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Finite-Dimensionality of Certain Banach Algebras

By

Tôzirô OGASAWARA

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The totality $\mathcal{B}$ of bounded linear operators T on a Hilbert space $\mathfrak{H}$ to itself is a Banach algebra (C*-algebra) under the norm $\|T\| = \underset{\|f\|=1}{l.u.b.} \|Tf\|$. It is known that $\mathcal{B}$ is reflexive if and only if $\mathcal{B}$ is finite-dimensional [6]. The main purpose of this paper is to show that this is also true for B*-algebras and certain other Banach *-algebras (Theorem 2). And we show that a completely continuous linear operator in $\mathfrak{H}$ is characterized as a weakly completely continuous element of the Banach algebra $\mathcal{B}$ (Theorem 4).

1. An algebra $\mathfrak{A}$ over the complex field C is called a *-algebra provided there is defined in $\mathfrak{A}$ an involution $x \rightarrow x^*$ which is a conjugate-linear anti-automorphism of period two. If $\mathfrak{A}$ is also a B-algebra, then $\mathfrak{A}$ is called a Banach *-algebra [15]. A subalgebra of a *-algebra is called a *-subalgebra provided it is closed under the involution. An element x of a *-algebra is said to be self-adjoint if $x=x^*$, normal if $xx^*=x^*x$.

Let $\mathfrak{A}$ be a *-algebra. Any commutative *-subalgebra is, by Zorn's lemma, contained in a maximal one $\mathfrak{B}$. A commutative *-subalgebra is maximal if and only if it coincides with its commutor. $\mathfrak{B}$ will be closed if $\mathfrak{A}$ is a Banach *-algebra.

LEMMA 1. *Let $\mathfrak{A}$ be a *-algebra such that every maximal commutative *-subalgebra of $\mathfrak{A}$ has a unit and no nilpotent self-adjoint elements. Then $\mathfrak{A}$ has a unit.*

PROOF. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be maximal commutative *-subalgebras of $\mathfrak{A}$. Let e, e' be a unit of $\mathfrak{B}$, $\mathfrak{B}'$ respectively. They are evidently self-adjoint. Since there exist no non zero self-adjoint elements annihilating a maximal commutative *-subalgebra with a unit, hence we obtain

$$e'=e'e+ee'-ee'e. \qquad (1)$$

$$e=ee'+e'e-e'ee'. \qquad (2)$$

From (1) we have

$$e'=e'e'e'=2e'ee'-e'ee'ee'. \qquad (3)$$

$$ee'e=2ee'ee'e-ee'ee'ee'e. \qquad (4)$$

If we put $u=ee'e$, then u is self-adjoint. (4) implies $(u-u^2)^2=0$. Hence by assumption we obtain $u=u^2$. In like manner $e'ee'$ is an idempotent. Then from (3) $e'=e'ee'$, therefore from (2) $e+e'-ee'-e'e=0$, that is, $(e-e')^2=0$, which implies by assumption that $e=e'$. Therefore maximal commutative *-subalgebras have the same unit e. We show that e is a unit of $\mathfrak{A}$. Since any $z\in\mathfrak{A}$ can be written $z=x+iy$, x, y self-adjoint it is sufficient to show that $xe=xe=x$ for every self-adjoint x. This follows from the existence of a maximal commutative *-subalgebra containing x, which completes the proof.

THEOREM 1. *Let $\mathfrak{A}$ be a *-algebra such that every maximal commutative *-subalgebra of $\mathfrak{A}$ is semi-simple, finite-dimensional, and $\mathfrak{A}$ has the property that $xx^*=0$ implies $x=0$. Then $\mathfrak{A}$ is *-isomorphic with a direct sum of full matrix algebras over C of finite orders. Therefore $\mathfrak{A}$ is semi-simple and finite-dimensional.*

PROOF. Let $\mathfrak{B}$ be a maximal commutative *-subalgebra of $\mathfrak{A}$. Since $\mathfrak{B}$ is finite-dimensional and semi-simple, it has a unit e which is also a unit of $\mathfrak{A}$ (Lemma 1) and it is a direct sum of fields Ce_i where each e_i is a primitive idempotent in $\mathfrak{B}$ and $e=\Sigma e_i$. Since by assumption $e_ie_i^*\neq 0$, it follows easily that e_i is self-adjoint. For any given self-adjoint element x, e_ixe_i is self-adjoint and commutative with every e_j. Maximality of $\mathfrak{B}$ implies that $e_ixe_i\in\mathfrak{B}$, whence $e_ixe_i=\lambda e_i$, λ being real. Any element $z\in\mathfrak{A}$ is represented as $z=x+iy$, x, y being self-adjoint. Therefore $e_i\mathfrak{A}e_i=Ce_i$. Since $xx^*=0$ implies $x=0$ for any x, $\mathfrak{A}$ has no nilpotent ideals. Hence $e_i\mathfrak{A}$ is a minimal right ideal [9, Theorem 5], and $\mathfrak{A}$ is a direct sum of $e_i\mathfrak{A}$. Thus $\mathfrak{A}$ is semi-simple. Put $e_ixx^*e_i=\lambda e_i$, then $\lambda\geqq 0$ [16, p. 33]. Therefore if $e_1\mathfrak{A}e_i\neq 0$ holds for some e_i, then there exists an $a_i\in\mathfrak{A}$ such that $e_1a_ie_ia_i^*e_1=e_1$. By making use of these facts it is easy to conclude that $\mathfrak{A}$ is *-isomorphic with a direct sum of full matrix algebras over C of finite orders. Therefore $\mathfrak{A}$ is finite-dimensional and semi-simple.

2. Throughout this section we assume that $\mathfrak{A}$ is a Banach *-algebra with norm $\|x\|$. $\mathfrak{A}$ is called a B*-algebra provided that $\|x\|^2=\|xx^*\|$ holds for every element x [15].

Suppose that $\mathfrak{A}$ is a Banach *-algebra in which

(5) $\qquad k\|x\|^2\leqq\|xx^*\|$, k being a positive constant,

for every normal x, and $xx^*=0$ implies $x=0$. Evidently B*-algebras satisfy these conditions. Any maximal commutative *-subalgebra of $\mathfrak{A}$ is semi-simple and equivalent to the Banach algebra $B(\Omega)$ of complex-valued continuous functions (vanishing at ∞) on some compact (locally compact) Hausdorff space Ω [2, 3]. If any such $B(\Omega)$ is finite-dimensional, then it follows by Theorem 1 that $\mathfrak{A}$ is finite-dimensional. The following lemma gives a criterion for finite-dimensionality of $B(\Omega)$.

LEMMA 2. *$B(\Omega)$ is finite-dimensional if and only if it satisfies any one of the following conditions:*

(*a*) *$B(\Omega)$ is reflexive.*

(*b*) *$B(\Omega)$ is weakly complete.*

(*c*) *The bi-conjugate space of $B(\Omega)$ is separable.*

PROOF. It is sufficient to show this lemma for the case where $B(\Omega)$ consists of real valued functions. Then the lemma holds as a special case of results established for abstract L^∞ spaces [13, pp. 87–89].

By making use of this lemma and Theorem 1 we obtain

THEOREM 2. *Let $\mathfrak{A}$ be a Banach *-algebra in which $k\|x\|^2 \leqq \|xx^*\|$, k being a positive constant, for every normal x, and $xx^*=0$ implies $x=0$. Then $\mathfrak{A}$ is finite-dimensional if and only if any one of the following conditions is satisfied:*

(*a*) *$\mathfrak{A}$ is reflexive.*
(*b*) *$\mathfrak{A}$ is weakly complete.*
(*c*) *The bi-conjugate space of $\mathfrak{A}$ is separable.*

PROOF. If $\mathfrak{A}$ satisfies any one of the conditions (*a*), (*b*), (*c*), then any maximal commutative *-subalgebra $\mathfrak{B}$ will satisfy the corresponding condition since $\mathfrak{B}$ is a closed subspace of $\mathfrak{A}$. It follows by Lemma 2 that $\mathfrak{B}$ is semi-simple and finite-dimensional. Hence by Theorem 1 we conclude that $\mathfrak{A}$ is finite-dimensional. The converse is evident.

From this theorem we have

COROLLARY. *If an infinite-dimensional Banach *-algebra A is reflexive or weakly complete, then we can introduce no auxiliary norm $|x|$ with $k|x|^2 \leqq |xx^*|$, k being positive constant, such that A becomes a Banach algebra with this norm.*

PROOF. Suppose that A is a Banach algebra with $|x|$. Then two norms are equivalent [10, 17]. Hence A is finite-dimensional by Theorem 2. This is a contradiction and completes the proof.

Example 1. Let $\mathscr{B}$ be the Banach algebra of bounded linear operators in a Hilbert space $\mathfrak{H}$, and let $\mathscr{I}$ be the Banach algebra of completely continuous linear operators in $\mathfrak{H}$. J. Dixmier has proved that $\mathscr{B}$ as a Banach space is isomorphic with the bi-conjugate space of $\mathscr{I}$. Since $\mathscr{I}$ is a Banach *-algebra with the above stated properties (C*-algebra) it follows by Theorem 2 that $\mathscr{B}$ is separable if and only if $\mathscr{B}$ is finite-dimensional.

Example 2. Let A be a proper H*-algebra of W. Ambrose [1, 11]. In other words A is a Banach*-algebra and a Hilbert space such that $(xy, z)=(y, x^*z)$ and $(yx, z)\ (y, zx^*)$, where the parentheses denote the Hilbert space inner product, and $xA=0$ implies $x=0$. Consider an auxiliary norm $|x|$ defined by

$$|x| = \underset{\|y\|=1}{l.u.b.} \|xy\|.$$

Then it is easy to see that $|x|$ satisfies, in addition to the usual multiplicative property, the condition $|x|^2=|xx^*|$. Denote by A_1 a normed algebra A with this norm. We show that the following conditions are equivalent;

(*1*) A_1 is a Banach algebra.
(*2*) A is finite-dimensional.
(*3*) A has a unit e.

PROOF. By the above Corollary, (*1*) and (*2*) are equivalent, and imply (*3*). If (*3*) holds, $\|x\|/\|e\| \leqq |x| \leqq \|x\|$ shows that two norms $\|x\|$, $|x|$ are equivalent, whence A_1 is complete.

3. Let $\mathfrak{A}$ be a Banach *-algebra. $x \in \mathfrak{A}$ is r. (l.) w. c. c. provided that right (left) multiplication by x is a weakly completely continuous operator. x is w. c. c. if x is both r. w. c. c. and l. w. c. c. Let $\mathfrak{J}$ be the set of w. c. c. elements of $\mathfrak{A}$. With slight modifications of Freundlich's proof for the commutative Banach algebra [8] we can conclude that $\mathfrak{J}$ is a closed two-sided ideal. The same is true for the set of r. (l.) w. c. c. elements. If the involution $x \to x^*$ is continuous, $\mathfrak{J}$ will be self adjoint. If $\mathfrak{A}$ is a B*-algebra or a Banach *-algebra with the condition (*5*) for every element x, then x is r. w. c. c. if and only if x is l. w. c. c. since every closed two-sided ideal is self-adjoint [10]. $\mathfrak{A}$ is called w. c. c, if every element of $\mathfrak{A}$ is w. c. c. For example an H*-algebra is w. c. c. since it is reflexive. It can be shown that a B*-algebra is w. c. c. if and only it is a dual algebra [14]. It's structure is characterized as the B*(∞)-sum of algebras, each of which is the algebra of all completely continuous operators in a Hilbert space [10]. Here we consider a case where $\mathfrak{A}$ has a unit e.

THEOREM 3. *Suppose that $\mathfrak{A}$ has a unit e and satisfies the same assumption as in Theorem 2. Then $\mathfrak{A}$ is w. c. c. if and only if $\mathfrak{A}$ is finite-dimensional.*

PROOF. Let $\mathfrak{A}$ be w. c. c. The unit sphere S of $\mathfrak{A}$ is werkly sequentially compact since $eS=S$. Hence $\mathfrak{A}$ is reflexive [7]. It follows by Theorem 2 that $\mathfrak{A}$ is finite-dimensional. The converse is evident.

4. Let $\mathscr{B}$ ($\mathscr{J}$) be the Banach algebra of bounded (completely continuous) linear operators in a Hilbert space $\mathfrak{H}$. Let $\mathfrak{J}$ be the set of w. c. c. elements of $\mathscr{B}$. As stated in Sec. **3**, $\mathfrak{J}$ is a self-adjoint closed two-sided ideal of $\mathscr{B}$. We shall show that $\mathfrak{J}=\mathscr{J}$.

THEOREM 4. *A bounded linear operator T in $\mathfrak{H}$ is completely continuous if and only if T is a w. c. c. element of $\mathscr{B}$.*

PROOF. Here we use notations due to Dixmier [6] without further reference. Let T be any element of $\mathscr{J}$. Let $\{B_n\}$ be a bounded sequence of elements of $\mathscr{B}$. By the theorem of the canonical decomposition [12] it is easily seen that we can write $T=T_1T_2$, $T_i \in \mathscr{J}$. Since the image $T_2{}^*S$ of the unit sphere S of $\mathfrak{H}$ is relatively compact, we can select a subsequence $\{B_{n_k}\}$ of $\{B_n\}$ such that $\{B^*_{n_k}T^*_2\}$ converges to an element $C^* \in \mathscr{B}$ under the weak topology in the sense of J. v. Neumann. Therefore $\{T_2B_{n_k}\}$ converges to C in the just stated sense, and also $\{TB_{n_k}\}$ converges to $T_1C \in \mathscr{J}$. Let θ be any bounded linear functional in $\mathscr{B}$. We can write $\theta=\varphi+\psi$, where $\varphi \in \mathscr{J}'$, $\psi \in \mathscr{J}^{\perp}$ [6, Théorême 3]. Then $\theta(TB_{n_k}-T_1C)=\varphi(TB_{n_k}-T_1C) \to 0$ [6 Proposition 8]. Thus T is an element of $\mathfrak{J}$.

Conversely let T be any element of $\mathfrak{J}$. First we consider the case where $\mathfrak{H}$ is separable. If $\mathfrak{H}$ is finite-dimensional, it is evident that $\mathfrak{J}=\mathscr{J}$. Let $\mathfrak{H}$ be infinite-dimensional. Calkin [5] has shown that $\mathscr{J}$ is a proper maximal ideal. It follows that either $\mathscr{I}=\mathfrak{J}$ or $\mathscr{B}=\mathfrak{J}$. The latter can not occur by Theorem 3. Therefore $\mathscr{I}=\mathfrak{J}$ if $\mathfrak{J}$ is separable. Now we turn to the general case. Let $\{f_n\}$ be any bounded sequence of elements of $\mathfrak{H}$. Let $\mathfrak{H}_0$ be the closed subspace spanned by the elements f_n, Tf_n, $n=1, 2, \cdots$. Then $\mathfrak{H}_0$ is separable. Let P be a projective operator with $\mathfrak{H}_0$ as its range. It is clear that $Tf_n=PTPf_n \in \mathfrak{H}_0$. And PTP is considered a w.c c. element of the Banach algebra of bounded linear operator in $\mathfrak{H}_0$. It follows from the above result that we can select a subsequence $\{f_{n_k}\}$ such that $\{PTPf_{n_k}\}$ converges strongly to an element g of $\mathfrak{H}_0$. Since $Tf_{n_k}=PTPf_{n_k}$, hence $\{Tf_{n_k}\}$ converges strongly to g. Thus $T \in \mathscr{I}$.

REMARK. If there exists a c.c. element $T(\neq 0)$, then $\mathfrak{H}$ is finite-dimensional. In fact, if $\mathfrak{H}$ is infinite-dimensional, then we put $B_n=\{f_n, g\}$, where $\{f_n\}$ is orthonormal and $\| Tg \|=1$. Since $\| TB_n-TB_m \|=\|\{f_n, Tg\}-\{f_m, Tg\}\|=\|f_n-f_m\|$, we can not select a subsequence $\{B_{n_k}\}$ such that $\{TB_{n_k}\}$ converges under the uniform topology.

5. In this section we consider a real Banach *-algebra $\mathfrak{A}$ in which the involution $x \to x^*$ is a linear anti-automorphism of period two. If the complexification [10, 17] of $\mathfrak{A}$ satisfies the assumption of Theorem 1, then $\mathfrak{A}$ is finite-dimensional if and only if $\mathfrak{A}$ satisfies the conditions stated in Theorem 1, 2. As an example [4] we give

THEOREM 5. *Let $\mathfrak{A}$ be a real Banach *-algebra in which* $\| x \|^2 \leqq \| xx^*+yy^* \|$. *Then $\mathfrak{A}$ is finite-dimensional if and only if one of the following conditions is satisfied:*

(*a*) *$\mathfrak{A}$ is reflexive.*
(*b*) *$\mathfrak{A}$ is weakly complete.*
(*c*) *The bi-conjugate space of $\mathfrak{A}$ is separable.*
(*d*) *$\mathfrak{A}$ is w. c. c. and has a unit.*

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