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A Characterization of Dual B^*-Algebras

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It is our purpose to give some remarks on the previous paper [**4**]. Let A be a B^*-algebra over the complex field. It has been shown by Kaplansky [**2**, Theorem 2.1] that the following conditions are equivalent: (1) A is dual, (2) A has a faithful $*$-representation by completely continuous operators on a Hilbert space, (3) the socle of A is dense. In our previous paper [**4**] we have proved that these conditions are also equivalent to the condition that (4) A is weakly completely continuous. It is easy to see from Kaplansky's proof and a result established in [**4**, Theorem 8] that the conditions (1)–(4) are also equivalent for Banach $*$-algebras with $\|x\|^2 \leq k\|x^*x\|$. In this paper we show that a B^*-algebra or Banach $*$-algebra with $\|x\|^2 \leq k\|x^*x\|$ is dual if and only if (5) every self-adjoint element has a spectrum without cluster points other than zero, and that for an A^*-algebra the condition (5) implies (2).

1. A Banach algebra A over the complex field is called a Banach $*$-algebra if A admits an involution $x \rightarrow x^*$, that is, a conjugate-linear anti-automorphism of period two. If A has also the property $\|x\|^2 = \|x^*x\|$, then A is called a B^*-algebra.

THEOREM 1. *Let A be a B^*-algebra or a Banach $*$-algebra with $\|x\|^2 \leq k\|x^*x\|$. Then the following statements are equivalent:*

(i) *A is dual.*

(ii) *Every maximal commutative $*$-subalgebra is dual.*

(iii) *The Gelfand space of every maximal commutative $*$-subalgebra is discrete.*

(iv) *Every self-adjoint element has a spectrum without cluster points other than zero.*

PROOF. (i) $\rightarrow$ (ii). This follows from the fact that every closed $*$-subalgebra of A is dual [**4**, Corollary of Theorem 8].

(ii) $\rightarrow$ (iii). Let B be a maximal commutative $*$-subalgebra. Since B is dual, it is weakly completely continuous, and therefore its Gelfand space is discrete [**4**, Lemma 5].

(iii) $\rightarrow$ (iv). Let x be any self-adjoint element. Embed x in a maximal commutative $*$-subalgebra B. B is $*$-isomorphic and equivalent to the algebra $C(\Omega)$ of continuous functions vanishing at infinity on the Gelfand space Ω. Since Ω is discrete, the spectrum of x has no cluster point other than zero.

(iv) $\to$ (iii). Let Ω be a Gelfand space of a maximal commutative $*$-subalgebra B. Consider any compact neighbourhood U of $p_0 \in \Omega$. There exists a real-valued continuous function $\hat{x}(p)$ such that $0 \leq \hat{x} \leq 1$, $\hat{x}(p_0) = 1$, and $\hat{x}(p) = 0$ for $p \notin U$. The range of $\hat{x}$ has no cluster point other than zero since $\hat{x}$ is a Gelfand representation of some $x \in B$, and therefore the set $\{p\,;\, \hat{x}(p) = 1\}$ is a compact open set contained in U. If p_0 is a cluster point of Ω, then we can take a sequence of compact open neighbourhoods U_n of p_0 in such a way that U_{n+1} is a proper subset of U_n. Let e_n be the self-adjoint element whose Gelfand representation is the characteristic function of U_n. Put $x = \sum \frac{1}{n^2} e_n$. It is easy to see that $\sum \frac{1}{n^2}$ is a cluster point of $\sigma(x)$. This is a contradiction.

(iii) $\to$ (i). Let x be any self-adjoint element. Embed x in a maximal commutative $*$-subalgebra B. Since the Gelfand space Ω of B is discrete, there exists a family of orthogonal self-adjoint primitive idempotents $e_\alpha \in B$ such that every element $z \in B$ can be expressed as $\sum \lambda_\alpha e_\alpha$, where $ez_\alpha = \lambda_\alpha e_\alpha$ and the number of λ_α such that $|\lambda_\alpha| > \varepsilon$, ε being any positive number, is finite. It follows from the maximality of B that $e_\alpha A e_\alpha =$ (the complex field) $\times\, e_\alpha$, and therefore $e_\alpha A$ is a minimal right ideal of A. Therefore x is contained in the closure of the socle S of A. Any element of A is a linear combination of self-adjoint elements, and therefore S is dense in A. This implies that A is dual. The proof is completed.

Corollary. *Let A be a B^*-algebra of a Banach $*$-algebra with $\|x\|^2 \leq k\|x^*x\|$. If every self-adjoint element has a finite spectrum, then A is finite-dimensional.*

Proof. Let Ω be the Gelfand space of a maximal commutative $*$-subalgebra B. Ω is finite and therefore compact since it is discrete and every self-adjoint element of B has a finite spectrum. Therefore B has a unit e. We show that e is also a unit of A. Let x be any element of A. Put $y = ex - x$. Then $yy^* = (e-1)xx^*(e-1)$. It follows from the maximality of B that $yy^* \in B$, and therefore $yy^* = 0$. This implies $ex = x$. Similarly we have $xe = x$. Therefore A is finite-dimensional [**4**].

2. A Banach *-algebra is called an A^*-algebra [**5**] provided A has an auxiliary norm $|x|$ which satisfies, in addition to the usual multiplicative property, the condition $|x|^2 \leq k|x^*x|$ (the completeness with respect to $|x|$ is not assumed).

Theorem 2. *Let A be an A^*-algebra in which every self-adjoint element has a spectrum without cluster points other than zero. Then*

(i) *A is symmetric.*

(ii) *A has a unique auxiliary norm (to within equivalence).*

(iii) *A is dense in a dual B^*-algebra.*

Proof. (i). Let x be any element of A. Embed x^*x in a maximal commutative $*$-subalgebra B. Let $\lambda \neq 0$ be in a spectrum $\sigma(x^*x)$. By hypothesis λ is an isolated point of $\sigma(x^*x)$. We can find by contour integration the existence of a non zero idempotent $e \in B$ such that $ex^*x = \lambda e$, and therefore $x^*xe^* = \bar{\lambda}e^*$. It follows from $\lambda ee^* = ex^*xe^* = \bar{\lambda}ee^*$ that λ is real. This implies that A is C-symmetric. By a theorem due to Kaplansky [**3**] λ can not be negative. Therefore A is symmetric.

(ii). Let B be any maximal commutative $*$-subalgebra. B is $*$-isomorphic with a dense subalgebra of the algebra $C(\Omega)$ of continuous functions vanishing at infinity on the Gelfand space Ω of B since B is semi-simple [**5**] and symmetric as proved in (i). Let U be any compact neighbourhood of a point $p_0 \in \Omega$. Let $\varphi(p)$ be a real-valued continuous function such that $0 \leq \varphi \leq 1$, $\varphi(p_0) = 1$, and $\varphi(p) = 0$ for $p \notin U$. Take a self-adjoint element x in such a way that its Gelfand representation $\hat{x}$ satisfies $|\varphi(p) - \hat{x}(p)| < \frac{1}{2}$ on Ω. The range of $\hat{x}$ has no cluster point other than zero, and therefore $\hat{x}(p_0)$ is an isolated point of $\sigma(x)$. We can find by contour integration a self-adjoint idempotent e whose Gelfand representation is the characteristic function of the set $\{p\,;\, \hat{x}(p) = \hat{x}(p_0)\}$. Thus any maximal commutative $*$-subalgebra is regular in a sense defined by Silov. This implies that A has a unique auxiliary norm to within equivalence [**4**].

(iii) We use the same notations as in (ii). By the same argument as in the proof of Theorem 1 we can show that p_0 is an isolated point of Ω. Therefore Ω is discrete and the characteristic function of one point set corresponds to the self-adjoint primitive idempotent of A. Let $\mathfrak{A}$ be the completion of A by its auxiliary norm $|x|$. Then e is also a primitive self-adjoint idempotent and $e\mathfrak{A}$ is a minimal right ideal. As in the proof of Theorem 1, A is containdd in the socle S of $\mathfrak{A}$, and therefore S is dense in $\mathfrak{A}$. This shows that $\mathfrak{A}$ is dual. The proof is completed.

We remark also that if A in the above theorem is algebraically embedded in a Banach $*$-algebra A', then the spectrum of $x \in A$ is the same both in A and A'. We can show this by the same line as in the proof given by Rickart [**6**] in case A is a B^*-algebra, in which the regularity of commutative B^*-algebras plays an essential role.

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