JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 17, NO. 2, DEC., 1953

A Theorem on the Unitary Groups over Rings

By

Takayuki NÔNO

(Received July 24, 1953)

1. It is well known(1) that the unitary group $U(n, Q)$ over the sfield Q of quaternions is isomorphic to the intersection of the unitary group $U(2n, C)$ and the symplectic group $Sp(2n, C)$ over the field C of complex numbers. We shall extend this theorem for the unitary groups over rings with unit element and with an involution.

Let R be an arbitrary ring with unit element ϵ, and with an involution I, that is, a one-to-one mapping $\xi \to \xi^I$ of R onto itself, distinct from the identity, such that $(\xi+\eta)^I=\xi^I+\eta^I$, $(\xi\eta)^I=\eta^I\xi^I$, and $(\xi^I)^I=\xi$. Let V_R be a right and left vector space over R; an hermitian form over V_R is a mapping $(x,y)\to f(x,y)$ of $V_R\times V_R$ into R, which for any x, is linear in y, and such that $f(y,x)=f(x,y)^I$. This implies that $f(x,y)$ is additive in x and such that $f(x\lambda, y)=\lambda^I f(x,y)$. And suppose that the form f is nondegenerate, or in other words that if $f(x,y)=0$ for all $y\in V_R$, then $x=0$. A unitary transformation u of V_R is a one-to-one linear mapping of V_R onto itself such that $f(u(x), u(y))=f(x,y)$ identically, these transformations constitute the unitary group $U(V_R, f)$. As for these definitions, we followed J. Dieudonné(2).

Let $\tilde{R}$ be the ring, which is the 2-dimensional right and left vectot space over R having e_0 and e_1 as a basis; in which the multiplication is defined by (i) the distributive law, (ii) $e_0e_0=e_0$, $e_0e_1=e_1e_0=e_1$, $e_1e_1=e_0\mu$ $(\mu\in R)$, and (iii) $e_0\alpha=\alpha e_0$, $e_1\alpha^*=\alpha e_1$ for every $\alpha\in R$, where the mapping $\alpha\to\alpha^*$ is a one-to-one mapping of R onto itself; and in which the involution J is defined by $e_0^J=e_0$, $e_1^J=e_1\nu$ $(\nu\in R)$, and $\alpha^J=\alpha^I$ for every $\alpha\in R$.

Moreover let $V_{\tilde{R}}$ be the extension of V_R for the extension $\tilde{R}$ of R, then $V_{\tilde{R}}$ is the 2-dimensional right and left vector space over V_R having e_0 and e_1 as a basis. And suppose that $e_0x=xe_0$, $e_1x=x^*e_1$ for every $x\in V_R$, where the mapping $x\to x^*$ is a one-to-one mapping of V_R onto itself.

2. We shall consider the condition for the $\tilde{R}$ stated in **1** to be a ring with an involution J.

LEMMA 1. *$\tilde{R}$ is a ring, if and only if $\mu^*=\mu$ and $\mu\xi^{**}=\xi\mu$ for every $\xi\in R$.*

PROOF. For the $\tilde{R}$ stated in **1**, the multiplication is defined by $(e_0\xi_0+e_1\xi_1)(e_0\eta_0+e_1\eta_1=e_0(\xi_0\eta_0+\mu\xi_1{}^*\eta_1)+e_1(\xi_1\eta_0+\xi_0{}^*\eta_1)$; and the axioms of ring, except for the

1) Cf. C. Chevalley, *Theory of Lie groups*, Princeton University Press, 1946, p. 22.

2) J. Diedonné, *On the structure of unitary groups*, Trans. Amer. Math. Soc. vol. 72 (1952), p. 367.

associativity of multiplication, are satisfied. The associativity of multiplication is written as $((e_0\xi_0+e_1\xi_1)(e_0\eta_0+e_1\eta_1))(e_0\zeta_0+e_1\zeta_1)=(e_0\xi_0+e_1\xi_1)((e_0\eta_0+e_1\eta_1)(e_0\zeta_0+e_1\zeta_1))$, that is, $\mu\xi_0^{**}\eta_1^*\zeta_1=\xi_0\mu\eta_1^*\zeta_1$ and $\mu^*\xi_1^{**}\eta_1^*\zeta_1=\xi_1\mu\eta_1^*\zeta_1$ for every ξ_0, ξ_1, η_1 and ζ_1. Therefore we have $\mu\xi^{**}=\xi\mu$ and $\mu^*=\mu$, as the necessary and sufficient condition to be found.

LEMMA 2. *In the ring $\tilde{R}$, the necessary and sufficient condition that the mapping $\tilde{\xi}\to\tilde{\eta}$ defined by $\tilde{\xi}e_1=e_1\tilde{\eta}$ be a one-to-one mapping of $\tilde{R}$ onto itself is that μ be neither zero element nor zero divisor of $\tilde{R}$.*

PROOF. If we write $\tilde{\xi}=e_0\xi_0+e_1\xi_1$ and $\tilde{\eta}=e_0\eta_0+e_1\eta_1$, then we have $\xi_0^*=\eta_0$ and $\mu(\xi_1^*-\eta_1)=0$. Since $\xi_1=0$ must imply $\eta_1=0$, $\mu\alpha=0$ must imply $\alpha=0$ for every $\alpha\in R$, and therefore $\mu\tilde{\alpha}=0$ implies $\tilde{\alpha}=0$ for every $\tilde{\alpha}\in\tilde{R}$. By Lemma 1, we have $\alpha\mu=\mu\alpha^{**}$; and so $\alpha\mu=0$ implies $\alpha=0$, and therefore $\tilde{\alpha}\mu=0$ implies $\tilde{\alpha}=0$ for every $\tilde{\alpha}\in\tilde{R}$. Thus we see that μ must be neither zero element nor zero divisor of $\tilde{R}$. And the converse is evident.

In the following we shall always suppose that the mapping $\tilde{\xi}\to\tilde{\eta}$ defined by $\tilde{\xi}e_1=e_1\tilde{\eta}$ is a one-to-one mapping of $\tilde{R}$ onto itself. Then we have $\tilde{\eta}=e_0\xi_0^*+e_1\xi_1^*$, and denote it by $\tilde{\xi}^*$. Obviously we have $(\tilde{\xi}\tilde{\eta})^*=\tilde{\xi}^*\tilde{\eta}^*$ and $(\tilde{\xi}^*+\tilde{\eta})^*=\tilde{\xi}^*+\tilde{\eta}^*$ for every $\tilde{\xi}$, $\tilde{\eta}\in\tilde{R}$.

LEMMA 3. *The one-to-one mapping J of $\tilde{R}$ onto itself such that $\xi^J=\xi^I$ for every $\xi\in R$, $e_0^J=e_0\nu\,(\nu\in R)$ is an involution, if and only if $\nu^{I*}\nu=\nu\nu^{I*}=\varepsilon$, and $\xi^{*I*}\nu=\nu\xi^I$ for every $\xi\in R$.*

And then ν has the inverse element; $\tilde{\xi}^J$ is expressed by $\tilde{\xi}^J=e_0\xi_0^I+e_1\xi_1^{I*}\nu$.

PROOF. If the mapping J is an involution, then $\tilde{\xi}^J$ is expressed by $\tilde{\xi}^J=(e_0\xi_0+e_1\xi_1)^J=\xi_0^I e_0^J+\xi_1^I e_1^J=e_0\xi_0^I+e_1\xi_1^{I*}\nu$. Obviously this mapping is additive. And $(e_0\xi_0+e_1\xi_1)^{JJ}=e_0\xi_0^{II}+e_1(\xi_1^{I*}\nu)^{I*}\nu=e_0\xi_0+e_1\xi_1$; that is, $\nu^{I*}\xi_1^{I*I*}\nu=\xi_1$ for every $\xi_1\in R$. In particular if we put $\xi_1=\varepsilon$, then we have $\nu^{I*}\nu=\varepsilon$. Furtheremore the mapping J satisfies $((e_0\xi_0+e_1\xi_1)(e_0\eta_0+e_1\eta_1))^J=(e_0\eta_0+e_1\eta_1)^J(e_0\xi_0+e_1\xi_1)^J$, that is, $\eta_1^I\xi_1^{*I}\mu^I=\mu\eta_1^{I**}\nu^*\xi_1^{I*}\nu$ and $\eta_1^{I*}\xi_0^{*I*}\nu=\eta_1^{I*}\nu\xi_0^I$; here if we put $\xi_1=\eta_1=\varepsilon$, then we have $\mu^I=\mu\nu^*\nu$, and $\xi_0^{*I*}\nu=\nu\xi_0^I$. Moreover if we take ξ_0 such as $\xi_0^*=\nu$, then we have $\varepsilon=\nu^{I*}\nu=\nu\xi_0^I$, hence ν has the inverse element. Thus, as the necessary condition that the mapping J be an involution, we obtain $\nu^{I*}\nu=\nu\nu^{I*}=\varepsilon$, $\mu^I=\mu\nu^*\nu$, and $\xi^{*I*}\nu=\nu\xi^I$ for every $\xi\in R$.

Conversely, from $\nu^{I*}\nu=\varepsilon$ and $(\xi_1^I)^{*I*}\nu=\nu\xi_1$ it follows $\nu^{I*}\xi_1^{I*I*}\nu=\xi_1$; and by means of $\xi_0^{*I*}\nu=\nu\xi_0^I$ we have $\eta_1^{I*}\xi_0^{*I*}\nu=\eta_1^{I*}\nu\xi_0^I$; and moreover from $\nu\xi_1^I=\xi_1^{*I*}\nu$, $\mu\nu^*\nu=\mu^I$, $\mu\eta^{I**}=\eta_1^I\mu$ and $\mu(\xi^{*I})^{**}=\xi_1^{*I}\mu$ (Lemma 1), we have $\mu\eta_1^{I**}\nu^*\xi_1^{I*}\nu=\eta_1^I\xi_1^{*I}\mu^I$. Therefore these conditions are also sufficient for the mapping to be an involution.

REMARK. By means of Lemmas 1 and 3, it is easily verified that $\mu\tilde{\xi}^{**}=\tilde{\xi}\mu$ and $\tilde{\xi}^{*J*}\nu=\nu\tilde{\xi}^J$ for every $\tilde{\xi}\in\tilde{R}$.

3. Let $\tilde{x}$ be an element of $V_{\tilde{R}}$, then $\tilde{x}=e_0x_0+e_1x_1$ where x_0, $x_1\in V_R$. We assign to $\tilde{x}$ the element x' of $V_R\times V_R$ such that $x'=(x_0,x_1)$. Let $(\tilde{x},\tilde{y})\to\tilde{f}(\tilde{x},\tilde{y})$ be a mapping of $V_{\tilde{R}} \times V_{\tilde{R}}$ into $\tilde{R}$, then $\tilde{f}(\tilde{x},\tilde{y})$ is expressed as $\tilde{f}(\tilde{x},\tilde{y})=e_0f_0(\tilde{x},\tilde{y})+e_1f_1(\tilde{x},\tilde{y})$, where $f_0(\tilde{x},\tilde{y})$, $f_1(\tilde{x},\tilde{y}) \in R$. If we put $f_0(\tilde{x},\tilde{y})=f_0(x',y')$ and $f_1(\tilde{x},\tilde{y})=f_1(x',y')$, then the mapping $(x',y') \rightarrow f_0(x',y')$ and $(x',y) \rightarrow f_1(x',y')$ are mappings of $(V_R \times V_R) \times (V_R \times V_R)$ into R.

In this section, we shall prove the following lemmas.

LEMMA 4. *The form $\tilde{f}(\tilde{x},\tilde{y})$ is nondegenerate, if and only if $f_0(x',y')$ and $f_1(x',y')$ are nondegenerate.*

PROOF. $\tilde{f}(\tilde{x},\tilde{y})=0$, if and only if $f_0(x',y')=0$ and $f_1(x',y')=0$, since $\tilde{x}=0$ is equivalent to $x'=0$, hence the lemma is proved.

LEMMA 5. *The form $\tilde{f}(\tilde{x},\tilde{y})$ is hermitian, if and only if $f_0(x',y')$ is an hermitian form, satisfying $f_0(x'^*,y'^*)=\mu\nu^*f_0(x',y')^*$, and $f_1(x',y')$ satisfies $\mu f_1(x',y')^*=f_0(x',y'^*)$, where $x'^*=(\mu x_1^*, x_0^*)$.*

PROOF. The condition $\tilde{f}(\tilde{x},\tilde{y}+\tilde{z})=\tilde{f}(\tilde{x},\tilde{y})+\tilde{f}(\tilde{x},\tilde{z})$ is equivalent to that $f_0(x',y'+z') =f_0(x',y')+f_0(x',z')$ and $f_1(x',y'+z')=f_1(x',y')+f_1(x',z')$. And the condition that $\tilde{f}(\tilde{x},\tilde{y}\tilde{\lambda})=\tilde{f}(\tilde{x},\tilde{y})\tilde{\lambda}$ for every $\tilde{\lambda} \in \tilde{R}$ is that $\tilde{f}(\tilde{x},\tilde{y}\lambda)=\tilde{f}(\tilde{x},\tilde{y})\lambda$ for every $\lambda \in R$ and $\tilde{f}(\tilde{x},\tilde{y}e_1)=\tilde{f}(\tilde{x},\tilde{y})e_1$. For, obviously the condition is necessary, and coversely, for $\tilde{\lambda}=\lambda_0e_0+\lambda_1e_1$, λ_0, $\lambda_1 \in R$, $\tilde{f}(\tilde{x},\tilde{y}\tilde{\lambda})=\tilde{f}(\tilde{x},\tilde{y}(\lambda_0e_0+\lambda_1e_1))=\tilde{f}(\tilde{x},\tilde{y}\lambda_0e_0)+\tilde{f}(\tilde{x},\tilde{y}\lambda_1e_1)$

$$=\tilde{f}(\tilde{x},\tilde{y})\lambda_0e_0+\tilde{f}(\tilde{x},\tilde{y}\lambda)e_1=\tilde{f}(\tilde{x},\tilde{y})\lambda_0e_0+\tilde{f}(\tilde{x},\tilde{y})\lambda_1e_1$$
$$=\tilde{f}(\tilde{x},\tilde{y})(\lambda_0e_0+\lambda_1e_1)=\tilde{f}(\tilde{x},\tilde{y})\tilde{\lambda}\,.$$

And $\tilde{f}(\tilde{x},\tilde{y}\lambda)=\tilde{f}(\tilde{x},\tilde{y})\lambda$ for every $\lambda \in R$, if and only if $f_0(x',y'\lambda)=f_0(x',y')\lambda$ and $f_1(x',y'\lambda)=f_1(x',y')\lambda$. For, $\tilde{y}\lambda$ corresponds to $y'\lambda$, for $\lambda \in R$. Next, since $\tilde{y}e_1=(e_0y_0 +e_1y_1)e_1=e_0\mu y_1^*+e_1y_0^*$ for $\tilde{y}=e_0y_0+e_1y_1$, $\tilde{y}e_1$ corresponds to $(\mu y_1^*, y_0^*)$, (we denote it by y'^*). That is, $y'^*=(\mu y_1^*, y_0^*)$, and hence $y'^{**}=(\mu y_0^{**}, \mu y_1^{**})=(y_0\mu, y_1\mu)=y'\mu$, (by Lemma 1). And then we have $\tilde{f}(\tilde{x},\tilde{y}e_1)=e_0f_0(x',y'^*)+e_1f_1(x',y'^*)$, and, on the other hand, $\tilde{f}(\tilde{x},\tilde{y})e_1=(e_0f_0(x',y')+e_1f_1(x',y'))e_1=e_0\mu f_1(x',y')^*+e_1f_0(x',y')^*$. Therefore we see that $\tilde{f}(\tilde{x},\tilde{y}e_1)=\tilde{f}(\tilde{x},\tilde{y})e_1$, is equivalent to

$$(1) \qquad f_0(x',y'^*)=\mu f_1(x',y')^*$$

and

$$(2) \qquad f_1(x',y'^*)=f_0(x',y')^*$$

Moreover we shall show that (1) implies (2) under the assumption that $f_0(x',y')$ is linear in y' over R[1]. If we substitute y'^* for y' in (1), then we have

$$\begin{aligned}\mu f_1(x',y'^*)^* &= f_0(x',y'^{**})\\ &= f_0(x',y'\mu) \qquad (\text{since } y'^{**}=y'\mu)\\ &= f_0(x',y')\mu \qquad (f_0(x',y') \text{ is linear in } y' \text{ over } R),\end{aligned}$$

and by means of Lemma 1, $f_0(x',y')^*\mu=\mu f_1(x',y'^*)^{**}=f_1(x',y'^*)\mu$. Therefore, by Lemma 2, we have $f_0(x',y')^*=f_1(x',y'^*)$.

Next we shall consider the condition for $\tilde{f}(\tilde{x},\tilde{y})^J=\tilde{f}(\tilde{y},\tilde{x})$.

1) (2) implies (1) under the assumption that $f_1(x',y')$ is linear in y' over R.

$$\tilde{f}(\tilde{x},\tilde{y})^J=(e_0f_0(x',y')+e_1f_1(x',y'))^J=e_0f_0(x',y')+e_1f_1(x',y')^{I*}\nu,$$

and, on the other hand, $\tilde{f}(\tilde{y},\tilde{x})=e_0f_0(y',x')+e_1f_1(y',x')$. Hence we see that $\tilde{f}(\tilde{x},\tilde{y})=\tilde{f}(\tilde{y},\tilde{x})$, if and only if

$$(3)\qquad f_0(x',y')^I=f_0(y',x')$$

and

$$(4)\qquad f_1(x',y')^{I*}\nu=f_1(y',x').$$

Furthermore we have

$$\begin{aligned} f_0(x'^*,y'^*)&=\mu f_1(x'^*,y')^* && \text{(by (1))}\\ &=\mu f_1(y',x'^*)^{I**}\nu^* && \text{(by (4))}\\ &=f_1(y',x'^*)^I\mu\nu^* && \text{(by Lemma 1)}\\ &=f_0(y',x')^{*I}\mu\nu^* && \text{(by (2))}\\ &=\mu\nu^*f_0(y',x')^{I*} && \text{(by Lemma 3)}\\ &=\mu\nu^*f_0(x',y')^* && \text{(by (3))}, \end{aligned}$$

that is, we have

$$(5)\qquad f_0(x'^*,y'^*)=\mu\nu^*f_0(x',y')^*,$$

or, in the other expression,

$$(5')\qquad \nu f_0(x',y')\mu=f_0(x'^*,y'^*).$$

And similarly we have

$$(6)\qquad f_1(x'^*,y'^*)=\nu\mu f_1(x',y')^*.$$

Conversely, it is easily seen that (4) follows from (1), (2), (3) and (5).

Moreover we remark that under the condition (1), if $f_0(x',y')$ is linear in y', then $f_1(x',y')$ is linear in y'. Indeed, if $f_0(x',y')$ is linear in y', then, by (1), it holds that $\mu f_1(x',y'+z')^*=f_0(x',(y'+z')^*)=f_0(x',y'^*+z'^*)=f_0(x',y'^*)+f_0(x',z'^*)=\mu(f_1(x',y')+f_1(x',z'))^*$, and $\mu f_1(x',y'\lambda)^*=f_0(x',(y'\lambda)^*)=f_0(x',y'^*\lambda^*)=f_0(x',y'^*)\lambda^*=\mu(f_1(x',y')\lambda)^*$. Since μ is neither zero element nor zero divisor of R and $*$-correspondence is a one-to-one mapping of R onto itself, $f_1(x',y')$ is linear in y'. Thus this lemma is completely proved.

4. As before, let $\tilde{x}$ correspond to the element x' of $V_R\times V_R$, then this correspondence is a one-to-one mapping of $V_{\tilde{R}}$ onto $V_R\times V_R$; and therefore there corresponds to every one-to-one mapping $\tilde{u}$ of $V_{\tilde{R}}$ onto itself a one-to-one mapping u' of $V_R\times V_R$ onto itself such that $\tilde{u}(\tilde{x})\mapsto u'(x')$ if $\tilde{x}\to x'$. Moreover, to the product $\tilde{u}\tilde{v}$ of two one-to-one mappings $\tilde{u}$ and $\tilde{v}$ of V_R onto itself, there corresponds the product $u'v'$of the corresponding one-to-one mappings of $V_R\times V_R$ onto itself.

We shall prove the lemma.

LEMMA 6. *The mapping $\tilde{x}\to\tilde{u}(\tilde{x})$ of $V_{\tilde{R}}$ onto itself is linear over $\tilde{R}$, if and only if the corresponding mapping $x'\to u'(x')$ is linear over R and satisfies the condition $u'(x'^*)=(u'(x'))^*$.*

PROOF[1]. If we write $\tilde{u}(\tilde{x})=e_0u_0(x')+e_1u_1(x')$, $u_0(x')$, $u_1(x')\in V_R$, then we have

1) This proof is similar as in Lemma 5.

$u'(x')=(u_0(x'), u_1(x'))$. It is obviously seen that $\tilde{u}$ is additive, if and only if u' is additive. The condition $\tilde{u}(\tilde{x}\tilde{\lambda})=\tilde{u}(\tilde{x})\tilde{\lambda}$ for every $\tilde{\lambda}\in\tilde{R}$ is equivalent to that $\tilde{u}(\tilde{x}\lambda)=\tilde{u}(\tilde{x})\lambda$ for every $\lambda\in R$ and $\tilde{u}(\tilde{x}e_1)=\tilde{u}(\tilde{x})e_1$. That $\tilde{u}(\tilde{x}\lambda)=\tilde{u}(\tilde{x})\lambda$ for every $\lambda\in R$ is equivalent to that u' is linear over R. And that $\tilde{u}(\tilde{x}e_1)=\tilde{u}(\tilde{x})e_1$ is equivalent to that $e_0u_0(x'^*)+e_1u_1(x'^*)=(e_0u_0(x')+e_1u_1(x'))e_1=e_0\mu u_1(x')^*+e_1u_0(x')^*$, hence we have $\mu u_1(x')^*=u_0(x'^*)$ and $u_0(x')^*=u_1(x'^*)$, i.e., $(u'(x'))^*=u'(x'^*)$. This proves Lemma 6. (Moreover we remark that under the assumption that $u_0(x')$ and $u_1(x')$ are linear over R, $u_0(x')^*=u_1(x'^*)$, if and only if $\mu u_1(x')^*=u_0(x'^*)$).

5. Let $\tilde{f}$ be a nondegenerate hermitian form over V_R, and let $\tilde{u}$ be a unitary transformation of $V_{\tilde{R}}$ with respect to $\tilde{f}$, that is, $\tilde{u}$ is a one-to-one linear mapping $V_{\tilde{R}}$ onto itself satisfying $\tilde{f}(\tilde{u}(\tilde{x}),\tilde{u}(\tilde{y}))=\tilde{f}(\tilde{x},\tilde{y})$ identically. Then the condition $\tilde{f}(\tilde{u}(\tilde{x}),\tilde{u}(\tilde{y}))=\tilde{f}(\tilde{x},\tilde{y})$ is equivalent to that $f_0(u'(x'),u'(y'))=f_0(x',y')$ and $f_1(u'(x'),u'(y'))=f_1(x',y')$; hence u' is a unitary transformation[1] with respect to f_0 of $V_R\times V_R$. Here we shall define that u' is a symplectic transformation with respect to f_0 if u' is a one-to-one linear mapping of $V_R\times V_R$ onto itself such that $f_1(u'(x'),u'(y')=f_1(x',y')$ where $f_0(x',y'^*)=\mu f_1(x',y')^*$. These transformations constitute the symplectic group $Sp(V_R\times V_R, f_0)$. And therefore we can say that if $\tilde{u}\in U(V_{\tilde{R}},\tilde{f})$, then $u'\in U(V_R\times V_R, f_0)\frown Sp(V_R\times V_R, f_0)$.

Conversely if $u'\in U(V_R\times V_R,f_0)\frown Sp(V_R\times V_R,f_0)$, then we have $f_0(u'(x'),u'(y'))=f_0(x',y')$ and $f_1(u'(x'),u'(y))=f_1(x',y')$, that is, $\tilde{f}(\tilde{u}(\tilde{x}),\tilde{u}(\tilde{y})=\tilde{f}(\tilde{x},\tilde{y})$. And moreover $\tilde{u}$ is linear over $\tilde{R}$; because, for any $\tilde{\lambda}\in\tilde{R}$,

$$\tilde{f}(\tilde{y},\tilde{u}(\tilde{x})\tilde{\lambda}-\tilde{u}(\tilde{x}\tilde{\lambda}))=\tilde{f}(\tilde{u}(\tilde{z}),\tilde{u}(\tilde{x})\tilde{\lambda}-\tilde{u}(\tilde{x}\tilde{\lambda}))$$

(since $\tilde{u}$ is a one-to-one mapping of $V_{\tilde{R}}$ onto itself any $\tilde{y}$ of $V_{\tilde{R}}$ is expressed in the form $\tilde{u}(\tilde{z})$)

$$=\tilde{f}(\tilde{u}(\tilde{z}),\tilde{u}(\tilde{x}))\tilde{\lambda}-\tilde{f}(\tilde{u}(\tilde{z}),\tilde{u}(\tilde{x}\tilde{\lambda})) \quad (\tilde{f}\text{ is linear over }\tilde{R})$$

$$=\tilde{f}(\tilde{z},\tilde{x})\tilde{\lambda}-\tilde{f}(\tilde{z},\tilde{x}\tilde{\lambda}) \quad (\tilde{u}\text{ is unitary})$$

$$=\tilde{f}(\tilde{z},\tilde{x})\tilde{\lambda}-\tilde{f}(\tilde{z},\tilde{x})\tilde{\lambda}=0 \quad (\tilde{f}\text{ is linear over }\tilde{R}),$$

that is, we have $f(\tilde{y},\tilde{u}(\tilde{x})\tilde{\lambda}-\tilde{u}(\tilde{x}\tilde{\lambda}))=0$ for every $\tilde{y}\in\tilde{R}$, and hence, as $\tilde{f}$ is nondegenerate, we have $\tilde{u}(\tilde{x})\tilde{\lambda}=\tilde{u}(\tilde{x}\tilde{\lambda})$, thus $\tilde{u}$ is linear over $\tilde{R}$.

Let $GL(V_{\tilde{R}})$ be the group of all one-to-one linear mapping of $V_{\tilde{R}}$ onto itself, then, by the correspondence $\tilde{u}\to u'$, this group is isomorphic to a subgroup $L(V_R\times V_R)$ of $GL(V_R\times V_R)$.

Now we shall prove that, under the condition $u'(x'^*)=(u'(x'))^*$, $f_0(u'(x'),u'(y'))=f_0(x',y')$ identically if and only if $f_1(u'(x'),u'(y'))=f_1(x',y')$ identically. By means of (1) in 3, we have $\mu f_1(x',y')^*=f_0(x',y'^*)$ and $\mu f_1(u'(x'),u'(y'))^*=f_0(u'(x'),u'(y')^*)$; by the hypothesis $u'(y'^*)=u'(y')^*$, we have $\mu f_1(u'(x'),u'(y'))^*=f_0(u'(x'),u'(y'^*))$. Hence $f_0(u'(x'),u'(y'^*))=f_0(u'(x',y'^*)$ implies $f_1(u'(x'),u'(y'))=f_1(x',y')$. Similarly

2) u' is linear over R by Lemma 6.

by means of (2) in **3**, $f_1(u'(x'), u'(y'))=f_1(x', y')$ implies $f_0(u'(x'), u'(y'))=f_0(x', y')$.

Summarizing these results, we obtain the following

THEOREM. *The unitary group* $U(V_{\tilde{R}}, \tilde{f})$ *is isomorphic to the group* $U(V_R\times V_R, f_0) \frown Sp(V_R\times V_R, f_0)$, *and moreover* $U(V_R\times V_R, f_0)\frown Sp(V_R\times V_R, f_0)=L(V_R\times V_R)\frown U(V_R\times V_R, f_0)=L(V_R\times V_R)\frown Sp(V_R\times V_R, f_0)$, *where* $\tilde{f}(\tilde{x}, \tilde{y})=e_0f_0(x', y')+e_1f_1(x', y')$ *and* $L(V_R\times V_R)$ *is a subgroup of* $GL(V_R\times V_R)$ *isomorphic to* $GL(V_{\tilde{R}})$ *by the correspondence* $\tilde{u}\rightarrow u'$.

6. As an example, we shall consider the Clifford algebra as the ring R. Let A be a commutative ring with unit element ε, and let $C_m(p_0, p_1, \cdots, p_m)$ be a Clifford algebra of dimension 2^m over A with $p_0, p_1, \cdots, p_m$ as a basis, that is, the basic elements satisfy

$$p_0^2=p_0,\quad p_0p_i=p_ip_0=p_i,\quad p_i^2=-p_0,\quad p_jp_i=-p_ip_j\ (i<j)\quad i,j=1,2,\cdots,m,$$

$$ap_0=p_0a,\quad ap_i=p_ia\ (i=1,2,\cdots,m)\qquad \text{for}\qquad a\in A.$$

Then the element $\tilde{\alpha}$ of $C_m(p_0, p_1, \cdots, p_m)$ is expressed as

$$\tilde{\alpha}=\sum a_{\alpha_1\alpha_2\cdots\alpha_m}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m},\qquad \text{where}\qquad a_{\alpha_1\alpha_2\cdots\alpha_m}\in A,\quad \alpha_i=0,1;$$

and we see that $\tilde{\alpha}=p_0\alpha_0+p_m\alpha_1$, where $\alpha_0,\ \alpha_1\in C_{m-1}(p_0, p_1, \cdots, p_{m-1})$. If we define the involution J of $C_m(p_0, p_1, \cdots, p_m)$ by the following conditions: $(a\alpha+b\beta)^J=a\alpha^J+b\beta^J$, $(a, b\in A)$, $(\alpha\beta)^J=\beta^J\alpha^J$ and $p_i^J=-p_i$, then we have $\tilde{\alpha}^J=\sum(-1)^{\sum_{i\geq j}\alpha_i\alpha_j}a_{\alpha_1\alpha_2\cdots\alpha_m}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$. And we define the $*$-mapping by the conditions $(a\alpha+b\beta)^*=a\alpha^*+b\beta^*$, $(a, b\in A)$, $(\alpha\beta)^*=\alpha^*\beta^*$, and $p_0^*=p_0$, $p_m^*=p_m$, $p_i^*=-p_i$ $(i=1, 2, \cdots, m-1)$, then we have $\tilde{\alpha}^*=\sum(-1)^{\alpha_1+\alpha_2+\cdots+\alpha_{m-1}}a_{\alpha_1\alpha_2\cdots\alpha_m}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ and $\tilde{\alpha}p_m=\tilde{\alpha}^*p_m$.

If we consider $C_m(p_0, p_1, \cdots, p_m)$ and $C_{m-1}(p_0, p_1, \cdots, p_{m-1})$ as $\tilde{R}$ and R in the preceding sections respectively, then we easily see that $\mu=\mu^*=-p_0$, $\nu=\nu^*=\nu^{J*}=-p_0$, $\mu\xi^{**}=\xi\mu$, $\mu^J=\mu\nu^*\nu$, and $\xi^{*J*}=\xi^J(\xi\in C_{m-1}(p_0, p_1, \cdots, p_{m-1}))$, and therefore that $C_m(p_0, p_1, \cdots, p_m)$ is a ring with an involution J over $C_{m-1}(p_0, p_1, \cdots, p_{m-1})$. Let C_m^n be an n-dimensional vector space over C_m, and let us define $\tilde{f}(\tilde{x}, \tilde{y})=\sum_{i=1}^n\tilde{\xi}_i^J\tilde{\eta}_i$, $\tilde{x}=(\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_n)$, $\tilde{y}=(\tilde{\eta}_1, \tilde{\eta}_2, \cdots, \tilde{\eta}_n)$, then $\tilde{f}(\tilde{x}, \tilde{y})$ is evidently hermitian, and then $f_0(x', y')=\sum_{i=1}^n x_{0i}^Jy_{0i}+\sum_{i=1}^n x_{1i}^Jy_{1i}$ and $f_1(x', y')=\sum_{i=1}^n(x_{0i}^{J*}y_{1i}-x_{1i}^{J*}y_{0i})$ where $\tilde{\xi}_i=e_0x_{0i}+e_1x_{1i}$, $\tilde{\eta}_i=e_0y_{0i}+e_1y_{1i}$, x_{0i}, x_{1i}, y_{0i} and $y_{1i}\in C_{m-1}(p_0, p_1, \cdots, p_{m-1})$.

As a special case of the theorem, we obtain the following theorem which contains the theorem stated at the beginning of this note as the case $m=2$.

THEOREM. *The unitary group* $U(C_m^n)$ *is isomorphic to the group* $U(C_{m-1}^{2n})\frown Sp(C_{m-1}^{2n})$, *and* $U(C_{m-1}^{2n})\frown Sp(C_{m-1}^{2n})=L(C_{m-1}^{2n})\frown U(C_{m-1}^{2n})=L(C_{m-1}^{2n})\frown Sp(C_{m-1}^{2n})$, *where* $L(C^{2n}_{m-1})$ *is a subgroup of* $GL(C^{2n}_{m-1})$ *isomorphic to* $GL(C^{n}_{m})$ *by the correspondence* $\tilde{u} \rightarrow u'$.

I wish to express my thanks to Professor Kakutaro Morinaga for suggesting this investigation as well as for constant guidance in the course of the work.

Mathematical Institute,
Hiroshima University