}{2}}\cdot(\underset{n}{E}\times\underset{k}{E}-2C\times M^{\frac{1}{2}}\Theta M^{\frac{1}{2}})^{-1}\cdot M^{\frac{1}{2}}\Theta\times C\Big\}\mathfrak{a}=0\,.$$

(1) is written

$$\sum_{\nu=1}^{\infty}\frac{tr\,(2M^{\frac{1}{2}}\Theta M^{\frac{1}{2}})^{\nu}}{\nu}r=\sum_{\nu=1}^{\infty}tr\,(2M^{\frac{1}{2}}\Theta M^{\frac{1}{2}})^{\nu}\,tr\,C^{\nu}\,.$$

Therefore (1) is equivalent to

(3) $$r=tr\,C^{\nu}\qquad\nu=1,2,\dots$$

As in the proof of Theorem 1, (3) is equivalent to

(4) $$r=tr\,C,\quad\text{and}\quad C\text{ is a projection.}$$

By making use of (4) and since $|M|\neq0$, we reduce (2) to

$$(C\times\Theta)\,\mathfrak{a}=0\,,$$

or

(5) $$C\mathfrak{a}_\alpha=0\,,\quad\alpha=1,2,\dots,n\,.$$

Thus we have the conclusion.

§ 7. Analogies to Cochran's theorem.

Corresponding to a symmetric matrix $D=(d_{\alpha\beta})$ of order $n\cdot n$, consider

$$m_{ij} = \sum_{\alpha\beta} d_{\alpha\beta} x_{i\alpha} x_{j\beta}.$$

If we regard $\sum \theta_{ij} l_{ij}$, $\sum \theta'_{ij} m_{ij}$, as quadratic quantities of $\mathfrak{z}$ we have

THEOREM 15. $(l_{11} \ldots l_{kk})$, $(m_{11} \ldots m_{kk})$ are independent if and only if $CD=0$.

Proof. On account of Theorem 3 $(l_{11} \ldots l_{kk})$, $(m_{11} \ldots m_{kk})$ are independent if and only if $(C\times\Theta)\cdot(D\times\Theta')=0$ for arbitrary Θ, Θ', that is $CD=0$.

Theorem 14, 15 enable us to extend the results obtained in §§4, 5 to the case of Wishart's distribution. We illustrate this by an example. For this purpose let $C=\sum_{\mu=1}^{m} \underset{\mu}{C}$ where $C=(C_{ij})$, $C_\mu=(\underset{\mu}{C}_{ij})$ are symmetric matrices of order $n\cdot n$ and put $\underset{\mu}{l}_{ij}=\sum_{\alpha\beta} \underset{\mu}{C}_{\alpha\beta} x_{i\alpha} x_{j\beta}$. Then Theorem 14, 15 and Lemma 2 give

THEOREM 16. If $(l_{11} \ldots l_{kk})$ is distributed in a Wishart's distribution, then the following conditions are equivalent:

(1) $(\underset{\mu}{l}_{11} \ldots \underset{\mu}{l}_{kk})$ $\mu=1, \ldots, m$ are in Wishart's distributions,

(2) $(\underset{\mu}{l}_{11} \ldots \underset{\mu}{l}_{kk})$ $\mu=1, \ldots, m$ are independent,

(3) C_μ $\mu=1, \ldots, m$ are projections,

(4) $\sum_\mu (\textit{rank of } C_\mu) \leq \textit{rank of } C$.

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