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On Lorentz Transformations and Continuity Equation of Angular Momentum in Relativistic Quantum Mechanics.

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§1. Introduction and outlines.

In special relativity and relativistic quantum mechanics, the fundamental laws of mechanics are formulated so as they are form-invariant under the transformations of the general Lorentz group. The transformation of the Lorentz group is obtained by means of a suitable combination of spatial rotations of the axes of coordinates in two systems together with a special Lorentz transformation of the form:

$$x'=\frac{x-ut}{\sqrt{1-u^2/c^2}},\quad y'=y,\quad z'=z,\quad t'=\frac{t-ux/c^2}{\sqrt{1-u^2/c^2}} \tag{1.1}$$

where x, y, z, t and x', y', z', t' are space-time coordinates in two systems K and K', the uniform velocity of K' relative to K being u along x-axis of K. The equations (1.1) represent the relations between the coordinates in K and K' where the relative velocity of K' to K is *parallel to the x-axis*. However, sometimes we shall need explicit expressions for the Lorentz transformations in a more general case where the relative velocity of K' to K is not parallel to the x-axis and where the Cartesian axes in K and K' have the same orientation (not arbitrary orientations relative to each other). Such a transformation is the so-called *Lorentz transformation without rotation* [1]*). The explicit expression for such Lorentz transformation without rotation is given by the following vector form: [1]

$$\begin{aligned} \boldsymbol{X}' &= \boldsymbol{X} + \boldsymbol{U}\left[\frac{(\boldsymbol{UX})}{u^2}\{(1-u^2/c^2)^{-\frac{1}{2}}-1\} - t\,(1-u^2/c^2)^{-\frac{1}{2}}\right] \\ t' &= (1-u^2/c^2)^{-\frac{1}{2}}\{t-(\boldsymbol{UX})/c^2\} \end{aligned} \tag{1.2}$$

where

$$\boldsymbol{X} = (x, y, z), \quad \boldsymbol{X}' = (x', y', z'), \quad \boldsymbol{U} = (u^1, u^2, u^3),$$

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*) The ciphers in the square brackets refer to the Bibliography attached to the end of this paper.

u^1, u^2, u^3 being the components of the velocity of the system K' relative to K, and $(\boldsymbol{UX}) \equiv u^1x + u^2y + u^3z$. For future consideration, we denote the coordinates x, y, z by $x^i (i=1, 2, 3)$ and rewrite the vector equations (1.2) in the following form:

$$x'^i = x^i + u^i \left[\frac{(ux)}{u^2} \{(1-u^2/c^2)^{-\frac{1}{2}} - 1\} - t\,(1-u^2/c^2)^{-\frac{1}{2}} \right]$$
$$t' = (1-u^2/c^2)^{-\frac{1}{2}} \{t - (ux)/c^2\} \tag{1.3}$$

where $(ux) \equiv u^1x^1 + u^2x^2 + u^3x^3$. Equations (1.3) represent the Lorentz transformation without rotation, u^1, u^2, u^3 being the components of the velocity of the system K' relative to K.

It is noticed, by several authors [**1, 2, 3,**]†, that the transformations defined by (1.3) regarding u^1, u^2, u^3 as parameters do not form a *3-parameter group*. Namely, if we combine two Lorentz transformations without rotation the resultant transformation will not be in general a Lorentz transformation without rotation but will in general correspond to a change of orientation of the Cartesian axes†). On the contrary, in non-relativistic kinematics the Galilean transformations without rotation of the Cartesian axes:

$$x'^i = x^i + u^i t \qquad (i = 1, 2, 3)$$
$$t' = t \tag{1.4}$$

form a *3*-parameter group regarding u^1, u^2, u^3 as parameters. So we consider the problem *to modify the Lorentz transformations defined by (1.3) such that they form a 3-parameter group (regarding $u^1, u^2\ u^3$ as parameters) satisfying the condition that the transformations make the form of $ds^2 \equiv (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dt)^2$ invariant.* By the theory of continuous groups of transformations, the form of the equations of such modified transformations can be obtained in the following form: [**5**]

$$x'^i = \sum_{j=1}^{3} x^j \left[\delta^i_j - \frac{d^i - u^i/c}{1-(du)/c} d_j - d^i \left\{ \frac{u^j/c}{\sqrt{1-(uu)/c^2}} - \frac{d_j\sqrt{1-(uu)/c^2}}{1-(du)/c} \right\} \right]$$
$$+ t \left[d^i \frac{(uu)/c - (du)\{1-\sqrt{1-(uu)/c^2}\}}{\{1-(du)/c\}\sqrt{1-(uu)/c^2}} - \frac{u^i}{1-(du)/c} \right] \tag{1.5}$$
$$t' = [t - (ux)/c^2]/\sqrt{1-(uu)/c^2} \qquad (i = 1, 2, 3)$$

†) L. H. Thomas has deduced the so-called Thomas precession concerning this circumstance. Phil. Mag. (7) **3**, **1** (1927). This is compared with the result deduced from our new transformations (1.5) (See the next peper [**7**]).

which is the equations (1.2) of the previous paper [**5**]. Here $d^i = d_i$ $(i=1,2,3)$ are arbitrary constants (but not parameters of transformations) provided that $\sum_{i=1}^{3} d^i d^i = 1$, δ^i_j denotes *1* or *0* according to $i=j$ or $i \neq j$, and the round bracket e. g. (du) means inner product: $(du) \equiv \sum_{i=1}^{3} d_i u_i$. Similarly $(ux) \equiv \sum_{i=1}^{3} u_i x_i$, $(uu) \equiv \sum_{i=1}^{3} u_i u_i = u^2$.

It is noticeable that new constants d^1, d^2, d^3 are introduced in our equations (1.5). We can easily show that the equations (1.5) coincide with the Lorentz transformation without rotation (1.3) in the special case where u^i is proportional to d^i i. e. $u^i = ud^i$.

The inverse transformation of (1.5) is given by the equations:

$$x^i = x'^i + u^i \frac{t' - (dx')/c}{\sqrt{1-(uu)/c^2}} + d^i \frac{(ux')/c - (dx')\{1 - \sqrt{1-(uu)/c^2}\}}{1-(du)/c}, \quad (i=1,2,3)$$

$$t = \frac{1}{\sqrt{1-(uu)/c^2}} \left[\frac{\{(du)/c - (uu)/c^2\}(dx')/c}{1-(du)/c} + t' \right] \tag{1.6}$$

$$+ [(ux') - (du)(dx')]/c^2 [1 - (du)/c]$$

which is the equations (1.3) of the previous paper [**5**]. From these equations we can see that dx^i/dt becomes u^i when $dx'^i/dt' = 0$ $(i=1,2,3)$, namely the components of the velocity of K' relative to K are equal to u^i. Hence we can take the equations (1.5) as representing the relations between the space-time coordinates in K and K' where the components of the velocity of K' relative to K are u^1, u^2, u^3. Then we may formulate the laws of mechanics such that they are form-invariant under the transformations defined by (1.5) instead of the transformations of the general Lorentz group.

In this paper we shall express the infinitesimal transformations of (1.5) in a tensor form in the *4*-dimensional space-time and compare this with the infinitesimal transformations of the general Lorentz group. And we shall indicate our intention by an example in relativistic quantum mechanics.

§2. The tensor form of the infinitesimal transformations in the 4-dimensional space-time.

It is well known that the infinitesimal transformations of the general Lorentz group are expressed as

$$x'_\mu = x_\mu + \delta w_{\mu\nu} x_\nu \text{ or } \delta x_\mu = \delta w_{\mu\nu} x_\nu \qquad (\mu, \nu = 1, \cdots, 4) \tag{2.1}$$

where x, y, z, ict are denoted by x_1, x_2, x_3, x_4 or briefly by x_μ ($\mu=1, \cdots, 4$), and $\delta w_{\mu\nu}$ ($\mu, \nu=1, \cdots, 4$) are arbitrary infinitesimal quantities subject to the condition $\delta w_{\mu\nu}=-\delta w_{\nu\mu}$. Here we have made use of the convention that when the same index appears twice in a term this term stands for the sum of the terms obtained by giving the index each of its values; thus the term $\delta w_{\mu\nu} x_\nu$ of the right hand side of (2.1) stands for the sum of *4* terms as ν takes the values *1* to *4*. The infinitesimal transformation (2.1) is also expressed by the following symbol:

$$T_{\mu\nu} = x_\nu \frac{\partial}{\partial x_\mu} - x_\mu \frac{\partial}{\partial x_\nu} \qquad (\mu, \nu = 1, \cdots, 4) \tag{2.2}$$

Corresponding to the above we intend to express the infinitesimal transformation of (1.5) in a tensor form in the *4*-dimensional space-time. In the previous paper [**4**] we have shown that the transformation defined by (1.5) is generated by the infinitesimal transformation with the symbol:

$$P_i = x^i \frac{\partial}{\partial (ct)} + ct \frac{\partial}{\partial x^i} - d_p x^p \frac{\partial}{\partial x^i} + x^i d^p \frac{\partial}{\partial x^p} \qquad (i, p = 1, 2, 3) \tag{2.3}$$

which is obtained from the expression (2.4) in the previous paprer [**4**] by putting $h=k=0$ and replacing d^p by $-d^p$. (The symbol (2.3) is verified by calculating the infinitesimal transformation of (1.5) regarding u^1, u^2, u^3 as parameters).

In order to compare the expression (2.3) with (2.2), we introduce the new *4*-dimensional vector (null vector or wave number vector) k_μ whose components (k_1, k_2, k_3, k_4) are (d_1, d_2, d_3, i):

$$k_\mu = (d_1, d_2, d_3, i), \tag{2.4}$$

then (2.3) is rewritten as

$$\mathrm{P}_i = x_i k_\mu \frac{\partial}{\partial x_\mu} - k_\mu x_\mu \frac{\partial}{\partial x_i} \qquad (\mu = 1, \cdots, 4; \quad i = 1, 2, 3)$$

Using (2.2), this is expressed as

$$P_i = k_\mu T_{\mu i} \qquad (i = 1, 2, 3; \quad \mu = 1, \cdots, 4) \tag{2.5}$$

In this expression the suffix i takes the values $1, 2, 3$. However, if we replace i by ν ($\nu=1, \cdots, 4$) and put

$$P_\nu = k_\mu T_{\mu\nu}, \qquad (\mu, \nu = 1, \cdots, 4) \tag{2.6}$$

P_4 is not independent from $P_i (i=1, 2, 3)$ since $k_\nu P_\nu = 0$ because of antisymmetrical property of $T_{\mu\nu}$. Hence, as the symbol of the infinitesimal transformation of (1.5), we can adopt (2.6) instead of (2.3). (In fact the commutator of P_ν and P_ω is exprssed as follows: $[P_\nu P_\omega]=k_\omega P_\nu - k_\nu P_\omega$ since $k_\mu k_\mu=0$) Then, by considering the symbol:

$$\delta w_\nu P_\nu = \delta w_\nu k_\mu \left(x_\nu \frac{\partial}{\partial x_\mu} - x_\mu \frac{\partial}{\partial x_\nu} \right) = \left(\delta w_\nu k_\mu x_\nu - \delta w_\mu k_\nu x_\nu \right) \frac{\partial}{\partial x_\mu} \tag{2.7}$$

the infinitesimal transformation corresponding to the symbol (2.7) is expressed as

$$x'_\mu = x_\mu + (k_\mu \delta w_\nu - k_\nu \delta w_\mu)\, x_\nu \qquad (\mu, \nu = 1, \cdots, 4) \tag{2.8}$$

where δw_μ ($\mu=1, \cdots, 4$) are arbitrary infinitesimal quantities (though it gives identical transformation when δw_μ is proportional to k_μ). This corresponds to the special case of (2.1) where $\delta w_{\mu\nu}$ has the form

$$\delta w_{\mu\nu} = k_\mu \delta w_\nu - k_\nu \delta w_\mu \tag{2.9}$$

So we can say that *the group of transformations defined by* (1.5) *is a subgroup of the general Lorentz group and the infinitesimal transformation of the former corresponds to the special case of* (2.1) *where* $\delta w_{\mu\nu}$ *has the special form*: $\delta w_{\mu\nu}=k_\mu \delta w_\nu - k_\nu \delta w_\mu$. (The tensor form of (1.5) in the 4-dimensional space is given by (3.5) of the next paper [7])

§3. Adoption of (2.8) in relativistic quantum mechanics.

According to the principle of relativity it is required that the fundamemtal equations representing the physical laws must be form-invariant under the Lorentz transformations (2.1). In the principle of relativity, if we adopt the transformations (2.8) instead of (2.1), the fundamental equations which are form-invariant under (2.8) instead of (2.1) may have another form. We shall indicate this circumstance by an example in relativistic quantum mechanics.

In the relativistic field theory of quantum mechanics, from variation principle: $\delta \int L dx_1 dx_2 dx_3 dx_4 = 0$, the equation of motion satisfied by field quantity $Q_\alpha(x)$ ($\alpha=1, 2, \cdots$) is given by the following form:

$$\frac{\partial L}{\partial Q_\alpha} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial Q_{\alpha:\mu}} \right) = 0 \qquad (\mu = 1, \cdots, 4;\ \ \alpha = 1, 2, \cdots) \tag{3.1}$$

where $L=L(Q_\alpha, Q_{\alpha:\mu})$ is Lagrangean of the system and $Q_{\alpha:\mu} \equiv \frac{\partial Q_\alpha}{\partial x_\mu}$. Corresponding to the infinitesimal transformation (2.1):

$$x_\lambda \rightarrow x'_\lambda = x_\lambda + \delta x_\lambda, \quad \delta x_\lambda \equiv \delta w_{\lambda\mu} x_\mu$$

it is assumed that the transformation of $Q_\alpha(x)$ is expressed as follows:

$$Q_\alpha(x) \rightarrow Q'_\alpha(x') = Q_\alpha(x) + \delta Q_\alpha(x), \quad \delta Q_\alpha(x) \equiv \tfrac{1}{2} S_{\mu\nu\alpha}{}^{\beta} Q_\beta \delta w_{\mu\nu}, \tag{3.2}$$

and according to the principle of relativity it is required that the equation of motion (or the integral $\int L\, dx_1 dx_2 dx_3 dx_4$) must be invariant under the Lorentz transformations (2.1). From this requirement, in the usual way, the following result has been deduced

$$\frac{\partial}{\partial x_\sigma}(M_{\mu\nu,\,\sigma})\,\delta w_{\mu\nu} = 0 \tag{3.3}$$

where $M_{\mu\nu,\,\sigma}$ is angular momentum tensor defined by

$$M_{\mu\nu,\,\sigma} \equiv x_\nu T_{\mu\sigma} - x_\mu T_{\nu\sigma} + \left\{\frac{\partial L}{\partial Q_{\alpha:\sigma}}\right\} S_{\mu\nu\alpha}{}^{\beta} Q_\beta \tag{3.4}$$

which is antisymmetric with respect to μ and ν, $T_{\mu\nu}$ being canonical energy-momentum tensor defined by

$$T_{\mu\nu} \equiv -\frac{\partial L}{\partial Q_{\alpha:\nu}} Q_{\alpha:\mu} + L\delta_{\mu\nu} \qquad (\mu, \nu = 1, \cdots, 4)$$

In the case where we adopt the Lorentz transformations (2.1), $\delta w_{\mu\nu}$ $(\mu, \nu = 1, \cdots, 4)$ are arbitrary subject to the condition $\delta w_{\mu\nu} = -\delta w_{\nu\mu}$. Hence in this case, from (3.3), we have

$$\frac{\partial}{\partial x_\sigma} M_{\mu\nu,\,\sigma} = 0. \tag{3.5}$$

But if we take the transformations (2.8) instead of (2.1), $\delta w_{\mu\nu}$ $(\mu, \nu = 1, \cdots, 4)$ are not arbitrary and have the form (2.9). Hence in this case from (3.3) it follows that

$$k_\mu \frac{\partial}{\partial x_\sigma} M_{\mu\nu,\,\sigma} = 0 \tag{3.6}$$

Therefore, in the case where we adopt the transformations (2.8) instead of (2.1), from the requirement of form-invariance under (2.8), we have the relation (3.6) and not (3.5). Here k_μ is the vector having the follwoing property: *Under the transformations generated by the infinitesimal transformations* (2.8), *the spatial direction* d_i *and the velocity* c *(in the* x, y, z, t*-coordinates) represented by the wave unmber vector proportional to* k_μ *are invariant and only the magnitude of the wave unmber is multiplied by a certain factor* ρ. For, by the infinitesimal transformations (2.8), the vector k_μ is transformed into k'_μ as follows: $k'_\mu = \rho k_\mu$ where $\rho \equiv 1 + k_\nu \delta w_\nu$. Under the finite transformations (1.5) or (1.6), k_μ (the covariant components being $d_1, d_2, d_3, -c$ in the x, y, z, t-coordinates) is transformed into k'_μ as follows:

$$k'_\mu = \frac{1-(du)/c}{\sqrt{1-(uu)/c^2}} k_\mu \qquad \left(\rho = \frac{1-(du)/c}{\sqrt{1-(uu)/c^2}}\right) \tag{3.7}$$

which is obtained from (1.6) by the formula : $k'_\mu = k_\lambda \partial x_\lambda / \partial x'_\mu$.

Next, if we add the further condition that (3.1) must be invariant under the spatial rotations about the axis whose direction cosines are d_1, d_2, d_3, we can deduce the following relations :

$$k_\mu \in_{\mu\nu\kappa\lambda} \frac{\partial}{\partial x_\sigma} M_{\kappa\lambda,\,\sigma} = 0 \tag{3.8}$$

where $\in_{\mu\nu\kappa\lambda}$ are defined by

$$\in_{\mu\nu\kappa\lambda} = \begin{cases} 1 \text{ if } \mu, \nu, \kappa, \lambda \text{ is an even permutation of } 1, 2, 3, 4, \\ -1 \text{ if } \mu, \nu, \kappa, \lambda \text{ is an odd permutation of } 1, 2, 3, 4, \\ 0 \text{ in any other cases} \end{cases} \tag{3.9}$$

The deduction of (3.8) is performed by considering the infinitesimal transformations with the symbol $Q_\nu \equiv k_\mu \in_{\mu\nu\kappa\lambda} T_{\kappa\lambda}$, namely the infinitesimal transfomations of the form :

$$x'_\kappa = x_\kappa + \delta w_\nu k_\mu \in_{\mu\nu\kappa\lambda} x_\lambda \qquad (\kappa, \lambda, \mu, \nu = 1, \cdots, 4) \tag{3.10}$$

Lastly, if we take the condition that (3.1) is moreover invariant under all the spatial rotations, (3.6) and (3.8) become (3.5). Corresponding to (3.6) and (3.8), we have the similar result concerning the total angular momentum $P_{\mu\nu}$ defined by the following integral on a surface σ :

$$P_{\mu\nu} = \int M_{\mu\nu,\,\rho}\, d\sigma_\rho \tag{3.11}$$

where $d\sigma_\rho \equiv (dx_2 dx_3 dx_4,\ dx_1 dx_3 dx_4,\ dx_1 dx_2 dx_4,\ dx_1 dx_2 dx_3)$. Namely, using Green's theorem, in accordance with (3.6) and (3.8) we have

$$k_\mu \frac{\delta}{\delta\sigma(x)} P_{\mu\nu} = 0 \tag{3.12}$$

and

$$k_\mu \in_{\mu\nu\kappa\lambda} \frac{\delta}{\delta\sigma(x)} P_{\kappa\lambda} = 0 \tag{3.13}$$

Here, following S. Tomonaga [**6**], we use the notation $\frac{\delta}{\delta\sigma(x)}$ to represent functional differentiation (at a point $(\boldsymbol{x})$ on σ) :

$$\frac{\delta F[\sigma]}{\delta\sigma(x)} = \lim_{\sigma(x)\to\sigma} \frac{F[\sigma(x)] - F[\sigma]}{d\omega}$$

$d\omega$ being 4-dimensional volume of the small world lying between surfaces $\sigma(x)$ (which overlap σ except in a small domain about the point (x)) and σ. ($F[\sigma]$ is a functional of the surface σ). In conclusion, dividing the total angular momentum into orbital angular momentum and spin angular momentum, further investigation may be performed.

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From the relations (3.7), if we consider that radiant energy is proportional to frequency (accordingly to wave number), we may deduce the following result: For the observer moving with velocity u^i, the radiant energy along the direction d^h is observed as multiplied by $[1-(du)/c]/\sqrt{1-u^2/c^2}$. Or, considering k'_μ fixed in K' and using $k_\mu = k'_\mu\sqrt{1-u^2/c^2}/[1-(du)/c]$, we may say that the radiant energy along the direction d^h (moving with velocity u^i) depends on the velocity u^i and is equal to the value at rest multiplied by $\sqrt{1-u^2/c^2}/[1-(du)/c]$. This corresponds to the fact that the vector $nck_\mu\sqrt{1-v^2/c^2}/[1-(dv)/c]$ is introduced in the momentum mass (energy) vector defined in the previous paper [5].