Embedding Theorem of Continuous Regular Rings

By

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Let L be a reducible continuous geometry, and Ω the set of all maximal neutral ideals J in L. Kawada-Matsushima-Higuchi [1][1] has proved that L is isomorphic to a sublattice of $\Pi(L/J; J\varepsilon\Omega)$, where L/J are irreducible continuous geometries. In this paper, I apply this result to a reducible continuous regular ring $\mathfrak{R}$, the set $\bar{R}_{\mathfrak{R}}$ of all principal right ideals of $\mathfrak{R}$ being a reducible continuous geometry. And I obtain an embedding theorem of $\mathfrak{R}$. (Cf. Theorem 3·2, below.)

§ 1. Dimension Functions of Reducible Continuous Geometries.

Let L be a continuous complemented modular lattice, i. e. a reducible continuous geometry, and Z the center of L. Then Z is a complete Boolean algebra. Denote by Ω the set of all maximal ideals $\mathfrak{p}$ of Z. For any $z\varepsilon Z$, let $E(z)$ be the set of all maximal ideals which do not contain z. Using $(E(z); z\varepsilon Z)$ as an additive basis for the open sets of Ω, Ω is a totally-disconnected bicompact Hausdoff space. T. Iwamura [1] proved that for any $a\varepsilon L$, there is a continuous functions $D(a)=\delta(a,\mathfrak{p})$ defined in Ω, which has the following properties;

(1°) $0\leqq D(a)\leqq 1$, $D(0)=0$, $D(1)=1$.

(2°) $a>0$ implies $D(a)>0$.

(3°) when $z\varepsilon Z$, $\delta(z,\mathfrak{p})=0$ or 1, according as $z\varepsilon\mathfrak{p}$ or not.

(4°) $D(a\vee b)+D(a\wedge b)=D(a)+D(b)$.

(5°) $a \gtreqless b$ are equivalent to $D(a)\gtreqless D(b)$ respectively.

LEMMA 1·1, *For any $a\varepsilon L$, let a real number $m(a)$ be defined as follows:*

(α) $0\leqq m(a)\leqq 1$, $m(0)=0$, $m(1)=1$,

(β) $z\varepsilon Z$ *implies* $m(z)=0$ *or* 1,

(γ) $m(a\vee b)+m(a\wedge b)=m(a)+m(b)$.

Put $\mathfrak{p}=(z;\, m(z)=0,\, z\varepsilon Z)$, $J=(a;\, m(a)=0)$. *Then $\mathfrak{p}$ is a maximal ideal in Z, and J is a maximal neutral ideal in L. And $a\varepsilon J$ when and only when* $r_n(A_a, A_1)\varepsilon\mathfrak{p}$ $(n=1,2....)$[2].

PROOF. Cf. Kawada-Matsushima-Higuchi [1].

THEOREM 1·1. *Let J be a maximal neutral ideal in L, and $\mathfrak{p}$ a maximal ideal in Z. Then*

(1°) $\mathfrak{p}(J)=(z;\, z\varepsilon J,\, z\varepsilon Z)$ *is a maximal ideal in* Z,

(2°) $J(\mathfrak{p})=(a;\, \delta(a,\mathfrak{p})=0)$ *is a maximal neutral ideal in* L,

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(1) The numbers in square brackets refer to the list given at the end of this paper.

(2) For the definition of $r_n(A_a, A_1)$ cf. v. Neumann [1] III 30.

(3°) $\mathfrak{P}(J(\mathfrak{P}))=\mathfrak{P}$, $J(\mathfrak{P}(J))=J$.[1]

PROOF. (i) It is evident that $\mathfrak{P}(J)$ is an ideal in Z which does not contain 1. If $\mathfrak{P}(J)$ is not maximal, then there exists an ideal $\mathfrak{z}$ such as $\mathfrak{P}(J)<\mathfrak{z}<Z$. Denote by $I(\mathfrak{z})$ the set of all $a\varepsilon L$ such as $a\leqq z$ for some $z\varepsilon\mathfrak{z}$. Then $I(\mathfrak{z})$ is a neutral ideal in L, and $J<I(\mathfrak{z})<L$. This contradicts to the fact that J is maximal. Hence $\mathfrak{P}(J)$ is maximal.

(ii) For a fixed $\mathfrak{P}$, $\delta(a,\mathfrak{P})$ has the same propeties as $m(a)$ in Lemma 1·1, hence $J(\mathfrak{P})$ is a maximal neutral ideal in L.

(iii) $z\varepsilon\mathfrak{P}\rightleftarrows\delta(z,\mathfrak{P})=0\rightleftarrows z\varepsilon J(\mathfrak{P})\rightleftarrows z\varepsilon\mathfrak{P}(J(\mathfrak{P}))$.

Hence $\mathfrak{P}=\mathfrak{P}(J(\mathfrak{P}))$.

(iv) By (i) and (ii), $J(\mathfrak{P}(J))$ is a maximal neutral ideal in L. If $J\neq J(\mathfrak{P}(J))$, then by Lemma 1·1, there are $a\varepsilon J$ and $n>0$ such that $r_n(A_a,A_1)\not\varepsilon\mathfrak{P}(J)$. Since

$$[r_n(A_a,A_1)]=n[r_n(A_a,A_1)\wedge a]+[p_n], \qquad [p_n]\ll[r_n(A_a,A_1)\wedge a],^{(2)}$$

and $r_n(A_a,A_1)\wedge a\varepsilon J$, we have $r_n(A_a,A_1)\varepsilon J$, which contradicts to the fact $r_n(A_a A_1)\not\varepsilon\mathfrak{P}(J)$. Therefore $J=J(\mathfrak{P}(J))$.

Theorem 1·1 shows that there is a one to one correspondence between the maximal ideals in Z and the maximal neutral ideals in L. Hence we denote by the same Ω the set of all maximal neutral ideals in L.

Denote $a\equiv b\ (J)$, when $a\vee b=(a\wedge b)\vee t$, $t\varepsilon J$. Then "$\equiv(J)$" is a congruence relation. Hence we can define $a/J=(x;x\equiv a(J))$, $L/J=(a/J;a\varepsilon L)$. Now we have the following embedding theorem of continuous complemented modular lattices, which has been obtained by Iwamura [1] and modified by Kawada-Matsushima-Higuchi [1].

THEOREM 1·2. *Every L/J is an irreducible continuous complemented modular lattice, and L is carried into (a part of) the product $\Pi(L/J;J\varepsilon\Omega)$ lattice-isomorphically by the transformation $a\rightarrow[a/J;J\varepsilon\Omega]$.*

Now we have the following theorem:

THEOREM 1·3.[3] *For every $a\varepsilon L$, if we define a real valued function $f_a(\mathfrak{P})$ $(\mathfrak{P}\varepsilon\Omega)$, such that*

(α) $0\leqq f_a(\mathfrak{P})\leqq 1$, *for all* $\mathfrak{P}\varepsilon\Omega$,

(β) *when* $z\varepsilon Z$, $f_z(\mathfrak{P})=0$ *or* 1 *according as* $z\varepsilon\mathfrak{P}$ *or not,*

(γ) $f_{a\vee b}(\mathfrak{P})+f_{a\wedge b}(\mathfrak{P})=f_a(\mathfrak{P})+f_b(\mathfrak{P})$,

then $f_a(\mathfrak{P})$ is uniquely determined and $f_a=\mathrm{D}(a)$.

PROOF. (i) Since $\mathfrak{P}=(z;f_z(\mathfrak{P})=0)=(z;\delta(z,\mathfrak{P})=0)$, by Lemma 1·1, $f_a(\mathfrak{P})=0$ when and only when $\delta(a,\mathfrak{P})=0$.

(ii) When $a\equiv b(J)$, let d be such that $a\vee b=(a\wedge b)\oplus d$,[4] then $d\varepsilon J$. Hence by Theorem 1·1 $\delta(d,\mathfrak{P}(J))=0$. Therefore by (i) $f_d(\mathfrak{P}(J))=0$. Consequently $f_{a\vee b}(\mathfrak{P}(J))=f_{a\wedge b}(\mathfrak{P}(J))$, that is, $f_a(\mathfrak{P}(J))=f_b(\mathfrak{P}(J))$.

(iii) From (ii), we can define $m(a/J)=f(\mathfrak{P}(J))$. Then $m(0/J)=0$, $m(1/J)=1$. Applying

(1) This theorem is obtained by Kawada-Matsushima-Higuchi [1], where $J(\mathfrak{P})$ is defined as $(a;r_n(A_a,A_1)\varepsilon\mathfrak{P},\ n=1,2,\ldots\ldots)$, which is equivalent to (2°) by Lemma 1·1. Using $\delta(a,\mathfrak{P})$ the proof is somewhat simplified.

(2) v. Neumann [1] III 30. Here $[a]$ means A_a.

(3) About this theorem, I am much indebted to Mr. U. Sasaki.

(4) $c=a_1\oplus\ldots\ldots\oplus a_n$ means $c=a_1\vee\ldots\ldots\vee a_n$ and $(a_1,\ldots\ldots,a_n)\perp$.

v. Neumann [1] I 72, Cor. 1 to L/J, we have $m(a/J)=\delta(a,\mathfrak{p}(J))$. Since this holds for all $J\varepsilon\Omega$, we have $f_a(\mathfrak{p})=\delta(a,\mathfrak{p})$, that is, $f_a=D(a)$.

REMARK 1·1. Let $a^*\varepsilon L$, then $L^*=L(0,a^*)$ is also a continuous complemented modular lattice. For $a\varepsilon L^*$, we have two dimensions, that is, $D(a)$ as an element in L, and $D^*(a)$ as an element in L^*. By v. Neumann [1] III Theorem 1·6, the center Z^* of L^* is the set $(z\wedge a^*;z\varepsilon Z)$. Hence between the maximal ideals $\mathfrak{p}$ in Z which do not contain $e(a^*)$ and the maximal ideals $\mathfrak{p}^*$ in Z^*, there is a one to one correspondence:

$$\mathfrak{p}\to\mathfrak{p}^*=(z\wedge a^*;z\varepsilon\mathfrak{p}),\qquad \mathfrak{p}^*\to\mathfrak{p}=(z;z\wedge a^*\varepsilon\mathfrak{p}^*).$$

Let

$$f_a(\mathfrak{p}^*)=\frac{\delta(a,\mathfrak{p})}{\delta(a^*,\mathfrak{p})}\qquad(\mathfrak{p}\varepsilon E(e(a^*))),$$

then $f_a(\mathfrak{p}^*)$ satisfies (α), (γ) in Theorem 1·3. For $z^*=z\wedge a^*\varepsilon Z^*$,

$$f_{z^*}(\mathfrak{p}^*)=\frac{\delta(z\wedge a^*,\mathfrak{p})}{\delta(a^*,\mathfrak{p})}=\frac{\delta(z,\mathfrak{p})\wedge\delta(a^*,\mathfrak{p})}{\delta(a^*,\mathfrak{p})}.$$

Hence $f_a(\mathfrak{p}^*)$ satisfies (β) in Theorem 1·3. Therefore $f_a(\mathfrak{p}^*)=\delta^*(a,\mathfrak{p}^*)$, that is,

$$\delta^*(a,\mathfrak{p}^*)=\frac{\delta(a,\mathfrak{p})}{\delta(a^*,\mathfrak{p})}.\qquad(e(a^*)\notin\mathfrak{p})$$

§ 2. Rank of Continuous Regular Rings.

Let $\mathfrak{R}$ be a continuous regular ring, that is, the set $\bar{R}_{\mathfrak{R}}$ of all principal right ideals in $\mathfrak{R}$ is a continuous complemented modular lattice. And the set $\bar{L}_{\mathfrak{R}}$ of all principal left ideals in $\mathfrak{R}$ is dual lattice-isomorphic to $\bar{R}_{\mathfrak{R}}$. Then there are unique dimension functions $D((a)_r)$ and $D'((a)_l)$ in $\bar{R}_{\mathfrak{R}}$ and $\bar{L}_{\mathfrak{R}}$ respectivly. The center $Z_{\mathfrak{R}}$ of $\bar{R}_{\mathfrak{R}}$ (and $\bar{L}_{\mathfrak{R}}$) is composed of all two-sided principal ideals $(\eta)_*$, where η are idempotents in the centrum $\mathfrak{Z}$ of $\mathfrak{R}$. But the set $\mathfrak{Z}_e$ of all idempotents in $\mathfrak{Z}$ is a Boolean ring, and it is lattice-isomorphic to $\mathfrak{Z}_{\mathfrak{R}}$ under the correspondence $\eta\leftrightarrow(\eta)_*$. Hence we may use the same symbols $\mathfrak{p}$ for the maximal ideals in $\mathfrak{Z}_e$ and in $Z_{\mathfrak{R}}$.

DEFINITION 2·1. For $a\varepsilon\mathfrak{R}$, we denote by $\eta(a)$ the smallest idempotent η in $\mathfrak{Z}_e$ such that $a=\eta a=\eta a$.

LEMMA 2·1. *In $\bar{R}_{\mathfrak{R}}$, we have*

(i) $e((a)_r)=(\eta(a))_*$,

(ii) $(a)_r\sim(\beta)_r$ *implies* $\eta(a)=\eta(\beta)$,

(iii) *a factor-correspondence between* $(a)_r$ *and* $(\beta)_r$ *implies* $\eta(a)=\eta(\beta)$.

Similarly for $\bar{L}_{\mathfrak{R}}$.

PROOF. (i) Since $(a)_r\leqq(\eta)_*$ holds when and only when $a=\eta a=a\eta$, we have $e((a)_r)=(\eta(a))_*$.

(ii) Since $(a)_r\sim(\beta)_r$ implies $e((a)_r)=e((\beta)_r)$, we have $\eta(a)=\eta(\beta)$.

(iii) If φ, ψ are special factors defining the factor-correspondence between $(a)_r$ and $(\beta)_r$, then by v. Neumann [1] II Lemma 15·2, $(a)_r=(\varepsilon)_r$, $(\beta)_r=(\varphi)_r$, $\varphi\varepsilon=\varphi$ where $\varepsilon=\psi\varphi$. Then $(\varepsilon)_l=(\varphi)_l$. Hence by (ii) $\eta(a)=\eta(\varepsilon)=\eta(\varphi)=\eta(\beta)$.

LEMMA 2·2. *The center of* $L((0),(a)_r)$ *is* $((\eta a)_r;\eta\varepsilon\mathfrak{Z}_e)$.

PROOF. The center of $L((0),(a)_r)$ is $((\eta_*\wedge(a)_r;\eta\varepsilon\mathfrak{Z}_e)$, and $(\eta)_*\wedge(a)_r=(\eta a)_r$.

LEMMA 2·3. *If there is a factor-correspondence between* $\mathfrak{a}$ *and* $\mathfrak{b}$ *in* $\bar{R}_{\mathfrak{R}}$, *then* $D(\mathfrak{a})=D(\mathfrak{b})$.

PROOF. (i) As in the proof (iii) of Lemma 2·1, φ,ψ being the special factors, $(a)_r=(\varepsilon)_r$,

$(\beta)_r=(\varphi)_r$, $\varphi\varepsilon=\varphi$, $\psi\varphi=\varepsilon$. And the given factor-correspondence generates a lattice-isomorphism of $L((0),(\varepsilon)_r)$ aud $L((0),(\varphi)_r)$. By Lemma 2·2, any element of the center of $L((0),(\alpha)_r)$ is expressed as $(\eta\varepsilon)_r$ $(\eta\varepsilon\mathfrak{Z}_e)$, then the element of the center of $L((0),(\varphi)_r)$ which corresponds to $(\eta\varepsilon)_r$ is $(\varphi\eta\varepsilon)_r=(\eta\varphi\varepsilon)_r=(\eta\varphi)_r$.

(ii) For $\mathfrak{a}_0\varepsilon L^*=L((0),\mathfrak{a})$, by Remark 1·1, the dimension of $\mathfrak{a}_0$ in L^* is $\dfrac{\delta(\mathfrak{a}_0,\mathfrak{p})}{\delta(\mathfrak{a},\mathfrak{p})}$ where $\mathfrak{p}$ are the maximal ideals in $Z_{\mathfrak{R}}$ which do not contain $e(\mathfrak{a})=e((\varepsilon)_r)$. But by Lemma 2·1. (i) we may consider $\mathfrak{p}$ as the maximal ideals in $\mathfrak{Z}_e$ which does not contain $\eta(\varepsilon)$.

(iii) Let $\mathfrak{b}_0$ be the element of $L^{**}=L((0),\mathfrak{b})$, which corresponds to $\mathfrak{a}$, by the lattice-isomorphism of L^* and L^{**}. By (i) the corresponding elements of the centers in L^* and L^{**} are expressed by the same $\eta\varepsilon\mathfrak{Z}_e$, and by Lemma 2·1 (iii), $\eta(\varepsilon)=\eta(\varphi)$. Hence by (ii)

$$\frac{\delta(\mathfrak{a}_0,\mathfrak{p})}{\delta(\mathfrak{a},\mathfrak{p})}=\frac{\delta(\mathfrak{b}_0,\mathfrak{p})}{\delta(\mathfrak{b},\mathfrak{p})}\qquad(\eta(\alpha)\notin\mathfrak{p}).$$

Thus there exists a real constant C such that $\delta(\mathfrak{b}_0,\mathfrak{p})=C\delta(\mathfrak{a}_0,\mathfrak{p})$, that is, $D(\mathfrak{b}_0)=CD(\mathfrak{a}_0)$ for every $\mathfrak{a}_0,\mathfrak{b}_0$ which correspond.

(iv) Suppose now that $\mathfrak{a}\not\leq\mathfrak{b}$. Then there exists $\mathfrak{a}_0$, such that $\mathfrak{a}=(\mathfrak{a}\wedge\mathfrak{b})\oplus\mathfrak{a}_0$, whence $\mathfrak{a}_0\wedge\mathfrak{b}\leqq\mathfrak{a}_0\wedge\mathfrak{a}\wedge\mathfrak{b}=(0)$, $\mathfrak{a}_0\neq(0)$. Let $\mathfrak{b}_0$ correspond to $\mathfrak{a}_0$, then $\mathfrak{b}_0\neq(0)$, and the factor correspondence of $\mathfrak{a}$, $\mathfrak{b}$ generates a lattice-isomorphism of $L((0),\mathfrak{a}_0)$, $L((0),\mathfrak{b}_0)$. Since $\mathfrak{a}_0\wedge\mathfrak{b}_0\leqq\mathfrak{a}_0\wedge\mathfrak{b}=(0)$, this isomorphism is a perspective isomorphism.[1] Therefore $\mathfrak{a}_0\sim\mathfrak{b}_0$, whence $D(\mathfrak{a}_0)=D(\mathfrak{b}_0)$, and $C=1$. Thus $\mathfrak{a}\not\leq\mathfrak{b}$ implies $D(\mathfrak{a})=D(\mathfrak{b})$. Similarely $\mathfrak{a}\not\geq\mathfrak{b}$ implies $D(\mathfrak{a})=D(\mathfrak{b})$.

LEMMA 2·4. *$(\alpha)_l=(\beta)_l$ implies $D((\alpha)_r)=D((\beta)_r)$.*

PROOF. Since $(\alpha)_l=(\beta)_l$, there exist φ, ψ with $\beta=\varphi\alpha$, $\alpha=\psi\beta$, whence $\alpha=\psi\varphi\alpha$, $\beta=\varphi\psi\beta$, and φ, ψ are factors of a factor-correspondence between $(\alpha)_r$ aud $(\beta)_r$. Hence by Lemma 2·3 we have $D((\alpha)_r)=D((\beta)_r)$.

THEOREM 2·1. $D((\alpha)_r)=D'((\alpha)_l)$.

PROOF. From Lemma 2·4, $(\alpha)_l=(\beta)_l$ implies $\delta((\alpha)_r,\mathfrak{p})=\delta((\beta)_r,\mathfrak{p})$. Hence for any element $(\alpha)_l\varepsilon\bar{L}_{\mathfrak{R}}$, we can define a function $f_{(\alpha)_l}(\mathfrak{p})=f((\alpha)_l,\mathfrak{p})$ such that $f((\alpha)_l,\mathfrak{p})=\delta((\alpha)_r,\mathfrak{p})$. Then $0\leqq f((\alpha)_l,\mathfrak{p})\leqq 1$, $f((0),\mathfrak{p})=0$, $f((1)_l,\mathfrak{p})=1$, for all $\mathfrak{p}\varepsilon\Omega$, and when $\eta\varepsilon\mathfrak{Z}_e$, $f((\eta)_*,\mathfrak{p})=0$ or 1 according as $\eta\varepsilon\mathfrak{p}$ or not.

Next, for any $(\alpha)_l$, $(\beta)_l$ let

$$(\alpha)_l=((\alpha)_l\wedge(\beta)_l)\oplus(\gamma)_l,\qquad(\beta)_l=((\alpha)_l\wedge(\beta)_l)\oplus(\delta)_l,$$

then $(\alpha)_l\vee(\beta)_l=((\alpha)_l\wedge(\beta))_l\oplus(\gamma)_l\oplus(\delta)_l$. Therefore there exist idempotents ε_1, ε_2, ε_3, such that $\quad(\alpha)_l\wedge(\beta)_l=(\varepsilon_1)_l,\quad(\gamma)_l=(\varepsilon_2)_l,\quad(\delta)_l=(\varepsilon_3)_l,\quad\varepsilon_i\varepsilon_j=0\quad(i\neq j)$,

and $\quad(\alpha)_l\vee(\beta)_l=(\varepsilon_1+\varepsilon_2+\varepsilon_3)_l,\quad(\alpha)_l=(\varepsilon_1+\varepsilon_2)_l,\quad(\beta)_l=(\varepsilon_1+\varepsilon_3)_l.$

Now

$$\begin{aligned}f((\alpha)_l\vee(\beta)_l,\mathfrak{p})&=f((\varepsilon_1+\varepsilon_2+\varepsilon_3)_l,\mathfrak{p})=\delta((\varepsilon_1+\varepsilon_2+\varepsilon_3)_r,\mathfrak{p})\\&=\delta((\varepsilon_1+\varepsilon_2)_r,\mathfrak{p})+\delta((\varepsilon_1+\varepsilon_3)_r,\mathfrak{p})-\delta((\varepsilon_1)_r,\mathfrak{p})\\&=f((\varepsilon_1+\varepsilon_2)_l,\mathfrak{p})+f((\varepsilon_1+\varepsilon_3)_l,\mathfrak{p})-f((\varepsilon_1)_l,\mathfrak{p})\\&=f((\alpha)_l,\mathfrak{p})+f((\beta)_l,\mathfrak{p})-f((\alpha)_l\wedge(\beta)_l,\mathfrak{p}).\end{aligned}$$

Applying Theorem 1·3 to $\bar{L}_{\mathfrak{R}}$, we have $f((\alpha)_l,\mathfrak{p})=\delta'((\alpha)_l,\mathfrak{p})$. Hence we have $D'((\alpha)_l)=D((\alpha)_r)$.

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(1) v. Neumenn [1] II Theorem 15·3.

DEFINITION 2·2. For every $\alpha\varepsilon\Re$ we define the *rank* of α by $R(\alpha)=D((\alpha)_r)=D'((\alpha)_l)$.

THOREEM 2·2. *The rank $R(\alpha)=r(\alpha, \mathfrak{P})$ of α is a continuous function defined on the set Ω of all maximal ideals $\mathfrak{P}$ of $\mathfrak{Z}_e$, and has the following properties:*

(1°) $0\leqq R(\alpha)\leqq 1$ *for every* $\alpha\varepsilon\Re$;

(2°) $R(\alpha)=0$ *if and only if* $\alpha=0$;

(3°) $R(\alpha)=1$ *if and only if α is non-singular;*

(4°) *for $\eta\varepsilon\mathfrak{Z}_e$, $r(\eta, \mathfrak{P})=0$ or 1, according as $\eta\varepsilon\mathfrak{P}$ or not;*

(5°) $R(\alpha)=R(\beta)$ *if and only if α is of the form $\alpha=\xi\beta\zeta$ where ξ, ζ are non-singular;*

(6°) $R(\alpha\beta)\leqq R(\alpha)$, $R(\beta)$;

(7°) $R(\alpha+\beta)\leqq R(\alpha)+R(\beta)$;

(8°) *if ε, η are idempotents and $\varepsilon\eta=\eta\varepsilon=0$, then* $R(\varepsilon+\eta)=R(\varepsilon)+R(\eta)$.

PROOF. All properties (1°)—(8°), except (4°), can be proved as in v. Nenmann [1] II Theorem 17·1. (4°) follows from the property of $D((\alpha)_r)=\delta((\alpha)_r, \mathfrak{P})$.

§ 3 Embedding Theorem of Continuous Regular Rings.

THEOREM 3·1. *Let $\mathfrak{a}$ be a maximal two-sided ideal in $\Re$, and J a maximal neutral ideal in $\bar{R}_{\Re}$. Then*

(1°) $J(\mathfrak{a})=((\alpha)_r;(\alpha)_r\leqq\mathfrak{a})$ *is a maximal neutral ideal in* $\bar{R}_{\Re}$,

(2°) $\mathfrak{a}(J)=(\alpha;(\alpha)_r\varepsilon J)$ *is a maximal two-sided ideal in* $\Re$,

(3°) $J(\mathfrak{a}(J))=J$, $\quad\mathfrak{a}(J(\mathfrak{a}))=\mathfrak{a}$.

PROOF. (i) Let $\mathfrak{a}$ be any two-sided ideal in $\Re$, then it is evident that $J(\mathfrak{a})=((\alpha)_r;(\alpha)_r\leqq\mathfrak{a})$ is an ideal in $\bar{R}_{\Re}$. Next when $(\alpha)_r\leqq\mathfrak{a}$, $(\alpha)_r\sim(\beta)_r$, let $\mathfrak{c}$ be a common complement of $(\alpha)_r$ and $(\beta)_r$. Then there exist idempotents ε, η such that

$$(\alpha)_r=(\varepsilon)_r, \qquad \mathfrak{c}=(1-\varepsilon)_r=(1-\eta)_r, \qquad (\beta)_r=(\eta)_r.$$

Now since $(1-\varepsilon)_r=(1-\eta)_r$, we have also $(\varepsilon)_l=(\eta)_l$. Hence $(\alpha)_r\leqq\mathfrak{a}$ implies ε, $\eta\varepsilon\mathfrak{a}$, and therefore $(\beta)_r\leqq\mathfrak{a}$. Consequeutly. $J(\mathfrak{a})$ is a neutral ideal in $\bar{R}_{\Re}$.

(ii) Let J be a maxmal neutral ideal in $\bar{R}_{\Re}$. Then from Theorem 1·1, $(\alpha)_r\varepsilon J$ if and only if $\delta((\alpha)_r, \mathfrak{P}(J))=0$. Therefore $\mathfrak{a}(J)=(\alpha;r(\alpha, \mathfrak{P}(J))=0)$. When $\alpha, \beta\varepsilon\mathfrak{a}(J)$, since

$$r(\alpha-\beta, \mathfrak{P}(J))\leqq r(\alpha, \mathfrak{P}(J))+r(\beta, \mathfrak{P}(J))=0,$$

we have $\alpha-\beta\varepsilon\mathfrak{a}(J)$, When $\alpha\varepsilon\mathfrak{a}(J)$, $\xi\varepsilon\Re$, since

$$r(\alpha\xi, \mathfrak{P}(J))\leqq r(\alpha, \mathfrak{P}(J))=0, \qquad r(\xi\alpha, \mathfrak{P}(J))\leqq r(\alpha, \mathfrak{P}(J))=0,$$

we have $\alpha\xi$, $\xi\alpha\varepsilon\mathfrak{a}(J)$. That is, $\mathfrak{a}(J)$ is a two-sided ideal, and $\mathfrak{a}(J)<\Re$.

(iii) Let J be a maximal neutral ideal in $\bar{R}_{\Re}$, then since

$$(\alpha)_r\varepsilon J\leftrightarrows\alpha\varepsilon\mathfrak{a}(J)\leftrightarrows(\alpha)_r\leqq\mathfrak{a}(J)\rightleftarrows(\alpha)_r\varepsilon J(\mathfrak{a}(J)),$$

we have $J=J(\mathfrak{a}(J))$.

(iv) Let $\mathfrak{a}$ be a maximal two-sided ideal in $\Re$, and if we assume that $J(\mathfrak{a})$ is not maximal in $\bar{R}_{\Re}$, then there exists a maximal neutral ideal I in $\bar{R}_{\Re}$ such that $J(\mathfrak{a})<I$. From (ii), $\mathfrak{a}(I)$ is a two-sided ideal in $\Re$ and $\mathfrak{a}(I)<\Re$. Since

$$\alpha\varepsilon\mathfrak{a}\rightarrow(\alpha)_r\leqq\mathfrak{a}\rightarrow(\alpha)_r\varepsilon J(\mathfrak{a})\rightarrow(\alpha)_r\varepsilon I\rightarrow\alpha\varepsilon\mathfrak{a}(I),$$

we have $\mathfrak{a}\leqq\mathfrak{a}(I)$. $\mathfrak{a}$ being maximal, we have $\mathfrak{a}=\mathfrak{a}(I)$. Hence by (iii) $J(\mathfrak{a})=J(\mathfrak{a}(I))=I$, which contradicts to $J(\mathfrak{a})<I$. Consequently $J(\mathfrak{a})$ is a maximal neutral ideal in $\bar{R}_{\Re}$.

(v) Let $\mathfrak{a}$ be a maximal two-sided ideal in $\mathfrak{R}$, then by (iv) $J(\mathfrak{a})$ is a maximal neutral ideal in $\bar{R}_{\mathfrak{R}}$. Since

$$a \varepsilon \mathfrak{a} \rightleftarrows (a)_r \leq \mathfrak{a} \rightleftarrows (a)_r \varepsilon J(\mathfrak{a}) \rightleftarrows a \varepsilon \mathfrak{a}(J(\mathfrak{a})),$$

we have $\mathfrak{a}=\mathfrak{a}(J(\mathfrak{a}))$.

(vi) Let J be a maximal ideal in $\bar{R}_{\mathfrak{R}}$, and assume that $\mathfrak{a}(J)$ is not maximal in $\mathfrak{R}$, then there exists a maximal two-sided ideal $\mathfrak{b}$ in $\mathfrak{R}$ such that $\mathfrak{a}(J) < \mathfrak{b}$. By (iii) $J=J(\mathfrak{a}(J)) \leq J(\mathfrak{b}) < \bar{R}_{\mathfrak{R}}$. J being maximal, we have $J=J(\mathfrak{b})$. Hence by (v) $\mathfrak{a}(J)=\mathfrak{a}(J(\mathfrak{b}))=\mathfrak{b}$, which is contradictory to $\mathfrak{a}(J) < \mathfrak{b}$. Consequently $\mathfrak{a}(J)$ is a maximal two-sided ideal in $\mathfrak{R}$.

LEMMA 3·1. *$\mathfrak{R}$ is irreducible if and only if $\mathfrak{R}$ is simple.*

PROOF. (i) If $\mathfrak{R}$ is a direct sum of two subrings $\mathfrak{R}_1$, $\mathfrak{R}_2$, then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are two-sided ideals in $\mathfrak{R}$.[1] Hence if $\mathfrak{R}$ is simple, then $\mathfrak{R}$ is irreducible.

(ii) Next assume that $\mathfrak{R}$ is irreducible. Since $\bar{R}_{\mathfrak{R}}$ is irreducible, (0) and $\mathfrak{R}$ are the only elements in $Z_{\mathfrak{R}}$.[2] Since there exists only one maximal ideal $\mathfrak{p}$ in $Z_{\mathfrak{R}}$, which is composed of only (0), by Theorem 1·1 there exists only one maximal neutral ideal $J=J(\mathfrak{p})$ in $\bar{R}_{\mathfrak{R}}$, which is composed of only (0). Hence by Theorem 3·1, there exists only one maximal two-sided ideal $\mathfrak{a}=\mathfrak{a}(J)$ in $\mathfrak{R}$, which is (0). Consequently $\mathfrak{R}$ is simple.

LEMMA 3·2. *Let $\mathfrak{a}$ be a maximal two-sided ideal in $\mathfrak{R}$, and $\mathfrak{R}/\mathfrak{a}$ the quotient ring whose elements are residue classes $a/\mathfrak{a} = (\xi;\xi \equiv a (mod\ \mathfrak{a}))$. Then $\bar{R}_{\mathfrak{R}/\mathfrak{a}}$ and $\bar{R}_{\mathfrak{R}}/J(\mathfrak{a})$ are lattice-isomorphic by the correspondence $(a/\mathfrak{a})_r \leftrightarrow (a)_r/J(\mathfrak{a})$. And $\mathfrak{R}/\mathfrak{a}$ is a simple continuous regular ring.*

PROOF. Let $a/\mathfrak{a}$ be any element in $\mathfrak{R}/\mathfrak{a}$. Since $\mathfrak{R}$ is regular, there exists an element ξ such that $a=a\xi a$.[3] Then $a/\mathfrak{a}=a/\mathfrak{a}\cdot\xi/\mathfrak{a}\cdot a/\mathfrak{a}$, hence $\mathfrak{R}/\mathfrak{a}$ is a regular ring. Consequently $\bar{R}_{\mathfrak{R}/\mathfrak{a}}$ is a complemented modular latticc. We can easily prove that $\bar{R}_{\mathfrak{R}/\mathfrak{a}}$ and $\bar{R}_{\mathfrak{R}}/J(\mathfrak{a})$ are lattice-isomorphic by the correspondence $(a/\mathfrak{a})_r \leftrightarrow (a)_r/J(\mathfrak{a})$. But since $\bar{R}_{\mathfrak{R}}/J(\mathfrak{a})$, is an irreducible continuous complemented modular lattice,[4] $\mathfrak{R}/\mathfrak{a}$ is a simple continuous regular ring.

DEFINITION 3·1. Let $(\mathfrak{R}_\lambda;\lambda\varepsilon I)$ be a system of rings. By the *direct product* $\Pi(\mathfrak{R}_\lambda;\lambda\varepsilon I)$ we mean the ring of classes $[a_\lambda;\lambda\varepsilon I]$ $(a_\lambda \varepsilon \mathfrak{R}_\lambda)$ having the following operations:

$$[a_\lambda;a\varepsilon I]+[\beta_\lambda;\lambda\varepsilon I]=[a_\lambda+\beta_\lambda;\lambda\varepsilon I],$$

$$[a_\lambda;a\varepsilon I]\cdot[\beta_\lambda;\lambda\varepsilon I]=[a_\lambda\beta_\lambda;\lambda\varepsilon I].$$

THEOREM 3·2. (*Embedding theorem*). *Let Ω be the set of all maximal two-sided ideals in the continuous regular ring $\mathfrak{R}$. Then $\mathfrak{R}$ is isomorphic to a subring of $\Pi(\mathfrak{R}/\mathfrak{a};\mathfrak{a}\varepsilon\Omega)$ where $\mathfrak{R}/\mathfrak{a}$ are simple continuous regular rings.*

PROOF. By Lemma 3·2, $\mathfrak{R}/\mathfrak{a}$ are simple continuous regular rings. Let $\mathfrak{R}_0$ be the set of all elements of $\Pi(\mathfrak{R}/\mathfrak{a};\mathfrak{a}\varepsilon\Omega)$, expressed in the form $[a/\mathfrak{a};\mathfrak{a}\varepsilon\Omega]$, then $\mathfrak{R}_0$ is homomorphic to $\mathfrak{R}$. Let $a\neq 0$ be any element in $\mathfrak{R}$, then there exists a maximal ideal $\mathfrak{p}$ of $Z_{\mathfrak{R}}$ such that $\delta((a)_r,\mathfrak{p})>0$. Then by Theorem 1·1, $(a)_r \bar{\varepsilon} J(\mathfrak{p})$; hence by Theorem 3·1 $a \bar{\varepsilon} \mathfrak{a}(J(\mathfrak{p}))$.

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(1) Cf. v. Neumann [1] II, 12.
(2) Cf. v. Neumaun [1] II Theorem 2·9.
(3) v. Neumann [1] II Theorem 2·2.
(4) Kawada-Matsuhims-Higuchi [1]

Put $\mathfrak{a}_0 = \mathfrak{a}(J(\mathfrak{F}))$, then $a/\mathfrak{a}_0 \neq 0/\mathfrak{a}_0$. Consequently $\mathfrak{R}_0$ is isomorphic to $\mathfrak{R}$.

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