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A Note on the Commutativity of Certain Rings

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In his recent paper Herstein [1] proved the following theorem: Let R be a ring with center Z. Suppose that every $x \in R$ satisfies $x^{n(x)} - x \in Z$, where $n(x) > 1$ is an integer depending on x, then R is commutative. This theorem is a generalization of a well-known theorem of Jacobson [2]: If in a ring R every $x \in R$ satisfies $x^n - x = 0$ with $n > 1$ depending on x, then R is commutative.

Throughout this note R denotes a ring with the identity element, Z is the center of R and every element of R satisfies $x^{n(x)} - h(x) \cdot x \in Z$ where $n = n(x) > 1$ is an integer bounded for all x, $h = h(x)$ is regular in Z for all $x \in R$ and the set H of elements $h(x)$ for all x is finite.

In this note, we shall prove the following theorem using Herstein's process and results [1].

Theorem. *Let R be a ring with the identity element in which every element x satisfies $x^n - hx \in Z$, where n, h depend on x, $n(>1)$ is an integer but bounded for all x, and $h \in H$ for all x, H being a finite set of regular elements contained in Z. Then R is commutative.*

In this theorem we assumed that the set H is finite. For in the division ring Q of all quaternions over the real number field every element $x = \alpha + \beta i + \gamma j + \delta k$ satisfies $x^2 - 2\alpha x \in Z$, where Z is the real number field, and when $\alpha = 0$, $x \neq 0$, x satisfies $x^3 - hx \in Z$ where $h = -(\beta^2 + \gamma^2 + \delta^2)$, and when $x = 0$, x satisfies $x^2 - x \in Z$. Therefore the ring Q, which is not commutative, does not satisfy the condition that H is finite, though it satisfies all the other conditions in the theorem.

Since we assumed that R has identity element and h is regular all the results of Herstein [1] can be obtained by a slight modification except for the division ring case and the final step of the proof of the main theorem. The proof can be divided into two cases: semi-simple ring case and general case. The first case can be reduced to the division ring case and the second can be reduced to the subdirectly irreducible case.

Division ring case. Let R be a division ring. In this case the center Z is a field.

When Z is an infinite field then $Z=R$. Indeed if $Z\neq R$ choose an element x of R such that $x\notin Z$. Since n is bounded and the set H is finite there exists at least one fixed pair (N, h_0) such that $(c_i x)^N - h_0(c_i x)\in Z$, $(N>1,\ h_0\in H)$ hold for infinitely many $c_i\in Z\,(i=0,1,2,\cdots)$. Suppose $c_0\neq 0$ then $h_0\left\{\left(\frac{c_i}{c_0}\right)^{N-1}-1\right\}x\in Z$. Since $x\notin Z, h_0\in Z$ and $h_0\neq 0$, $\left(\frac{c_i}{c_0}\right)^{N-1}-1=0$. This shows that the equation $y^{N-1}-1=0$ would have infinitely many roots in Z. This is impossible. Hence if Z is infinite $Z=R$ i. e. R is commutative.

When Z is finite, then any element x of R is a root of an equation of the form $y^n - hy = c$ where $h, c\in Z$. The field $Z(x)$ is a finite extension of a finite field and itself finite. Therefore x satisfies the equation of the form $x^{m(x)}-x=0$, $m(x)>0$. By the above theorem of Jacobson R is commutative.

Subdirectly irreducible case. Let R be subdirectly irreducible and not commutative, and S be the intersection of the non-zero ideals of R. Then $A(S)=\{x\in R,\ Sx=(0)\}$ is an ideal of R and $A(S)\subsetneqq Z$. Moreover $R/A(S)$ is a field and its characteristic p is finite ([1] p. 110)

Let an element $x\in R$ reflect into $\bar{x}\in R/A(S)$ and $x\notin Z$, then we have $(n-1)\bar{x}=\bar{0}$ and $n\bar{x}^{n-1}=\bar{h}$ for some $h\notin H$ ([1] p. 110. Instead of $n\bar{x}^{n-1}=1$ we have $n\bar{x}^{n-1}=\bar{h}$ for some $h\in H$.) Hence $n\equiv 1\ (mod\ p)$, therefore $\bar{x}^{n-1}=\bar{h}$. Put $n-1=n_1$, and $\bar{h}=\bar{h}_1$, then $\bar{x}^{n_1}=\bar{h}_1$, where $n_1\geqq 1$ and $h_1\in H$. Choose an integer m such that $m>n_1$ and $x^m\notin Z$ then $(\bar{x}^m)^{m_2}=\bar{h}_2$ for some integer $m_2\geqq 1$ and $h_2\in H$. Put $mm_2=n_2$ then $n_1<n_2$, $\bar{x}^{n_2}=\bar{h}_2$. Continuing this way we have an increasing sequence $\{n_i\}$ of natural numbers and a sequence $\{h_i\}$ of elements of H such that $\bar{x}^{n_i}=\bar{h}_i$ hold for $i=1, 2, \cdots$. Since H is finite there exists a pair of integers i, j such that $i>j$ and $h_i=h_j$. Therefore $\bar{x}^{n_i}=\bar{x}^{n_j}$ hence $\bar{x}^q=\bar{x}$ where $n_i-n_j+1=q>1$. Thus we have $x^q-x\in A(S)\subsetneqq Z$ for $x\notin Z$. Therefore for any $x\in R$ we have $x^q-x\in Z$. Hence by the above theorem of Herstein R is commutative.

References

[1] I. N. Herstein, A generalization of a theorem of Jacobson III Amer. J. Math. **75** (1953) 105-111.
[2] N. Jacobson, Structure theory for algebras of bounded degree, Ann. of Math. **46** (1945) 695-707.