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A Note on Lattice Segment

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W. D. Duthie introduced the concept of a segment in a lattice and characterized the modularity and the distributivity of a lattice by it[1]. And M. Sholander has, from the axiomatic standpoint, investigated the segments and obtained the three axioms which characterize the segments of a distributive lattice with O and I[2].

The purpose of this paper is to generalize the M. Sholander's result and to obtain the axioms which characterize the segments of a lattice with O.

§ 1. Segment of a Lattice L.

In this section, we consider the properties of the segments of a lattice L.

Here, we use the definition of a segment which was used by W. D. Duthie, that is, for any pair a, b of the elements of a lattice L, the set of all elements $x \in L$ which satisfies the condition $ab \leqq x \leqq a+b$ is called the segment joining a and b, and is denoted by the symbol (a, b).

From the above definition of the segment, we have the following lemmas.

(1.1) $$(a, b) \cup (c, d) \subset (abcd, a+b+c+d)$$

PROOF. Suppose $x \in (a, b) \cup (c, d)$. Then element x satisfies the conditions $ab \leqq x \leqq a+b$ or $cd \leqq x \leqq c+d$. So, we have $abcd \leqq x \leqq a+b+c+d$.

Note. Briefly, we write the set $(abcd,\ a+b+c+d)$ by the symbol $(a,b) \overset{*}{\cup} (c,d)$.

(1.2) Let L be a lattice with O. $(a, b) \subset (p, q)$ if and only if $(O, a) \cap (O, b) \supset (O, p) \cap (O, q)$ and $(O, a) \overset{*}{\cup} (O, b) \subset (O, p) \overset{*}{\cup} (O, q)$.

PROOF. First we prove the necessity. W.D.Duthie has shown that $(a, b) \cap (c, d) =$

1) W. D. Duthie, "Segments of ordered sets," Trans. Am. Math. Soc. vol. 51 (1942) pp. 1–14.

2) M. Sholander, "Tree, lattice, order and betweenness," Proc. Am. Math. Soc. vol. 3 (1952) pp. 369–381.

$(ab+cd, (a+b)(c+d))$ if the intersection is not empty. From this result and (1.1), we have $(O, x) \cap (O, y) = (O, xy)$ and $(O, x) \overset{*}{\cup} (O, y) = (O, x+y)$. Then from the assumption of this lemma, we have $(O, ab) \supset (O, pq)$ and $(O, a+b) \subset (O, p+q)$ $\xrightarrow{(3)}$ $ab \geqq pq$ and $a+b \leqq p+q \longrightarrow pq \leqq ab \leqq a+b \leqq p+q$. So, we have $(a, b) \subset (p, q)$.

Similarly, we can prove the sufficiency.

(1.3) $$b \in (a, c) \text{ if and only if } (a, b) \subset (a, c).$$

Proof. The necessity is evident, so we will prove the sufficiency. Suppose $b \in (a, c)$. $\longrightarrow ac \leqq b \leqq a+c. \longrightarrow ac \leqq ab$ and $a+b \leqq a+c. \longrightarrow ac \leqq ab \leqq a+b \leqq a+c$.

Hence, we have $(a, b) \subset (a, c)$.

§ 2. Axiom of Segment.

In this section, we define the segment by the three axioms and investigate the properties of the segments which will result from these axioms.

We consider a set S of elements $a, b, c, \cdots\cdots$ such that for each pair a, b of elements of S there corresponds a unique subset of S denoted by (a, b).

If the collection of these subsets (a, b) satisfy the following three conditions, the subset (a, b) is called the segment joining a and b.

S_1 : $(a, a) = \{a\}$ *for every element* $a \in S$.

S_2 : *For each pair* a, b *of elements, there correspond a fixed element* O *and a unique element* r *such that* $(O, a) \cap (O, b) = (O, r)$. *Further, there corresponds a unique element* s *such that the set* (O, s) *is the minimum set which contains the sets* (O, a) *and* (O, b).

S_3 : $(a, b) \subset (p, q)$ *if and only if* $(O, a) \cap (O, b) \supset (O, p) \cap (O, q)$ *and* $(O, a) \overset{*}{\cup} (O, b) \subset (O, p) \overset{*}{\cup} (O, q)$.

where $(O, x) \overset{*}{\cup} (O, y)$ *represents the minimum set which contains the sets* (O, x) *and* (O, y).

From the above definition of the segment, we have the following properties.

(2.1) $$a, b \in (a, b)$$

Proof. If we put a for b and p, and b for q in the axiom S_3, we have $(a, a) \subset (a, b)$.

3) The notation "$A \longrightarrow B$" means that since the condition (or relation) A is satisfied, so the condition (or relation) B is satisfied.

From the axiom S_1, we have $a \in (a, b)$. Similarly, we have $b \in (a, b)$.

(2. 2) $$(a, b) = (b, a).$$

We can easily see it from the axiom S_3.

(2. 3) $$(O, a) = (O, b) \quad \text{implies} \quad a = b.$$

Proof. By the axiom S_2, there is a unique element r such that $(O, a) \frown (O, b) = (O, r)$. From the assumption, $(O, a) \frown (O, b) = (O, a) = (O, b)$. So, we have $a = b$.

(2. 4) $$b \in (a, c) \text{ if and only if } (a, b) \subset (a, c).$$

Proof. The necessity follows from (2. 1). So we will prove the sufficiency. Suppose $b \in (a, c)$. $\longrightarrow$ $(b, b) \subset (a, c)$ $\xrightarrow{S_3}$[4] $(O, b) \supset (O, a) \frown (O, c)$ and $(O, b) \subset (O, a) \overset{*}{\smile} (O, c)$. $\xrightarrow{S_3}$ $(a, b) \subset (a, c)$.

(2. 5) $$b \in (O, a) \text{ and } a \in (O, b) \text{ imply } a = b.$$

Proof. Suppose $b \in (O, a)$ and $a \in (O, b)$ $\xrightarrow{(2.4)}$ $(O, b) \subset (O, a)$ and $(O, a) \subset (O, b)$. $\longrightarrow$ $(O, a) = (O, b)$. $\xrightarrow{(2.3)}$ $a = b$.

(2. 6) $(O, a) \frown (O, b) = (O, r)$ and $(O, a) \overset{*}{\smile} (O, b) = (O, s)$ imply $(O, r) \subset (O, s)$.

This follows from the definitions of $(O, a) \frown (O, b)$ and $(O, a) \overset{*}{\smile} (O, b)$.

(2. 7) $(O, a) \frown (O, b) = (O, r)$ and $(O, a) \overset{*}{\smile} (O, b) = (O, s)$ imply $(a, b) = (r, s)$.

Proof. From the assumption and (2. 6), we have $(O, a) \frown (O, b) = (O, r) = (O, r) \frown (O, s)$ and $(O, a) \overset{*}{\smile} (O, b) = (O, s) = (O, r) \overset{*}{\smile} (O, s)$. So, from the axiom S_3 we have $(a, b) = (r, s)$.

§ 3. Characterization of lattice segment.

In this section we prove that our axioms S_1, S_2 and S_3 characterize the segment of a lattice with O.

Now, we consider the set $S = \{a, b, c, \ldots\ldots\ldots\}$ such that to each pair a, b of elements of S there corresponds a unique subset (a, b) of S and the collection of these subsets satisfy the conditions S_1, S_2 and S_3. We denote such a set S by the symbol $S\ [S_1, S_2, S_3]$.

Then we have the following lemmas.

(3. 1) The set $S\ [S_1, S_2, S_3]$ is a lattice with O.

Proof. Now, for each pair a, b of elements of S we define "meet" ab and "join"

4) The notation "$A \xrightarrow{S_3} B$" means that since the condition (or relation) A is satisfied, so by S_3 the condition (or relation) B is satisfied.

$a+b$ as follows:

$$r=ab \quad \text{if and only if} \quad (O, r)=(O, a)\cap(O, b),$$

$$s=a+b \quad \text{if and only if} \quad (O, s)=(O, a)\overset{*}{\cup}(O, b).$$

Then the meet and join satisfy the lattice conditions L1 (idempotent law), L2 (commutative law), L3 (associative law) and L4 (absorption law). For:

L1, L2 and L4 result from the definitions of $(O, a)\cap(O, b)$ and $(O, a)\overset{*}{\cup}(O, b)$. So, we will prove L3, that is, $x(yz)=(xy)z$ and $x+(y+z)=(x+y)+z$. The former results from the definition of $(O, a)\cap(O, b)$. For the proof of the latter, it is sufficient to show that $(O, x)\overset{*}{\cup}\{(O, y)\overset{*}{\cup}(O, z)\}=\{(O, x)\overset{*}{\cup}(O, y)\}\overset{*}{\cup}(O, z)$. Now, put $(O, y)\overset{*}{\cup}(O, z)=(O, s)$, $(O, x)\overset{*}{\cup}(O, s)=(O, p)$, $(O, x)\overset{*}{\cup}(O, y)=(O, r)$ and $(O, r)\overset{*}{\cup}(O, z)=(O, q)$. Then we have $(O, p)\supset(O, x)$ and (O, s), and $(O, s)\supset(O, y)$ and (O, z), $\longrightarrow$ $(O, p)\supset(O, x)$ and (O, y). $\longrightarrow$ $(O, p)\supset(O, r)$ and (O, z). $\longrightarrow$ $(O, p)\supset(O, q)$. Similarly we have $(O, p)\subset(O, q)$. Hence we have $(O, p)=(O, q)$.

Furthermore, there is a subset (O, x) for every element x of S. And from (2.1) we have $O\in(O, x)$. So, we have $O\cdot x=O$.

Thus, $S\ [S_1, S_2, S_3]$ is a lattice with O.

Now, we introduce the order in the lattice $S\ [S_1, S_2, S_3]$ by the ordinary method, that is, for a pair x, y of elements of S we define $x\leqq y$ if and only if $xy=x$. Then we have:

(3. 2) $\qquad x\in(a, b)$ implies $ab\leqq x\leqq a+b$.

Proof. Now, put $r=ab$ and $s=a+b$.

Suppose $x\in(a, b)$. $\xrightarrow{(2.7)}$ $x\in(r, s)$. $\xrightarrow{(2.4)}$ $(r, x)\subset(r, s)$. $\xrightarrow{S_3,(2.6)}$ $(O, r)\cap(O, x)\supset(O, r)$ and $(O, r)\overset{*}{\cup}(O, x)\subset(O, s)$. $\longrightarrow$ $(O, r)=(O, r)\cap(O, x)$ and $(O, x)=(O, s)\cap(O, x)$. $\longrightarrow$ $r\leqq x$ and $x\leqq s$. Hence we have $ab\leqq x\leqq a+b$.

(3. 3) $\qquad ab\leqq x\leqq a+b$ implies $x\in(a, b)$.

Proof. Put $r=ab$ and $s=a+b$.

Suppose $ab\leqq x\leqq a+b$. $\longrightarrow$ $(O, x)\supset(O, r)$ and $(O, x)\subset(O, s)$. $\xrightarrow{S_3}$ $(r, x)\subset(r, s)$. $\xrightarrow{(2.4)}$ $x\in(r, s)$.

Hence we have $x\in(a, b)$ from (2. 7).

From the above results, we have the theorem:

Theorem. *The axioms S_1, S_2 and S_3 characterize the segment of a lattice with O.*

Proof. If in a lattice with O we define the segment (a, b) as the set of all x such that $ab\leqq x\leqq a+b$, the axioms S_1 and S_2 are easily derived. And the axiom S_3 follows from (1. 2).

Conversely, that these axioms S_1, S_2 and S_3 give the segment of a lattice with O follows from (3. 1), (3. 2) and (3. 3).

Remark. From the above theorem and the W. D. Duthie's results, we can see that the axioms S_1, S_2, S_3 and the following S_m characterize the segment of a modular lattice with O.

S_m : *If* $(O, r)=(O, a)\cap(O, b)$, $(O, s)=(O, a)\overset{*}{\cup}(O, b)$ *and* $(r, c)=(s, c)$, *then* $a=b$.

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