ON SOLUTIONS OF THE LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION IN THE VICINITY OF THE SINGULARITY. I.

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§ 1. Introduction.

We consider the differential equation

$$(1.1)\qquad \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i}=0,$$

where X_i are regular in the vicinity of $x_i=0$. When X_i do not all vanish for $x_i=0$, it is evident that there exist $n-1$ independent integrals which are regular in the vicinity of $x_i=0$. In this paper we consider the case where all X_i vanish for $x_i=0$. Expanding X_i in the vicinity of $x_i=0$, we write them as follow:

$$(1.2)\qquad X_i=\sum_{j=1}^{n} a_{ij}\, x_j+\cdots\cdots$$

Put $\|a_{ij}\|=A$. We consider the following substitution

$$(1.3)\qquad y_i=\sum_{j=1}^{n} s_{ij}\, x_j,$$

where det. $|s_{ij}|\neq 0$. If we put $\|s_{ij}\|=S$, then, from (1.3), we have

$$(1.4)\qquad x_j=\sum_{k=1}^{n} S_{jk}\, y_k,$$

where $\|S_{jk}\|=S^{-1}$. By means of the substitution (1.3), the equation (1.1) is transformed to the equation of the same form as follows:

$$(1.5)\qquad \sum_{i=1}^{n} Y_i \frac{\partial f}{\partial y_i}=0,$$

where $Y_i=\sum_{k} s_{ik}X_k=\sum_{k,l,j} s_{ik}a_{kl}S_{lj}y_j+\cdots\cdots\cdots$ If we put $\sum_{k,l} s_{ik}a_{kl}S_{lj}=b_{ij}$ and $\|b_{ij}\|=B$, then $B=SAS^{-1}$. Now, by the theory of matrices, we know that there exists a matrix S, by means of which the matrix B is made to have the Jordan's form, namely the following form:

$$(1.6)\qquad B=A_1\dot{+}A_2\dot{+}\cdots\cdots\dot{+}A_s,$$

where $A_i=A_{i1}\dot{+}A_{i2}\dot{+}\cdots\dot{+}A_{it}$ and $A_{ij}=\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_i & 1 \\ 0 & \cdots & & 0 & \lambda_i \end{pmatrix}$, λ_i being eigen values of the matrix A. Then we see that, by means of the substitution (1.3) induced by such a matrix S, the quation (1.1) is transformed to the equation of the following form:

(1.7) $$\sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} = 0,$$

where $X_i = \sum_j b_{ij} x_j + \dots\dots$, and $B = \| b_{ij} \|$ is of the form (1.6).

When the eigen values of the matrix A are all distinct, the matrix B has the diagonal form. H. Poincaré has discussed this case[1] and he has obtained $n-1$ independent integrals under the following assumption:

The eigen values λ_i of the matrix A satisfy the following two conditions:

(I) $\lambda_1 p_1 + \lambda_2 p_2 + \dots\dots + \lambda_n p_n - \lambda_i \neq 0$ for all non-negative integers $p_1, p_2, \dots, p_n$, satisfying $p = p_1 + p_2 + \dots + p_n \geqq 2$,

(II) when $\lambda_1, \lambda_2, \dots, \lambda_n$ are marked on a complex plane, there exists a convex domain which contains the points $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ and does not contain the origin.[2]

In this paper, as Poincaré, assuming that the eigen values λ_i of the matrix A satisfy the above two conditions, we discuss the general case where the matrix B is not in general of the diagonal form.[3]

First, we consider the differential equation

(1.8) $$\sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} = \lambda f + \varphi,$$

where λ is one of the eigen values of the matrix A and φ is a given arbitrary function which is regular in the vicinity of $x_i = 0$ and vanishes there, and moreover all the derivatives of the first order of which vanish there. Next, making use of the integrals of the equation (1.8), we shall obtain $n-1$ independent integrals of the equation (1.7), i.e. integrals of the original equation.

§ 2. The case where all the eigen values are distinct.

In this case the matrix B has the form $\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$, therefore the equation (1.8) can be written as follows:

(2.1) $$(\lambda_1 x_1 + v_1)\frac{\partial f}{\partial x_1} + (\lambda_2 x_2 + v_2)\frac{\partial f}{\partial x_2} + \dots\dots + (\lambda_n x_n + v_n)\frac{\partial f}{\partial x_n} = \lambda_i f + \varphi,$$

where $v_1, v_2, \dots\dots, v_n$ denote the sums of the terms of the second and higher orders. After having differentiated both sides of (2.1) with respect to $x_1, x_2, \dots\dots, x_n$, put $x_1 = x_2 = \dots = x_n = 0$, then we have

$$(\lambda_1 - \lambda_i)\frac{\partial f}{\partial x_1} = 0, \quad (\lambda_2 - \lambda_i)\frac{\partial f}{\partial x_2} = 0, \dots\dots\dots, \quad (\lambda_n - \lambda_i)\frac{\partial f}{\partial x_n} = 0,$$

therefore, for $x_1 = x_2 = \dots = x_n = 0$, $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_{i-1}} = \frac{\partial f}{\partial x_{i+1}} = \dots = \frac{\partial f}{\partial x_n} = 0$ and $\frac{\partial f}{\partial x_i}$

1) H. Poincaré, Sur les propriétés de fonctions définies par les équations aux difference partielles. (Paris, 1879). See also E. Picard, Traité d'Analyse, t. I, p. 5.

2) Picard has discussed also the case when the weaker conditions are satisfied. Picard, ibid., p. 17.

3) We can discuss also the case where the weaker conditions which Picard has assumed are satisfied. In the next paper we shall describe the discussions in that case.

is indeterminate. Then, if we put $f=Ax_i+v$ where A denotes an arbitrary constant and v denotes the sum of the terms of the second and higher orders, the equation (2.1) is transformed to the equation as follows:

$$\lambda_1 x_1\frac{\partial v}{\partial x_1}+\ldots+\lambda_n x_n\frac{\partial v}{\partial x_n}-\lambda_i v=-v_1\frac{\partial v}{\partial x_1}-\ldots-v_n\frac{\partial v}{\partial x_n}+(\varphi-Av_i). \tag{2.2}$$

Now, the eigin values λ_i satisfy Poincaré's two conditions. Then there exists a positive numer ε such that

$$\left|\frac{\lambda_1 p_1+\lambda_2 p_2+\ldots\ldots+\lambda_n p_n-\lambda_i}{p_1+p_2+\ldots\ldots+p_n-1}\right|>\varepsilon$$

for all non-negative integers $p_1, p_2, \ldots, p_n$ satisfying $p=p_1+p_2+\ldots+p_n\geqq 2.$[1]

Corresponding to the equation (2.2), we consider the equation

$$\varepsilon\left(x_1\frac{\partial V}{\partial x_1}+x_2\frac{\partial V}{\partial x_2}+\ldots+x_n\frac{\partial V}{\partial x_n}-V\right)=W\left(\frac{\partial V}{\partial x_1}+\frac{\partial V}{\partial x_2}+\ldots+\frac{\partial V}{\partial x_n}+1\right), \tag{2.3}$$

where $W=\dfrac{M}{1-\dfrac{x_1+x_2+\ldots+x_n}{r}}-M-M\dfrac{x_1+x_2+\ldots+x_n}{r}.$ Here r denotes a positive number such that $v_1, v_2, \ldots, v_n$ and φ are regular for $|x_1|, |x_2|, \ldots, |x_n|\leqq r$, and M denotes the maximum value of $|v_1|, |v_2|, \ldots, |v_n|$ and $|\varphi-Av_i|$ in the above closed domain. If we put $x_1+x_2+\ldots+x_n=u$ and consider V as the function of u, then the equation (2.3) can be written as follows:

$$\varepsilon\left(u\frac{dV}{du}-V\right)=W\left(n\frac{dV}{du}+1\right), \tag{2.4}$$

where $W=\dfrac{M}{1-\dfrac{u}{r}}-M-M\dfrac{u}{r}.$ The equation (2.4) is linear, consequently it can be easly integrated, and actually we obtain the integral

$$V=u\left[-\frac{1}{n}+c\{r^2-(r+nM')u\}^{-\frac{nM'}{r+nM'}}\right],$$

where $M'=M/\varepsilon$ and c is an arbitrary constant. If we determine the value of c so that $-\dfrac{1}{n}+cr^{-\frac{2nM'}{r+nM'}}=0,$ i. e. $c=\dfrac{1}{n}r^{\frac{2nM'}{r+nM'}}$, then we obtain the integral $V=u^2\Psi(u)$ where $\Psi(u)$ is regular in the vicinity of $u=0$. Substituting $u=x_1+x_2+\ldots+x_n$ into this integral, we obtain the integral of (2.3) which is regular in the vicinity of $x_1=\ldots=x_n=0$ and vanishes there, and moreover all the derivatives of the first order of which vanish there.

Now, after having differentiated both sides of (2.2) $p_1, p_2, \ldots\ldots, p_n$ -times with respect to $x_1, x_2, \ldots, x_n$ respectively, put $x_1=x_2=\ldots=x_n=0$, then we have the following relation between the values of the derivatives of v for $x_1=x_2=\ldots=x_n=0$,

$$(\lambda_1 p_1+\lambda_2 p_2+\ldots+\lambda_n p_n-\lambda_i)\frac{\partial^p v}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}$$

$=$ linear combination of the derivatives of the orders $p-1$ at most $+$ const.,

where $p=p_1+p_2+\ldots+p_n$, and the constant in the right-hand side vanishes for the case

1) Picard, ibid., p.6.

$p=1$. For $p\geqq 2$, from Poincaré's condition, $\lambda_1 p_1+\lambda_2 p_2+\dots+\lambda_n p_n-\lambda_i \neq 0$, therefore the values of the derivatives of the second and higher orders are determined successively. Let the values of the derivatives be c_1, c_2, $\dots$, arranged in order as they are determined, then the recurring formula determining them is as follows:

$$c_{m+1}=\frac{1}{D}F(c_1, c_2, \dots, c_m), \tag{2.5}$$

where $D=\lambda_1 p_1+\lambda_2 p_2+\dots+\lambda_n p_n-\lambda_i$ and F is a linear expression of c_1, c_2, $\dots$, c_m.

We perform the same process to the equation (2.3). Then the values of the derivatives of V of the second and higher orders are successively determined in the same manner as before. Let the values of the derivatives of V be C_1, C_2, $\dots$, arranged in order as they are determined and corresponding to c_1, c_2, $\dots$. Then the recurring formula determining C_k is as follows:

$$C_{m+1}=\frac{1}{\varepsilon(p-1)}\Phi(C_1, C_2, \dots, C_m). \tag{2.6}$$

Now, the coefficients of Φ are all positive, consequently C_k are all positive.

Comparing the right-hand sides of the equation (2.2) and (2.3), the coefficients of (2.3) are not less than the absolute values of the coefficients of (2.2) Therefore the coefficients of Φ are not less than the absolute values of the coefficients of F. Moreover $|D|>\varepsilon(p-1)$. It is evident that, for the values of the derivatives of the second order, $C\geqq|c|$. If $C_1\geqq|c_1|$. $\dots$, $C_m\geqq|c_m|$, then

$$|c_{m+1}|=\frac{1}{|D|}|F(c_1, c_2, \dots, c_m)|<\frac{1}{\varepsilon(p-1)}\Phi(|c_1|, |c_2|, \dots, |c_m|)\leqq\frac{1}{\varepsilon(p-1)}\Phi(C_1, C_2, \dots,$$

$C_m)=C_{m+1}$, i. e. $C_{m+1}>|c_{m+1}|$. Therefore, for all k, $C_k\geqq|c_k|$. Now fhe integral $V=u^2\Psi(u)$ of (2.3), where $u=x_1+x_2+\dots+x_n$, is expanded in the power series and the coefficients of that integral are of the forms $\frac{1}{N_k}C_k$ where N_k are positive integers, namely the power series with the coefficients $\frac{1}{N_k}C_k$ converges in the vicinity of $x=x_2=\dots=x_n=0$. Therefore the power series with the coefficients $\frac{1}{N_k}c_k$ also converges in the same domain. Namely the equation (2.2) has a regular integral in the vicinity of $x_1=x_2=\dots=x_n=0$.

Thus the equation (2.1) has an integral which is regular in the vicinity of $x_1=x_2=\dots=x_n=0$ and has the form $f=Ax_i+\dots$, where the unexpressed terms are of the second and higher orders.

Specially when $\varphi\equiv 0$, the constant term of F of (2.5) has a factor A. Therefore all c_k contain A as a linear factor. Thus the integral f is of the form $f=A(x_i+\dots\dots)$.

§ 3. The case when all the eigen values are equal.

Let the eigen value be λ, then, from Poincaré's condition, $\lambda\neq 0$. If we divide both sides of the equafion (1.8) by λ, then the equation is of the form as follows:

$$\begin{aligned}(3.1)\qquad &(x_{11}+ax_{12}+\dots)\frac{\partial f}{\partial x_{11}}+(x_{12}+ax_{13}+\dots)\frac{\partial f}{\partial x_{12}}+\dots+(x_{1k}+\dots)\frac{\partial f}{\partial x_{1k}}\\ &+(x_{21}+ax_{22}+\dots)\frac{\partial f}{\partial x_{21}}+\dots\dots\dots\dots \qquad +(x_{2l}+\dots)\frac{\partial f}{\partial x_{2l}}\\ &+\dots\dots\dots\dots\dots\dots\end{aligned}$$

$$+(x_{r1}+ax_{r2}+\dots)\frac{\partial f}{\partial x_{r1}}+\dots \qquad +(x_{rm}+\dots)\frac{\partial f}{\partial x_{rm}}-f=\varphi,$$

where $a=1/\lambda\neq 0$. This can be written as follows:

$$\begin{aligned}(3.2)\qquad & x_{11}\frac{\partial f}{\partial x_{11}}+x_{12}\frac{\partial f}{\partial x_{12}}+\dots \qquad +x_{1k}\frac{\partial f}{\partial x_{1k}}\\
&+x_{21}\frac{\partial f}{\partial x_{21}}+\dots\dots \qquad +x_{2l}\frac{\partial f}{\partial x_{2l}}\\
&+\dots\dots\dots\dots\dots\dots\\
&+x_{r1}\frac{\partial f}{\partial x_{r1}}+\dots\dots \qquad +x_{rm}\frac{\partial f}{\partial x_{rm}}-f\\
&=-ax_{12}\frac{\partial f}{\partial x_{11}}-ax_{13}\frac{\partial f}{\partial x_{12}}-\dots \qquad -ax_{1k}\frac{\partial f}{\partial x_{1k-1}}\\
&-ax_{22}\frac{\partial f}{\partial x_{21}}-\dots \qquad -ax_{2l}\frac{\partial f}{\partial x_{2l-1}}\\
&-\dots\dots\dots\dots\dots\dots\\
&-ax_{r2}\frac{\partial f}{\partial x_{r1}}-\dots \qquad -ax_{rm}\frac{\partial f}{\partial x_{rm-1}}\\
&+v_{11}\frac{\partial f}{\partial x_{11}}+v_{12}\frac{\partial f}{\partial x_{12}}+\dots \qquad +v_{1k}\frac{\partial f}{\partial x_{1k}}\\
&+\dots\dots\dots\dots\dots\dots\\
&+v_{r1}\frac{\partial f}{\partial x_{r1}}+\dots \qquad +v_{rm}\frac{\partial f}{\partial x_{rm}}+\varphi,\end{aligned}$$

where v_{ij} denote the sums of the terms of the second and higher orders. Let $|a|=A$ and ρ be a positive number such that, for $|x_{ij}|\leqq\rho$, v_{ij} and φ are regular. Let the maximum values of $|v_{ij}|$ and $|\varphi|$ for $|x_{ij}|\leqq\rho$ be M and N respectively. Corresponding to the equation (3.2), we consider the equation:

$$\begin{aligned}(3.3)\qquad & \rho_{11}x_{11}\frac{\partial F}{\partial x_{11}}+\rho_{12}x_{12}\frac{\partial F}{\partial x_{12}}+\dots \qquad +\rho_{1k}x_{1k}\frac{\partial F}{\partial x_{1k}}\\
&+\rho_{21}x_{21}\frac{\partial F}{\partial x_{21}}+\dots \qquad +\rho_{2l}x_{2l}\frac{\partial F}{\partial x_{2l}}\\
&+\dots\dots\dots\dots\dots\dots\\
&+\rho_{r1}x_{r1}\frac{\partial F}{\partial x_{r1}}+\dots \qquad +\rho_{rm}x_{rm}\frac{\partial F}{\partial x_{rm}}-\rho_{ij}F\\
&=Ax_{12}\frac{\partial F}{\partial x_{11}}+Ax_{13}\frac{\partial F}{\partial x_{12}}+\dots+Ax_{1k}\frac{\partial F}{\partial x_{1k-1}}\\
&+Ax_{22}\frac{\partial F}{\partial x_{11}}+\dots \qquad +Ax_{2l}\frac{\partial F}{\partial x_{2l-1}}\\
&+\dots\dots\dots\dots\dots\dots\\
&+Ax_{r2}\frac{\partial F}{\partial x_{r1}}+\dots+Ax_{rm}\frac{\partial F}{\partial x_{rm-1}}\\
&+V\left(\frac{\partial F}{\partial x_{11}}+\dots+\frac{\partial F}{\partial x_{1k}}+\frac{\partial F}{\partial x_{21}}+\dots+\frac{\partial F}{\partial x_{rm}}\right)+W,\end{aligned}$$

where $V=\dfrac{M}{1-\dfrac{x_{11}+\dots+x_{rm}}{\rho}}-M-M\dfrac{x_{11}+\dots+x_{rm}}{\rho}$ and $W=\dfrac{N}{1-\dfrac{x_{11}+\dots+x_{rm}}{\rho}}-N-$

$-N\dfrac{x_{11}+\dots+x_{rm}}{\rho}$. After having differentiated both sides of (3.3) p_{st}-times with respect to x_{st}, put $x_{st}=0$, then we have the following relation between the values of the derivatives for $x_{st}=0$:

$$(3.4)\qquad (\rho_{11}p_{11}+\rho_{12}p_{12}+\dots+\rho_{1k}p_{1k}+\rho_{21}p_{21}+\dots+\rho_{rm}p_{rm}-\rho_{ij})\frac{\partial^p F}{\partial x_{11}^{p_{11}}\dots\partial x_{1k}^{p_{1k}}\partial x_{21}^{p_{21}}\dots\partial x_{rm}^{p_{rm}}}$$
$$=Ap_{12}\frac{\partial^p F}{\partial x_{11}^{p_{11}+1}\partial x_{12}^{p_{12}-1}\dots\partial x_{rm}^{p_{rm}}}+Ap_{13}\frac{\partial^p F}{\partial x_{11}^{p_{11}}\partial x_{12}^{p_{12}+1}\partial x_{13}^{p_{13}-1}\dots\partial x_{rm}^{p_{rm}}}+\dots$$
$$+Ap_{1k}\frac{\partial^p F}{\partial x_{11}^{p_{11}}\dots\partial x_{1k-1}^{p_{1k-1}+1}\partial x_{1k}^{p_{1k}-1}\dots\partial x_{rm}^{p_{rm}}}$$
$$+Ap_{22}\frac{\partial^p F}{\partial x_{11}^{p_{11}}\dots\partial x_{21}^{p_{21}+1}\partial x_{22}^{p_{22}-1}\dots\partial x_{rm}^{p_{rm}}}+\dots$$
$$+Ap_{2l}\frac{\partial^p F}{\partial x_{11}^{p_{11}}\dots\partial x_{2l-1}^{p_{2l-1}+1}\partial x_{2l}^{p_{2l}-1}\dots\partial x_{rm}^{p_{rm}}}$$
$$+\dots\dots\dots\dots$$
$$Ap_{r2}\frac{\partial^p F}{\partial x_{11}^{p_{11}}\dots\partial x_{r1}^{p_{r1}+1}\partial x_{r2}^{p_{r2}-1}\dots\partial x_{rm}^{p_{rm}}}+\dots+Ap_{rm}\frac{\partial^p F}{\partial x_{11}^{p_{11}}\dots\partial x_{rm-1}^{p_{rm-1}+1}\partial x_{rm}^{p_{rm}-1}}$$
$$+\text{linear combination of the derivatives of the order } p-1 \text{ at most } + \text{const.},$$

where $p=p_{11}+p_{12}+\dots+p_{1k}+p_{21}+\dots+p_{rm}$. The constant of the right-hand side vanishes when $p=1$. From (3.2), we have the analogous relation between the values of the derivatives of f.

When $p=1$, we have:

$$(3.5)\qquad \begin{cases}(\rho_{11}-\rho_{ij})\dfrac{\partial F}{\partial x_{11}}=0,\\ (\rho_{12}-\rho_{ij})\dfrac{\partial F}{\partial x_{12}}=A\dfrac{\partial F}{\partial x_{11}},\\ \quad\vdots\\ (\rho_{1k}-\rho_{ij})\dfrac{\partial F}{\partial x_{1k}}=A\dfrac{\partial F}{\partial x_{1k-1}},\end{cases}\quad \begin{cases}(\rho_{21}-\rho_{ij})\dfrac{\partial F}{\partial x_{21}}=0,\\ (\rho_{22}-\rho_{ij})\dfrac{\partial F}{\partial x_{22}}=A\dfrac{\partial F}{\partial x_{21}},\\ \quad\vdots\\ (\rho_{2l}-\rho_{ij})\dfrac{\partial F}{\partial x_{2l}}=A\dfrac{\partial F}{\partial x_{2l-1}},\end{cases}\dots \begin{cases}(\rho_{r1}-\rho_{ij})\dfrac{\partial F}{\partial x_{r1}}=0,\\ (\rho_{r2}-\rho_{ij})\dfrac{\partial F}{\partial x_{r2}}=A\dfrac{\partial F}{\partial x_{r1}},\\ \quad\vdots\\ (\rho_{rm}-\rho_{ij})\dfrac{\partial F}{\partial x_{rm}}=A\dfrac{\partial F}{\partial x_{rm-1}}.\end{cases}$$

For (3.2), all $\rho_{st}=1$ and $A=-a\neq 0$. Therefore, for $x_{st}=0$, we have:
$\dfrac{\partial f}{\partial x_{11}}=\dots=\dfrac{\partial f}{\partial x_{1k-1}}=\dfrac{\partial f}{\partial x_{21}}=\dots=\dfrac{\partial f}{\partial x_{2l-1}}=\dots=\dfrac{\partial f}{\partial x_{r1}}=\dots=\dfrac{\partial f}{\partial x_{rm-1}}=0$, and $\dfrac{\partial f}{\partial x_{1k}}$, $\dfrac{\partial f}{\partial x_{2l}}$, $\dots$, $\dfrac{\partial f}{\partial x_{rm}}$ are indeterminate. For (3.3), if we assume that all ρ_{st} are distinct, then, for $\rho_{ij}=\rho_{1k}$, $\dfrac{\partial F}{\partial x_{1k}}$ is indeterminate and all the other derivatives vanish. Likewise we have: for $\rho_{ij}=\rho_{2l}$, $\dfrac{\partial F}{\partial x_{2l}}$ is indeterminate and all the others vanish, ..., for $\rho_{ij}=\rho_{rm}$, $\dfrac{\partial F}{\partial x_{rm}}$ is indeterminate and all the others vanish. We take a positive number ε_0 which is less than 1/3 and choose ρ_{st} so that $\rho_{st}=1-\varepsilon_{st}$ where all ε_{st} are distinct and $2\varepsilon_0>\varepsilon_{st}>\varepsilon_0$. Then, for $p\geqq 2$, it follows that

$$D=\rho_{11}p_{11}+\dots+\rho_{rm}p_{rm}-\rho_{ij}=\Sigma(1-\varepsilon_{st})p_{st}-(1-\varepsilon_{ij})=(\Sigma p_{st}-1)-(\Sigma\varepsilon_{st}p_{st}-\varepsilon_{ij}).$$

If we put $\Sigma p_{st}-1=d$, then $d=p-1\geqq 1$, therefore $d-D=\Sigma\varepsilon_{st}p_{st}-\varepsilon_{ij}>2\varepsilon_0-2\varepsilon_0=0$, i. e. $d>D$. Now $D=\Sigma(1-\varepsilon_{st})p_{st}-(1-\varepsilon_{ij})>2(1-2\varepsilon_0)-(1-\varepsilon_0)=1-3\varepsilon>0$. Thus we have $d>D>0$. Then, from (3.4) and the analogous relation deduced from (3.2), the values of the derivatives of F and f of the second and higher oders are determined successively. For $\rho_{ij}=\rho_{1k}$, let the values of the derivatives of F be C_1, C_2, ..., arranged in order as they are determined, then the recurring formula determining them is of the form:

(3.6) $$C_{s+1}=\frac{1}{D}\Phi(C_1, C_2, \ldots, C_s),$$

where Φ is a linear expression of C_1, C_2, ..., C_s and its coefficients are all positive. In this case F is of the form $F=C_1x_{1k}+\ldots$, therefore, when we choose C_1 so that $C_1>0$, all C_s are positive. For (3.2), if we choose the values of the derivatives of f of the first order so that $\frac{\partial f}{\partial x_{1k}}=c_1$, $\frac{\partial f}{\partial x_{2l}}=\ldots\frac{\partial f}{\partial x_{rm}}=0$, then the values of the derivatives of f are determined successively in the same manner as C_s. Namely, if we write the values of the derivatives of f as c_1, c_2, ..., corresponding to C_1, C_2, ..., arranged in order as they are determined, then the recurring formula determining c_s is of the form

(3.7) $$c_{s+1}=\frac{1}{d}\Psi(c_1, c_2, \ldots, c_s),$$

where Ψ is of the same form as Φ. However, comparing the right-hand sides of (3.2) and (3.3), we see that the coefficients of Φ are not less than the absolute values of the coefficients of Ψ. We choose C_1 so that $C_1\geqq|c_1|$. Then, by means of the relation $d>D>0$, we can easily prove as in § 2 that $C_s\geqq|c_s|$ for all s.

Now ρ_{st} are eigen values of the matrix, the elements of which are the coefficients of the terms of the first order of the coefficients of $\frac{\partial F}{\partial x_{st}}$ in (3.3), and it is evident that our ρ_{st} satisfy Picard's two conditions. Then, from § 2, we see that the equation (3.3) has regular integrals. Those regular integrals can be expanded in the power series of x_{st}. For $\rho_{ij}=\rho_{1k}$, when we choose a regular integral so that the value of the derivative of the first order which is arbitrary should be equal to our C_1, then the coefficients of the expansion of that regular integral are of the form $\frac{1}{N_s}C_s$ where N_s are positive integers. Namely the power series with the coefficients $\frac{1}{N_s}C_s$ converges in the vicinity of $x_{st}=0$. Then, from $|c_s|\leqq C_s$, we see that the power series with the coefficients $\frac{1}{N_s}c_s$ also converges in the same domain of x_{st}. Namely the equation (3.2) has a regular integral which is of the form $f=c_1x_{1k}+\ldots\ldots$. If we take ρ_{2l}, ..., ρ_{rm} as ρ_{ij}, then we have the following regular integrals respectively: $f=cx_{2l}+\ldots$, ..., $f=cx_{rm}+\ldots\ldots$

Specially when $\varphi\equiv0$ in (3.1), then Ψ of (3.7) becomes a linear homogeneous expression, consequently the c_s $(s\geqq2)$ contain c_1 as a linear factor. Thus in this case the above regular integrals have the forms: $f=k_1(x_{1k}+\ldots)$, $f=k_2(x_{2l}+\ldots)$, ..., $f=k_r(x_{rm}+\ldots)$. Then it is evident that $f=k_1(x_{1k}+\ldots)+k_2(x_{2l}+\ldots)+\ldots+k_r(x_{rm}+\ldots)$ also satisfies the equation (3.1). Here the values of the derivatives $\frac{\partial f}{\partial x_{1k}}$, $\frac{\partial f}{\partial x_{2l}}$, ..., $\frac{\partial f}{\partial x_{rm}}$ are arbitrary, therefore this is a general regular integral of (3.1) in which $\varphi\equiv0$.

§ 4. General case.

The equation (1.8) can be written in a general form as follows:

(4.1) $$(\lambda_1x_{11}+x_{12}+\ldots)\frac{\partial f}{\partial x_{11}}+(\lambda_1x_{12}+x_{13}+\ldots)\frac{\partial f}{\partial x_{12}}+\ldots+(\lambda_1x_{1k}+\ldots)\frac{\partial f}{\partial x_{1k}}$$
$$+(\lambda_1x_{21}+x_{22}+\ldots)\frac{\partial f}{\partial x_{21}}+\ldots\qquad+(\lambda_1x_{2l}+\ldots)\frac{\partial f}{\partial x_{2l}}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(\lambda_2 x_{r1}+x_{r2}+\cdots)\frac{\partial f}{\partial x_{r1}}+\cdots\cdots \quad +(\lambda_2 x_{rm}+\cdots)\frac{\partial f}{\partial x_{rm}}$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\lambda_i f=\varphi.$$

Here the eigen values λ_s satisfy Poincaré's two conditions. Then it is readily seen as in § 2 that there exists a positive number ε such that

$$\left|\frac{\lambda_1 p_{11}+\cdots+\lambda_1 p_{1k}+\lambda_1 p_{21}+\cdots+\lambda_1 p_{2l}+\cdots+\lambda_2 p_{r1}+\cdots+\lambda_2 p_{rm}+\cdots-\lambda_i}{p_{11}+\cdots+p_{1k}+p_{21}+\cdots+p_{2l}+\cdots+p_{r1}+\cdots+p_{rm}\cdots-1}\right|>\varepsilon \tag{4.2}$$

for all non-negative integers p_{st} satisfying $p=\Sigma p_{st}\geqq 2$. The equation (4.1) is transformed into the equation of the form as follows:

$$\begin{aligned}
&\lambda_1 x_{11}\frac{\partial f}{\partial x_{11}}+\lambda_1 x_{12}\frac{\partial f}{\partial x_{12}}+\cdots \quad +\lambda_1 x_{1k}\frac{\partial f}{\partial x_{1k}}\\
&+\lambda_1 x_{21}\frac{\partial f}{\partial x_{21}}+\cdots \quad +\lambda_1 x_{2l}\frac{\partial f}{\partial x_{2l}}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+\lambda_2 x_{r1}\frac{\partial f}{\partial x_{r1}}+\cdots \quad +\lambda_2 x_{rm}\frac{\partial f}{\partial x_{rm}}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots-\lambda_i f\\
=&-x_{12}\frac{\partial f}{\partial x_{11}}-x_{13}\frac{\partial f}{\partial x_{12}}-\cdots \quad -x_{1k}\frac{\partial f}{\partial x_{1k-1}}\\
&-x_{22}\frac{\partial f}{\partial x_{21}}-\cdots \quad -x_{2l}\frac{\partial f}{\partial x_{2l-1}}\\
&-\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-x_{r2}\frac{\partial f}{\partial x_{r1}}-\cdots \quad -x_{rm}\frac{\partial f}{\partial x_{rm-1}}\\
&-\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+v_{11}\frac{\partial f}{\partial x_{11}}+\cdots+v_{1k}\frac{\partial f}{\partial x_{1k}}+\cdots+v_{r1}\frac{\partial f}{\partial x_{r1}}+\cdots+v_{rm}\frac{\partial f}{\partial x_{rm}}+\cdots \quad +\varphi,
\end{aligned} \tag{4.3}$$

where v_{st} denote the sums of the terms of the second and higher orders. We take a positive number ρ such that, for $|x_{st}|\leqq\rho$, v_{st} and φ are regular. Let the maximum values of $|v_{st}|$ and $|\varphi|$ for $|x_{st}|\leqq\rho$ be M and N respectively. Corresponding to the equation (4.3), we consider the equation

$$\begin{aligned}
&\varepsilon(x_{11}\frac{\partial F}{\partial x_{11}}+x_{12}\frac{\partial F}{\partial x_{12}}+\cdots \quad +x_{1k}\frac{\partial F}{\partial x_{1k}}\\
&+x_{21}\frac{\partial F}{\partial x_{21}}+\cdots \quad +x_{2l}\frac{\partial F}{\partial x_{2l}}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\\
&+x_{r1}\frac{\partial F}{\partial x_{r1}}+\cdots \quad +x_{rm}\frac{\partial F}{\partial x_{rm}}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots-F)\\
=&x_{12}\frac{\partial F}{\partial x_{11}}+x_{13}\frac{\partial F}{\partial x_{12}}+\cdots \quad +x_{1k}\frac{\partial F}{\partial x_{1k-1}}\\
&+x_{22}\frac{\partial F}{\partial x_{21}}+\cdots \quad +x_{2l}\frac{\partial F}{\partial x_{2l-1}}
\end{aligned} \tag{4.4}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+x_{r2}\frac{\partial F}{\partial x_{r1}}+\cdots \qquad +x_{rm}\frac{\partial F}{\partial x_{rm-1}}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+V\left(\frac{\partial F}{\partial x_{11}}+\cdots+\frac{\partial F}{\partial x_{1k}}+\frac{\partial F}{\partial x_{21}}+\cdots+\frac{\partial F}{\partial x_{2l}}+\cdots+\frac{\partial F}{\partial x_{r1}}+\cdots+\frac{\partial F}{\partial x_{rm}}+\cdots\right)+W,$$

where $V=\dfrac{M}{1-\dfrac{x_{11}+\cdots+x_{1k}+\cdots}{\rho}}-M-M\dfrac{x_{11}+\cdots+x_{1k}+\cdots}{\rho}$ and $W=\dfrac{N}{1-\dfrac{x_{11}+\cdots+x_{1k}+\cdots}{\rho}}$ $-N-N\dfrac{x_{11}+\cdots+x_{1k}\cdots}{\rho}$. As in § 3, we determine the values of the derivatives of f and F for $x_{st}=0$. After having differentiated both sides of (4.3) p_{st}-times with respect to x_{st}, put $x_{st}=0$, then we have:

$$(4.5)\qquad (\lambda_1 p_{11}+\cdots+\lambda_1 p_{1k}+\lambda_1 p_{21}+\cdots+\lambda_2 p_{r1}+\cdots-\lambda_i)\frac{\partial^p f}{\partial x_{11}^{p_{11}}\cdots\partial x_{21}^{p_{21}}\cdots}$$

$$=-p_{12}\frac{\partial^p f}{\partial x_{11}^{p_{11}+1}\partial x_{12}^{p_{12}-1}\cdots}-p_{13}\frac{\partial^p f}{\partial x_{11}^{p_{11}}\partial x_{12}^{p_{12}+1}\partial x_{13}^{p_{13}-1}\cdots}-\cdots-p_{1k}\frac{\partial^p f}{\partial x_{11}^{p_{11}}\cdots\partial x_{1k-1}^{p_{1k-1}+1}\partial x_{1k}^{p_{1k}-1}\cdots}$$

$$-p_{22}\frac{\partial^p f}{\partial x_{11}^{p_{11}}\cdots\partial x_{21}^{p_{21}+1}\partial x_{22}^{p_{22}-1}\cdots}-\cdots\cdots-p_{2l}\frac{\partial^p f}{\partial x_{11}^{p_{11}}\cdots\partial x_{2l-1}^{p_{2l-1}+1}\partial x_{2l}^{p_{2l}-1}\cdots}$$

$$-\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$+$ linear combination of the derivatives of the orders $p-1$ at most $+$ const.,

where $p=\Sigma p_{st}$. Here the constant of the right-hand side vanishes when $p=1$. When $p=1$, (4.5) becomes as follows:

$$(4.6)\qquad \left\{\begin{array}{l|l|l}
(\lambda_1-\lambda_i)\dfrac{\partial f}{\partial x_{11}}=0, & (\lambda_1-\lambda_i)\dfrac{\partial f}{\partial x_{21}}=0. & (\lambda_2-\lambda_i)\dfrac{\partial f}{\partial x_{r1}}=0, \\
(\lambda_1-\lambda_i)\dfrac{\partial f}{\partial x_{12}}=-\dfrac{\partial f}{\partial x_{11}}, & (\lambda_1-\lambda_i)\dfrac{\partial f}{\partial x_{22}}=-\dfrac{\partial f}{\partial x_{21}},\cdots & (\lambda_2-\lambda_i)\dfrac{\partial f}{\partial x_{r2}}=-\dfrac{\partial f}{\partial x_{r1}},\cdots \\
\quad\vdots & \quad\vdots & \quad\vdots \\
(\lambda_1-\lambda_i)\dfrac{\partial f}{\partial x_{1k}}=-\dfrac{\partial f}{\partial x_{1k-1}}, & (\lambda_1-\lambda_i)\dfrac{\partial f}{\partial x_{2l}}=-\dfrac{\partial f}{\partial x_{2l-1}}, & (\lambda_2-\lambda_i)\dfrac{\partial f}{\partial x_{rm}}=-\dfrac{\partial f}{\partial x_{rm-1}},
\end{array}\right.$$

For $\lambda_i=\lambda_1$, $\dfrac{\partial f}{\partial x_{11}}=\cdots=\dfrac{\partial f}{\partial x_{1k-1}}=\dfrac{\partial f}{\partial x_{21}}=\cdots=\dfrac{\partial f}{\partial x_{2l-1}}=\cdots=0$, $\dfrac{\partial f}{\partial x_{r1}}=\cdots=\dfrac{\partial f}{\partial x_{rm}}=\cdots=0$, and $\dfrac{\partial f}{\partial x_{1k}}$, $\dfrac{\partial f}{\partial x_{2l}}$, ..., are indeterminate. For the equation (4.4), we can make the analogous equation as (4.5), however for (4.4) all the eigen values are equal. Therefore $\dfrac{\partial F}{\partial x_{1k}}$, $\dfrac{\partial F}{\partial x_{2l}}$, ..., $\dfrac{\partial F}{\partial x_{rm}}$, ..., are indeterminate and all the other derivatives of the first order vanish.

For the derivatives of the second and higher orders, from (4.5) and the analogous relation deduced from (4.4), the values of the derivatives of f and F are determined successively. For $\lambda_i=\lambda_1$, we consider the sets of values in which the derivatives with respect to x_{1k} are arbitrary and all the others vanish. Then the values of both sets are determined in the same manner. Let the values of the derivatives of f and F be c_1, c_2, ..., and C_1, C_2, ..., respectively, arranged in order as they are determined, so that C_s corresponds to c_s. If we choose C_1 so that $C_1 \geqq |c_1|$ then, by means of (4.2), we can easily prove as in § 2 that $C_s \geqq |c_s|$ for all s.

Now, from the result of § 3, the equation (4.4) has a regular intergral which has the form $F=Cx_{1k}+\ldots\ldots$ Then, by the same reasonings as in § 3, we see that the equation (4.3) has a regular integral which has the form $f=c_1x_{1k}+\ldots\ldots$ Likewise, for $\lambda_i=\lambda_1$, we have regular integrals $f=cx_{2l}+\ldots\ldots$, $\ldots\ldots$, and, for $\lambda_i=\lambda_2$, $f=cx_{rm}+\ldots\ldots$, $\ldots\ldots$ For $\lambda_i=\lambda_3$, $\ldots\ldots$, we have analagously regular integrals.

As in § 3, when $\varphi\equiv 0$, the above regular integrals have the following forms: $f=k_1(x_{1k}+\ldots\ldots)$, $f=k_2(x_{2l}+\ldots\ldots)$, $\ldots\ldots$ Then, for $\lambda_i=\lambda_1$, $f=k_1(x_{1k}+\ldots)+k_2(x_{2l}+\ldots)+\ldots\ldots$ evidently satisfies the equation (4.1). Here the values of the derivatives $\frac{\partial f}{\partial x_{1k}}$, $\frac{\partial f}{\partial x_{2l}}$, $\ldots$ are arbitrary, therefore this is a general regular integral of (4,1) for $\lambda_i=\lambda_1$, in which $\varphi\equiv 0$.

§ 5. Solution of the linear homogeneous partial differential equation.

In this paragraph we consider the differential equation

$$Xf=\sum_{ij} X_{ij}\frac{\partial f}{\partial x_{ij}}=0, \tag{5.1}$$

where X_{ij} are of the same form as the coefficients of $\frac{\partial f}{\partial x_{ij}}$ in (4.1). Corresponding to (5.1) we consider the differential equation

$$X\varphi=\lambda\varphi, \tag{5.2}$$

where λ is an eigen value of the matrix, the elements of which are the coefficients of the terms of the first order in X_{ij}. We assume that the eigen values λ_i satisfy Poincaré's two conditions. Then, by the result of § 4, there exist regular integrals which have the following forms: for $\lambda=\lambda_1$, $\varphi_{11}=x_{1k}+\ldots$, $\varphi_{12}=x_{2l}+\ldots$, $\ldots$, for $\lambda=\lambda_2$, $\varphi_{21}=x_{rm}+\ldots\ldots$, $\ldots\ldots$, $\ldots\ldots$ Here, for simplicity, we take the intergrals in which arbitrary constants are unity.

We take one of the above integrals, for example, $\varphi=\varphi_{11}$ for $\lambda=\lambda_1$. We consider the system of the equations

$$\begin{cases} X\varphi_1=\lambda\varphi_1+\varphi, \\ X\varphi_2=\lambda\varphi_2+2\varphi_1, \\ \quad\vdots \\ X\varphi_s=\lambda\varphi_s+s\varphi_{s-1}, \\ \quad\vdots \end{cases} \tag{5.3}$$

Then there exist regular integrals φ_s which have the forms $\varphi_s=s!\,x_{1k-s}+\psi_s$ where ψ_s are the sums of the terms of the second and higher orders. The proof is as follows:

Put $\varphi_1=x_{1k-1}+\psi_1$, then substituting this into the first equation of (5.3) we have

$$X_{11}\frac{\partial\psi_1}{\partial x_{11}}+\ldots+X_{1k-1}\left(1+\frac{\partial\psi_1}{\partial x_{1k-1}}\right)+X_{1k}\frac{\partial\psi_1}{\partial x_{1k}}+X_{21}\frac{\partial\psi_1}{\partial x_{21}}+\ldots=\lambda_1(x_{1k-1}+\psi_1)+x_{1k}+\psi,$$

where $\psi=\varphi-x_{1k}$ is the sum of the terms of the second and higher orders. Then the above equation becomes as follows: $X\psi_1=\lambda_1\psi_1+\psi'$, where ψ' is the sum of the terms of the second and higher orders. Therefore, from the result of § 4, there exist regular integrals ψ_1 which have the forms $\psi_1=cx_{1k}+\ldots$, $cx_{2l}+\ldots\ldots$, $\ldots\ldots$ From these integrals we take an integral ψ_1 in which an arbitrary constant c is zero. Thus we have a regular integral φ_1 which has the form $\varphi_1=x_{1k-1}+\psi_1$.

We assume that there exists a regular integral φ_{s-1} which has the form $\varphi_{s-1}=(s-1)!x_{1k-s+1}+\psi_{s-1}$. Then, putting $\varphi_s=s!x_{1k-s}+\psi_s$, we have $X\varphi_s=X\psi_s+s!X_{1k-s}=\lambda_1(s!x_{1k-s}+\psi_s)+s!x_{1k-s+1}+s\psi_{s-1}$, therefore $X\psi_s=\lambda_1\psi_s+\psi'_{s-1}$, where ψ'_{s-1} is the sum of the terms of the second and higher orders. Therefore there exist regular integrals ψ_s which have the forms $\psi_s=cx_{1k}+\ldots\ldots$, $cx_{2l}+\ldots\ldots$, ... From these integrals we take an integral ψ_s in which an arbitrary constant c is zero. Thus we have a regular integral φ_s which has the form $\varphi_s=s!x_{1k-s}+\psi_s$. q. e. d.

Then the solutions of (5.3) are as follows:

(5.4) $$\varphi=\varphi_{11}=x_{1k}+\psi,\ \varphi_1=x_{1k-1}+\psi_1,\ \ldots\ldots,\ \varphi_s=s!x_{1k-s}+\psi_s,\ \ldots\ldots,\ \varphi_{k-1}=(k-1)!x_{11}+\psi_{k-1}.$$

Put $\varphi_1/\varphi=U_0$, then $\varphi_1=U_0\varphi$. According to the formula of Leibnitz upon differentiation, we define U_s by means of the equation

(5.5) $$\varphi_{s+1}=U_s\varphi+\binom{s}{1}U_{s-1}\varphi_1+\binom{s}{2}U_{s-2}\varphi_2+\ldots\ldots+\binom{s}{t}U_{s-t}\varphi_t+\ldots\ldots+U_0\varphi_s,$$

where $k-2\geqq s\geqq 1$. Now $XU_0=X(\varphi_1/\varphi)=\frac{1}{\varphi}X\varphi_1-\frac{\varphi_1}{\varphi^2}X\varphi=\frac{1}{\varphi}(\lambda\varphi_1+\varphi)-\frac{\varphi_1}{\varphi^2}\lambda\varphi=1$, namely

(5,6) $$XU_0=1.$$

Then we can prove that

(5.7) $$XU_s=0$$

where $s\geqq 1$. The proof is as follows:

From (5.5), $\varphi_2=U_1\varphi+U_0\varphi_1$. Operating X on both sides, by means of (5.3) we have:

$$\lambda\varphi_2+2\varphi_1=XU_1\cdot\varphi+U_1\cdot\lambda\varphi+\varphi_1+U_0(\lambda\varphi_1+\varphi)=XU_1\cdot\varphi+\lambda(U_1\varphi+U\varphi_1)+2\varphi_1.$$

Thereofre $XU_1=0$. We assume that $XU_1=0$, ..., $XU_{s-1}=0$. Operating X on both sides of (5.5), by means of (5.3) we have:

$$\begin{aligned}\lambda\varphi_{s+1}+(s+1)\varphi_s=XU_s\cdot\varphi+U_s\cdot\lambda\varphi&\\ +\binom{s}{1}U_{s-1}\cdot(\lambda\varphi_1+\varphi)&\\ \vdots\quad&\\ +\binom{s}{t}U_{s-t}\cdot(\lambda\varphi_t+t\varphi_{t-1})&\\ \vdots\quad&\\ +\quad U_0\cdot(\lambda\varphi_s+s\varphi_{s-1})+\varphi_s.&\end{aligned}$$

Now $\binom{s}{t}t=\binom{s-1}{t-1}s$. Therefore the above equation becomes as follows:

$$\lambda\varphi_{s+1}+(s+1)\varphi_s=XU_s\cdot\varphi+\lambda\varphi_{s+1}+s\varphi_s+\varphi_s.$$

Then it follows that $XU_s=0$. q. e. d.

From the equation (5.2), we have $X\left(\frac{1}{\lambda}log\varphi\right)=1$. Comparing this to (5.6), it follows that

(5.8) $$X\left(\frac{1}{\lambda}log\varphi-U_0\right)=0.$$

Thus, as the solutions of (5.1), we have the following functions:

(5.9) $$\frac{1}{\lambda}log\,\varphi-U_0,\ U_1,\ U_2,\ \ldots\ldots,\ U_{k-2}.$$

The integrals (5.9) are obtained starting from the equation (5.2) in which $\lambda=\lambda_1$ and

$\varphi=\varphi_{11}$. If we give to the equation (5.2) various possible values λ_s and functions φ_{st} corresponding to them and perform the same process as above, then, as the integrals of (5.1), we have the following functions:

$$(5.10)\qquad \begin{cases} \frac{1}{\lambda_1}\log\varphi-U_0,\ U_1,\ U_2,\ \ldots\ldots,\ U_{k-2}, \\ \frac{1}{\lambda_1}\log\theta-V_0,\ V_1,\ \ldots\ldots,\ V_{l-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{1}{\lambda_2}\log\omega-W_0,\ W_1,\ \ldots\ldots,\ W_{m-2}, \\ \ldots\ldots\ldots\ldots, \end{cases}$$

Moreover $X\left(\frac{1}{\lambda_1}\log\varphi\right)=X\left(\frac{1}{\lambda_1}\log\theta\right)=\cdots=X\left(\frac{1}{\lambda_2}\log\omega\right)=\ldots.=1.$ Therefore, besides the integrals indicated in (5.10), we have the integrals as follows:

$$(5.11)\qquad \theta/\varphi,\ \ldots.,\ \omega^{\frac{1}{\lambda_2}}/\varphi^{\frac{1}{\lambda_1}},\ \ldots\ldots\ldots$$

The total number of the integrals indicated in (5.10) and (5.11) is $(k-1)+l+\ldots.+m+\ldots.=n-1$ where n denotes the total number of the variables. Moreover the functions indicated in (5.10) and (5.11) are independent to each other. For, from (5,5), U_s $(s=0,1,\ldots,k-2)$ is a function of $\varphi,\ \varphi_1,\ \ldots\ldots,\ \varphi_{s+1}$ and depends actually upon φ_{s+1}; V_s is a function of $\theta,\ \theta_1,\ \ldots\ldots,\ \theta_{s+1}$ and depends actually upon θ_{s+1}; $\ldots.$, W_s is a function of $\omega,\ \omega_1,\ \ldots.,\ \omega_{s+1}$ and depends actually upon ω_{s+1}; $\ldots.$ Therefore, if the functions indicated in (5.10) and (5.11) are not independent to each other, then $\varphi,\ \varphi_1,\ \ldots.;\ \theta,\ \theta_1,\ \ldots\ldots;\ \ldots.;\ \omega,\ \omega_1\ \ldots.,\ \ldots\ldots$ are not independent to each orher. However, for $x_{ij}=0$, the Jacobian

$$\frac{\partial(\varphi,\ \varphi_1,\ \ldots.,\ \varphi_{k-1};\ \theta,\ \theta_1,\ \ldots;\ \ldots;\ \omega,\ \omega_1,\ \ldots.;\ \ldots.)}{\partial(x_{1k},\ x_{1k-1},\ \ldots,\ x_{11};\ x_{2l},\ x_{2l-1},\ \ldots;\ \ldots.;\ x_{rm},\ x_{rm-1},\ \ldots;\ \ldots.)}$$

is $1!2!\ldots(k-1)!1!2!\ldots(l-1)!\ldots.\neq 0$. This is a contradiction.

Thus we have $n-1$ independent integrals of the equation (5.1), which are of the forms indicated in (5.10) and (5.11).

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