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Theory of the Spherically Symmetric Space-Times. VII. Space-Times with Corresponding Geodesics[1)]

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§ 1. Introduction

Let n dimensional Riemannian spaces V_n and V_n^* be defined by the fundamental forms

$$ds^2=g_{ij}\,dx^i\,dx^j \quad \text{and} \quad ds^{*2}=g_{ij}^*\,dx^i\,dx^j$$

respectively with common coordinates x^i. A necessary and sufficient condition that their geodesics correspond with each other is given by

$$\left\{ {i \atop jk} \right\}^* = \left\{ {i \atop jk} \right\} + \delta_j^i\,\psi_k + \delta_k^i\,\psi_j\,, \qquad (\psi_i = \nabla_i\,\psi)\,, \tag{1.1}$$

where ψ is a scalar, $\left\{ {i \atop jk} \right\}$ and $\left\{ {i \atop jk} \right\}^*$ are the Christoffel symbols formed with respect to g_{ij} and g_{ij}^* respectively and ∇_i is the covariant derivative with respect to g_{ij}.

In this paper we shall prove that when V_4 is an S_0, V_4^* is also s. s. This is a generalization of the well known theorem concerning spaces of constant curvature.[2)] Further some other properties concerning the S_0's with corresponding geodesics will be made clear.

§ 2. Fundamental theorem

In this section, first we shall prove the following fundamental theorem:

Theorem [2.1] *The only four dimensional Riemannian spaces with fundamental forms of signature -2 whose geodesics correspond to the geodesics of an S_0 are s. s. space-times.*

Proof. Let the geodesics of a V_4^* correspond to the geodesics of an S_0. We shall take a standard coordinate system for g_{ij} of the S_0. Then the fundamental form of the S_0 is

$$ds^2=-A\,(r,t)\,dr^2-B\,(r,t)\,(d\theta^2+\sin^2\theta\,d\phi^2)+C\,(r,t)\,dt^2 \tag{2.1}$$

and
$$\gamma=0. \tag{2.2}$$

We have from (1.1)

$$K^{*m}_{\cdot\,ijl}=K^{m}_{\cdot\,ijl}+\delta^m_l\psi_{ij}-\delta^m_j\psi_{il}, \qquad (\psi_{ij}=\nabla_i\nabla_j\psi-\nabla_i\psi\cdot\nabla_j\psi) \tag{2.3}$$

where $K^{m}_{\cdot\,ijl}$ and $K^{*m}_{\cdot\,ijl}$ are the curvature tensors of the S_0 and the V_4^* respectively. If we raise and lower the indices of an asterisked tensor by means of g^{*ij} and g^*_{ij} respectively, it holds identically that

$$K^*_{ijkl}+K^*_{jikl}=0, \qquad K^*_{ijkl}=K^*_{klij}. \tag{2.4}$$

When (2.1), (2.2) and (2.3) are substituted in (2.4), a part of the resulting equations is reducible to

$$g^*_{1a}(\alpha-\eta)=0,\ g^*_{1a}(\beta-\xi)=0,\ g^*_{14}(\alpha-\beta)=0,\ g^*_{4a}(\alpha-\xi)=0,\ g^*_{4a}(\beta-\eta)=0, \tag{2.5}$$

where $a=2,3$. As the theorem is established for spaces of constant curvature, we shall assume that the S_0 is not an [A] (i. e. de Sitter space-time) in the following. Then, from (2.5) the surviving components of g^*_{ij} are as follows:

Case I When the S_0 is an S_a: g^*_{ii}, g^*_{23}, g^*_{14}.
Case II When the S_0 is an S_b and $\alpha=\xi$, $\beta=\eta$: g^*_{ii}, g^*_{23}, g^*_{4a}.
Case III ,,, ,,, $\alpha=\eta$, $\beta=\xi$: g^*_{ii}, g^*_{23}, g^*_{1a}.
Case IV ,,, ,,, excluding cases II and III: g^*_{ii}, g^*_{23}.

In cases II and III, after some calculations, we can obtain

$$g^*_{4a}=0 \qquad \text{and} \qquad g^*_{1a}=0 \tag{2.6}$$

respectively in consequence of the remaining equations obtained from (2.4). Then, in all cases, from

$$\frac{\partial g^*_{ij}}{\partial x^l}=\left\{{s \atop il}\right\}^* g^*_{sj}+\left\{{s \atop jl}\right\}^* g^*_{si} \tag{2.7}$$

and (2.6) we have $\psi_2=\psi_3=0$ and consequently

$$\psi_{33}=\psi_{22}\sin^2\theta, \qquad \psi_{23}=\psi_{1a}=\psi_{4a}=0. \tag{2.8}$$

From (2.6) and (2.8), (2.4) becomes

$$\left.\begin{aligned}
&K^*_{1a1a}=g^*_{11}(\alpha g_{aa}-\psi_{aa})=g^*_{aa}(\alpha g_{11}-\psi_{11}),\\
&K^*_{4a4a}=g^*_{44}(\beta g_{aa}-\psi_{aa})=g^*_{aa}(\beta g_{44}-\psi_{44}),\\
&K^*_{1aa4}=-g^*_{14}(\beta g_{aa}-\psi_{aa})=-g^*_{14}(\alpha g_{aa}-\psi_{aa})=g^*_{aa}\psi_{14},\\
&K^*_{2323}=g^*_{22}(\eta g_{33}-\psi_{33})=g^*_{33}(\eta g_{22}-\psi_{22}),\\
&K^*_{1414}=g^*_{11}(\xi g_{44}-\psi_{44})+g^*_{14}\psi_{14}=g^*_{44}(\xi g_{11}-\psi_{11})+g^*_{14}\psi_{14};
\end{aligned}\right\} \tag{2.9}$$

$$g^*_{14}(\xi g_{11}-\psi_{11})+g^*_{11}\psi_{14}=0\,,\qquad g^*_{14}(\xi g_{44}-\psi_{44})+g^*_{44}\psi_{14}=0\,, \tag{2.10}$$

$$g^*_{23}(\alpha g_{11}-\psi_{11})=0\,,\ g^*_{23}(\beta g_{44}-\psi_{44})=0\,,\ g^*_{23}(\eta g_{aa}-\psi_{aa})=0\,,\ g^*_{23}\psi_{14}=0\,. \tag{2.11}$$

If $g^*_{33}-g^*_{22}\sin^2\theta$ or g^*_{23} does not vanish, we have $\alpha=\beta=\eta$ by means of (2.9) or (2.11) respectively and therefore the S_0 is an S_{15}. Then we can take $B=r^2$ from the beginning and consequently $\alpha=\beta=\eta$ reduces to $\xi=\alpha$. Hence we have finally $g^*_{33}-g^*_{22}\sin^2\theta=0$ and $g^*_{23}=0$. At last by making use of (2.7) we know that g^*_{11}, g^*_{22}, g^*_{44} and g^*_{14} are all functions of r and t. Thus the theorem is proved.

In the above proof we have used a standard coordinate system for g_{ij} of the S_0. But, as g^*_{14} does not necessarily vanish, it is impossible to say with certainty that this coordinate system is simultaneously standard for g^*_{ij} of the S^*_0 or even that this coordinate system is s. s. for the S^*_0. (As it has been proved that the V^*_4 is s. s., we shall denote it by S^*_0 in the following.) This circumstance will become clear later. ([2.4], [2.5])

Theorem [2.2] *The only S^*_0 whose geodesics correspond to the geodesics of an S_a (or S_b) is S^*_a (or S^*_b).*

Proof. From (2.9) we have

$$K^{*1a}_{\cdot\cdot 1a}=g^{*22}(\alpha g_{22}-\psi_{22})\,,\quad K^{*4a}_{\cdot\cdot 4a}=g^{*22}(\beta g_{22}-\psi_{22})\,,\quad K^{*1a}_{\cdot\cdot a4}=0 \tag{2.12}$$

and therefore it follows that $\alpha^*=\beta^*$ or not according as $\alpha=\beta$ or not. (**I,** [5.2])

Incidentally, from the proof of [2.1] we can easily obtain

Theorem [2.3] *When an S_0 is not of constant curvature and the geodesics of the S_0 correspond to the geodesics of an S^*_0, ψ in (1.1) is s. s. for a c. s. of the S_0.*

The coordinate system of (2.12) is standard for g_{ij} of the S_0 but is not necessarily s. s. for the S^*_0. This fact was already noticed after [2.1] and further in this connection we have the following two theorems.

Theorem [2.4] *When the geodesics of an S_a correspond to the geodesics of an S^*_a and the coordinate system is standard for g_{ij} of the S_a, this coordinate system is not necessarily standard for g^*_{ij} of the S^*_a simultaneously. Nevertheless, it becomes standard for g^*_{ij} by carrying out a transformation of r and t at most. Of course, the new obtained coordinate system is not necessarily standard for g_{ij} of the S_a.*

Theorem [2.5] *When the geodesics of an S_b correspond to the geodesics of* *an S_b^* and the coordinate system is standard for g_{ij} of the S_b, this coordinate system is always standard for g_{ij}^* of the S_b^* simultaneously.*

These theorems are obvious from the classification (Case I,$\cdots$, IV) given in the proof of [2.1].

§ 3. Correspondence between s. s. coordinate systems

Let the fundamental form of an S_0 be given by (2.1) using a s. s. coordinate system. In this section we shall study under what condition there exists an S_0^* whose geodesics correspond to those of the S_0 and for which the coordinate system of the S_0 is simultaneously s. s. It is to be noted that the above restriction for the coordinate system is weaker than that of [2.4] and [2 5], because (2.2) does not necessarily hold in a s. s. coordinate system.

We shall assume that an S_0^* is defined by the fundamental form

$$ds^{*2}=-A^*(r,t)\,dr^2-B^*(r,t)\,(d\theta^2+\sin^2\theta\,d\phi^2)+C^*(r,t)\,dt^2 \tag{3.1}$$

in the same coordinate system as (2.1). If the geodesics of the S_0 and the S_0^* correspond with each other, (1.1) reduces to

$$\left.\begin{aligned}4\psi_1&=\frac{\partial}{\partial r}\left(\log\frac{A^*}{A}\right)=2\frac{\partial}{\partial r}\left(\log\frac{B^*}{B}\right)=2\frac{\partial}{\partial r}\left(\log\frac{C^*}{C}\right),\quad \psi_2=\psi_3=0,\\ 4\psi_4&=2\frac{\partial}{\partial t}\left(\log\frac{A^*}{A}\right)=2\frac{\partial}{\partial t}\left(\log\frac{B^*}{B}\right)=\frac{\partial}{\partial t}\left(\log\frac{C^*}{C}\right),\end{aligned}\right\} \tag{3.2}$$

$$A^*:B^{*\prime}:C^{*\prime}=A:B':C',\qquad C^*:\dot{A}^*:\dot{B}^*=C:\dot{A}:\dot{B}. \tag{3.3}$$

By solving (3.2) we obtain

$$A^*=\bar{a}\,A\,R^2(r)\,T(t),\quad B^*=B\,R(r)\,T(t),\quad C^*=\bar{c}\,C\,R(r)\,T^2(t), \tag{3.4}$$

where $R(r)$ and $T(t)$ are arbitrary functions of r and t respectively, and $\bar{a}$ and $\bar{c}$ are arbitrary constants. If we substitute (3.4) in (3.2) and (3.3) we have

$$\left.\begin{aligned}A&=h(r)\{af(r)+bg(t)\}, & B&=f(r)\,g(t),\\ C&=k(t)\{af(r)+bg(t)\}, & (a&=-\bar{c}),\end{aligned}\right\} \tag{3.5}$$

and

$$\psi=\frac{1}{2}\log R(r)\,T(t)+\text{const.},\quad R(r)^{-1}=\bar{a}-f(r),\quad T(t)^{-1}=\bar{c}-b\,g(t)/\bar{a}, \tag{3.6}$$

where $h(r)$ and $k(t)$ are arbitrary functions of r and t respectively and b is an arbitrary constant.

Conversely, when the fundamental form of an S_0 is given by (2.1) with

(3.5), the coordinate system of the S_0 is simultaneously s. s. for an S_0^* which corresponds to the S_0 by (1.1) with (3.6). Thus we have

Theorem [3.1] *Let the fundamental form of an S_0 be given by (2.1) using a s. s. coordinate system. A necessary and sufficient condition that there exist an S_0^* whose geodesics correspond to those of the S_0 and for which the coordinate system of the S_0 is s. s. simultaneously is that the fundamental from of the S_0 have the form (3.5). In this case ψ in (1.1) is given by (3.6).*

As a matter of course, it is easily seen that A^*, B^* and C^* have the same forms as in (3.5) by substituting (3.5) and (3.6) in (3.4). Further the fundamental forms of $[A]$, $[B]$, $[C]$, $[D]$ and $[E]$ are reducible to the same forms as (2.1) with (3.5), and therefore these space-times satisfy the above condition of [3.1].

From (3.5) we can easily obtain

Theorem [3.2] *Let the fundamental form of an S_0 be given by (2.1) using a s. s. coordinate system. In order that there exist an S_0^* whose geodesics correspond to those of the S_0 and for which the coordinate system of the S_0 is simultaneously s. s., this coordinate system must be standard for g_{ij}.*

Theorem [3.3] *Let the fundamental form of an S_0 be given by (2.1) using a s. s. coordinate system. A necessary and sufficient condition that there exist an S_0^* whose geodesics correspond to those of the S_0 and for which the coordinate system of the S_0 is simultaneously s. s, is that there exist a c. s. of the S_0 which satisfies*

$$\sigma\kappa+\kappa\bar{\kappa}-\bar{\sigma}\bar{\kappa}=0\,, \tag{3.7}$$

and
$$\nabla_i\{\log(\sigma+\bar{\kappa})/\sigma\}=\nabla_i\{\log\bar{\sigma}/(\bar{\sigma}-\kappa)\}=-2(\kappa\alpha_i+\bar{\kappa}\beta_i)\,. \tag{3.8}$$

In (3.8), when a denominator or a numerator under log vanishes, the member containing it is to be omitted. (For example, when $\bar{\sigma}$ vanishes, the latter condition (3.8) becomes merely $\nabla_i\{\log(\sigma+\bar{\kappa})/\sigma\}=-2(\kappa\alpha_i+\bar{\kappa}\beta_i)$.)

Proof. If the coordinate system of (2.1) with (3.5) is standard for a c. s., from the formulae concerning c. s. given in (**I**) it follows that

$$af(r)\sigma+bg(t)(\sigma+\bar{\kappa})=0\,,\qquad af(r)(\bar{\sigma}-\kappa)+bg(t)\bar{\sigma}=0\,. \tag{3.9}$$

When all of a, b, $f'(r)$ and $\dot{g}(t)$ do not vanish, (3.7) must hold by virtue of (3.9). Then by solving (3.9) with $B=\exp(-2F)$, we have

$$f(r)=\sqrt{\{-b\bar{\kappa}\bar{\sigma}/a\kappa\sigma\}}\exp(-F)\,,\qquad g(t)=\sqrt{\{-a\kappa\dot{\sigma}/b\bar{\kappa}\bar{\sigma}\}}\exp(-F)\,. \tag{3.10}$$

From this and the formulae concerning c. s. it follows that

$$\nabla_i\{\log(\bar{\kappa}\,\bar{\sigma}/\kappa\,\sigma)\} = -2(\kappa\,\alpha_i + \bar{\kappa}\,\beta_i) \tag{3.11}$$

and this reduces to (3.8) by using (3.7).

When one of a, b, $f'(r)$ and $\dot{g}(t)$ vanishes, (3.7) holds in consequence of (3.9). On the other hand (3.8) also holds by means of (3.9). For example, when $f'(r)$ vanishes it follows that $\kappa=\bar{\sigma}=0$ from (3.9) and consequently the second member of (3.8) can be omitted. Next, from (3.9) it follows directly that the first and third members are identical with each other. In the same way we can deal with the case of $a=0$ or $b=0$ or $\dot{g}(t)=0$.

Conversely, if (3.7) and (3.8) are satisfied by a c. s., we have from the formulae cencerning c. s.

$$\alpha^i\,\beta^j\,\nabla_i\,\nabla_j\,F = \kappa\,\bar{\kappa}\,, \qquad \alpha^i\,\beta^j\,\nabla_i\,\nabla_j\,F = -\dot{F}'/\sqrt{A\,C} - \kappa\,\sigma + \bar{\kappa}\,\bar{\sigma} \tag{3.12}$$

in the standard coordinate system for the c. s. By using this and (3.7) it follows that $\dot{F}'=0$, and therefore B is of the form given in (3.5).

When $\sigma(\sigma+\bar{\kappa})$ or $\bar{\sigma}(\bar{\sigma}-\kappa)$ does not vanish, A or C has the form given by (3.5) respectively in consequence of (3.8) and the formulae of c. s. When σ vanishes, it follows that $\bar{\kappa}=0$ or $\kappa-\bar{\sigma}=0$ by (3.7) and accordingly $\{A=A(r),\ g(t)=\text{const.}\}$ or $\{A=A(r),\ C=B\,k(t)\}$ respectively, where $k(t)$ is an arbitrary function of t. Thus if σ vanishes, A and C are of the form given in (3.5). Similarly, we can deal with the case of $\bar{\sigma}=0$ or $\sigma+\bar{\kappa}=0$ or $\bar{\sigma}-\kappa=0$.

Theorem [3.3] *Let S_0 and S_0^* be two s. s. space-times with common s. s. coordinate system. When their geodesics correspond with each other, we can define the correspondence between the c. s.'s of both space-times so as to make the above common coordinate system standard for both c. s.'s simultaneously.*

In fact, we may define the correspondence as follows:

$$\alpha_i^* = \delta_i^1\sqrt{A} = \sqrt{(\bar{a}\,R^2\,T)}\,\alpha_i\,, \qquad \beta_i^* = \delta_i^4\sqrt{C} = \sqrt{(\bar{c}\,R\,T^2)}\,\beta_i\,,$$

$$\sigma^* = -\dot{A}^*/2A^*\sqrt{C^*} = (\sigma - \beta^i\,\psi_i)/\sqrt{(\bar{c}\,R\,T^2)}\,,$$

$$\bar{\sigma}^* = -C^{*\prime}/2C^*\sqrt{A^*} = (\bar{\sigma} + \alpha^i\,\psi_i)/\sqrt{(\bar{a}\,R^2\,T)}\,,$$

$$\kappa^* = -B^{*\prime}/2B^*\sqrt{A^*} = (\kappa + \alpha^i\,\psi_i)/\sqrt{(\bar{a}\,R^2\,T)}\,,$$

$$\bar{\kappa}^* = \dot{B}^*/2B^*\sqrt{C^*} = (\bar{\kappa} + \beta^i\,\psi_i)/\sqrt{(\bar{c}\,R\,T^2)}\,,$$

$$\overset{1}{\rho}{}^* = \frac{1}{2}\sum_{\lambda} g^{*\lambda\lambda}\,g_{\lambda\lambda}\,\overset{1}{\rho} + \frac{1}{2}\sum_{\lambda,a}(g^{*\lambda\lambda}\,g_{\lambda\lambda} - g^{*aa}\,g_{aa})(\overset{2}{\rho} + \overset{3}{\rho} + 2\overset{4}{\rho})$$
$$-2\sum_{\lambda,a}(g^{*\lambda\lambda}\,\psi_{\lambda\lambda} - g^{*aa}\,\psi_{aa})\,,$$

$$\overset{2}{\rho}{}^*=\frac{1}{2}\sum_a g^{*aa}\, g_{aa}\, \overset{2}{\rho}\,, \qquad \overset{3}{\rho}{}^*=\frac{1}{2}\sum_a g^{*aa}\, g_{aa}\, \overset{3}{\rho}\,,$$

$$\overset{4}{\rho}{}^*=\frac{1}{2}\sum_a g^{*aa}\,(\overset{4}{\rho}\, g_{aa}-2\,\psi_{aa})\,,$$

$$F^*=F-\frac{1}{2}\log R\, T\,, \qquad (R=R(r),\quad T=T(t),\quad \lambda=1,4.)$$

Lastly, we add that we can generalize the results obtained in this paper to the case of the n dimensional s. s. space-times.

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Notes.

1) This paper is a continuation of (I) H. Takeno, Journ. Math. Soc. Japan, 3 (1951), 317; (II) ——, this Journal, 16 (1952), 67; (III) ——, this Journal, 16 (1952), 291; (IV) ——, this Journal, 16 (1952), 299; (V) ——, this Journal, 16 (1953), 497; (VI) ——, this Journal, 16 (1953), 507. The same notations as in these papers are used throughout the present paper. See also H. Takeno, Prog. Theor. Phys. 8 (1952), 317.

2) L. P. Eisenhart, *Riemannian Geometry*, Princeton, 1949, p. 134.