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Convergence of Numerical Iteration in Solution of Equations

By

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§ 1. Introduction

Let R be a *linear normed space* and F be a *complete subset of* R. Let T be a functional defined on F such that $T(F) \subset R$. We assume that T *is Lipschitz bounded, namely that there exists a positive constant* K *such that*

$$\| Tf_1 - Tf_2 \| \leqq K \| f_1 - f_2 \|$$

for any $f_1, f_2 \in F$.

In F, let us consider the equation

$$x = T x. \tag{1.1}$$

We assume that

(i) $K < 1$;

(ii) *for a selected* $x_0 \in F$, $x_1 = Tx_0$ *belongs to* F;

(iii) *the sphere* $S\{h : \|h - x_1\| \leqq \frac{K}{1-K}\|x_1 - x_0\|\}$ *is contained in* F.

L. Collatz [1] has shown that the iteration $x_{n+1} = Tx_n$ $(n = 0, 1, 2, \cdots)$ can be continued idefinitely and the sequence $\{x_n\}$ converges to a certain limit $\bar{x}$ which gives a unique solution of (1.1) in F. But, when x is a numerical quantity, there arises an error in computation of Tx, consequently, in numerical iteration, the obtained sequence is not the sequence $\{x_n\}$ determined by $x_{n+1} = Tx_n$, but the numerical sequence $\{x_n^*\}$ determined by $x_{n+1}^* = T^* x_n^*$, where T^* is a certain approximate functional of T. Then, as is shown in this paper, *the numerical sequence* $\{x_n^*\}$ *does not necessarily converge contrary to convergence of the true sequence* $\{x_n\}$. Then, in order to seek for the solution of (1.1), *at what step the iteration process should be stopped? When the iteration process is stopped at the favorable step in this sense, with how*

1) L. Collatz, *Einige Anwendungen funktionalanalytischer Methoden in der praktischen Analysis*, Z. Angew. Math. Phys., **4**, 327-357 (1953).

much error is the obtained approximate solution attended? In this paper, we discuss on these problems.

On these problems, Sibagaki[1] has studied the case where R is a real line. He has said that *the numerical sequence* $\{x_n^*\}$ *is in the state of numerical convergence* when the iteration process reaches the step at which $x_{n+1}^* = x_n^*$ occurs. He has shown that, when the numerical sequence $\{x_n^*\}$ reaches the state of numerical convergence by sufficiently accurate computation and rounding thereafter, the approximate solution is obtained from that state of numerical convergence, and he has given the bound of the error of that approximate solution. But his discussions are confined to so simple and favorable cases that they are not sufficient for practical application.

In this paper, in order to obtain the theory applicable to almost all practical cases, in general form, we study the problems proposed above for general R. First, we investigate the behavior of the numerical sequence $\{x_n^*\}$ and define anew *the state of numerical convergence*. When R is an Euclidean space, the state of numerical convergence in new sense becomes an oscillating state where $x_{n+m}^* = x_n^*$ for a certain fixed positive integer m. For general R, it is shown that the numerical sequence $\{x_n^*\}$ always reaches the state of numerical convergence in new sense after finite times of repetition of iteration. If the sequence $\{x_n^*\}$ reaches the state of numerical convergence in Sibagaki's sense, it evidently converges actually, but, as is remarked just now, in general, the sequence $\{x_n^*\}$ does not necessarily converge although it reaches the state of numerical convergence in new sense.

Next we determine the approximate solution from this state of numerical convergence and seek for the bound of error of the approximate solution thus determined.

Lastly we show that, specially when R is a real line, the state of numerical convergence becomes an oscillation where x_n^* takes only two values at most, provided that the computation is carried on sufficiently minutely and accurately.

§ 2. The behavior of numerical sequence

Let the composite error committed in computing Tx be η. η is a sum of the truncation error and the rounding errors. Let the bound of η be $\mathcal{E}$, namely

1) W. Sibagaki, *On the idea of "numerical convergence" and its some applications.* Mem. Fac. Sci. Kyûshû Univ., Ser. A, **5**, 89–97 (1950).

suppose that

$$\|\eta\| \leqq \varepsilon. \tag{2.1}$$

Put

$$\delta = \varepsilon/(1-K). \tag{2.2}$$

We consider the sphere $\sum\{h : \|h - x_1^*\| \leqq \frac{K}{1-K}\|x_1^* - x_0\| + \delta\}$. Since $\|x_1^* - x_1\| = \|\eta\| \leqq \varepsilon$, for any point $h \in S$, it holds that

$$\begin{aligned} \|h - x_1^*\| &\leqq \|h - x_1\| + \|x_1 - x_1^*\| \\ &\leqq \frac{K}{1-K}\|x_1 - x_0\| + \varepsilon \\ &\leqq \frac{K}{1-K}[\|x_1 - x_1^*\| + \|x_1^* - x_0\|] + \varepsilon \\ &\leqq \frac{K}{1-K}\|x_1^* - x_0\| + \delta, \end{aligned}$$

consequenty $S \subset \sum$. We assume that *the δ-neighborhood U of $\sum$ is contained in F.* Then evidently the δ-neighborhood of S is contained in $U \subset F$.

If $x_1^*, x_2^*, \cdots, x_m^*$ belong to U, it is valid that

$$\begin{cases} x_1^* = T^* x_0 = T x_0 + \eta_1, \\ x_2^* = T^* x_1^* = T x_1^* + \eta_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{m+1}^* = T^* x_m^* = T x_m^* + \eta_{m+1}, \end{cases} \tag{2.3}$$

where

$$\|\eta_n\| \leqq \varepsilon \qquad (n = 1, 2, \cdots, m+1).$$

Since $x_n \in S$ $(n = 1, 2, \cdots)$ by Collatz[1], it is valid that

$$x_{n+1} = T x_n \qquad (n = 0, 1, 2, \cdots). \tag{2.4}$$

Due to Lipschitz boundedness of T, from (2.3) and (2.4), it follows successively that

$$\begin{aligned} \|x_n^* - x_n\| &\leqq K\|x_{n-1}^* - x_{n-1}\| + \varepsilon \leqq \varepsilon(1 + K + \cdots + K^{n-1}) \\ &< \delta^{2)} \qquad (n = 1, 2, \cdots, m+1). \end{aligned} \tag{2.5}$$

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1) L. Collatz, ibid.

2) $x_0^* = x_0$.

Then, since $x_{m+1} \in S$, it follows that $x^*_{m+1} \in U$. Thus, by induction, it is seen that *the numerical iteration*

$$(2.6) \qquad x^*_{n+1} = T^* x^*_n = T x^*_n + \eta_{n+1} \qquad (n = 0, 1, 2, \cdots)$$

can also be continued indefinitely and all x^*_n*'s* $(n = 1, 2, \cdots)$ *belong to* $U \subset F$.

Likewise as (2.5) is deduced, from (2.6) follows

$$(2.7) \qquad \begin{aligned} \| x^*_{n+1} - x^*_n \| &\leqq K \| x^*_n - x^*_{n-1} \| + 2\varepsilon \\ &\leqq K^n \| x^*_1 - x_0 \| + 2\varepsilon (1 + K + \cdots + K^{n-1}) \\ &< K^n \| x^*_1 - x_0 \| + 2\delta. \end{aligned}$$

Consequently we have

$$\overline{\lim_{n\to\infty}} \| x^*_{n+1} - x^*_n \| \leqq 2\delta,$$

from which it is evident that the numerical sequence $\{x^*_n\}$ does not necessarily converge contrary to convergence of the true sequence $\{x_n\}$.

From (2.7), it follows that

$$\begin{aligned} \|x^*_n - x^*_{n-1}\| - \|x^*_{n+1} - x^*_n\| &\geqq \|x^*_n - x^*_{n-1}\| - K\|x^*_n - x^*_{n-1}\| - 2\varepsilon \\ &= (1 - K)(\|x^*_n - x^*_{n-1}\| - 2\delta). \end{aligned}$$

Consequently, so long as $\|x^*_n - x^*_{n-1}\| > 2\delta$, the quantity $\|x^*_{n+1} - x^*_n\|$ decreases monotonely. Also, from (2.7), if we take a positive number N so that

$$(2.8) \qquad K^N \| x^*_1 - x_0 \| < 2e$$

where e is an arbitrary positive number, then, for $n \geqq N$, it holds that

$$(2.9) \qquad \| x^*_{n+1} - x^*_n \| < 2\delta_1$$

where $\delta_1 = \delta + e$. In other words, *when we continue the iteration process starting from* x_0, *after certain finite times of repetition*, (2.9) *holds always.*

§ 3. The state of numerical convergence

1° *The case where U is totally bounded.* In this case, it is shown that the relation

$$(3.1) \qquad x^*_{n+m} = x^*_n$$

surely occurs for certain $n \geqq N$ and $m > 0$. For, if not so, the points x^*_n's $(n = N, N+1, \cdots)$ are all distinct. Then, from the totally boundedness of U, there exists a subsequence $\{x^*_{n_m}\}$ forming a Cauchy sequence, namely a subsequence such that, for any small positive number ξ, there holds the relation

$$\| x^*_{n_p} - x^*_{n_q} \| < \xi \tag{3.2}$$

for $p, q > G$ G being a sufficiently large positive number. Since x^*_n's are the numerical quantities, if ξ is chosen sufficiently small, then, from (3.2), $x^*_{n_p}$ must coincide with $x^*_{n_q}$. This contradicts the initial assumption. Thus we see that the relation (3.1) surely occurs.

The relaion (3.1) shows that the iteration returns to the state of x^*_n after m-times of repetition thereafter, consequently it holds that

$$x^*_{n+m+i} = x^*_{n+i}, \qquad (i = 0, 1, 2, \cdots, m-1) \tag{3.3}$$

namely the sequence $\{x^*_n\}$ oscillates taking m values. We call this state of oscillation of the sequence $\{x^*_n\}$ *the state of numerical convergence.* When $m = 1$, this state of numerical convergence coincides with that of Sibagaki.

Since $\lim_{n\to\infty} x_n = \bar{x}$ is a true solution of (1.1), from (2.5), it is evident that, when the numerical iteration is used for solution of (1.1), the most accurate values are offered by those of x^*_n in the state of numerical conervgence. Let us find the bound of errors of x^*_n in the state of numerical convergence. From (3.3), in the state of numerical convergence, it holds evidently that

$$x^*_{n+mi} = x^*_n \qquad (i = 1, 2, \cdots).$$

Then, from (2.5) follows

$$\| x^*_n - x_{n+mi} \| = \| x^*_{n+mi} - x_{n+mi} \| < \delta,$$

consequently, letting $i \to \infty$, we see that

$$\| x^*_n - \bar{x} \| \leqq \delta, \tag{3.4}$$

namely the bound of error of any x^*_n in the state of numerical convergence is given by δ.

When R is an Euclidean space, since U is bounded, U becomes totally bounded, consequently the above results are valid. The case where the algebraic or transcendetal equations are solved by the method of iteration falls under this case.

Example.

$$z = \frac{8}{9} e^{2\pi i/3} z.$$

Adopting $-0.444+0.770\,i$ as the approximate value of $\frac{8}{9} e^{2\pi i/3}$ and rounding off all numbers to three decimal places, let us compute the root of the above equation. If we take 0.010 as z_0, then the results of computation are as follows:

n	z_n^*	n	z_n^*	n	z_n^*
0	0.010	9	0.002	18	0.001−0.002 i
1	−0.004+0.008 i	10	−0.001+0.002 i	19	0.002+0.002 i
2	−0.004−0.007 i	11	−0.002−0.002 i	20	−0.003+0.001 i
3	0.007	12	0.003−0.001 i	21	−0.002 i
4	−0.003+0.005 i	13	+0.002 i	22	0.002+0.001 i
5	−0.003−0.004 i	14	−0.002−0.001 i	23	−0.002+0.002 i
6	0.004	15	0.002−0.002 i	**24**	**−0.001−0.003 i**
7	−0.002+0.003 i	16	0.001+0.003 i		
8	**−0.001−0.003 i**	17	−0.002		

Thus z_n^*'s in the state of numerical convergence are $z_8^*, z_9^*, \cdots, z_{23}^*$. In this example, $\varepsilon \approx \sqrt{2} \times 10^{-3}$ and $K \approx 8/9$, consequently $\delta \approx 9\sqrt{2} \times 10^{-3}$. The maximum of errors of $z_8^*, z_9^*, \cdots, z_{23}^*$ is $\sqrt{10} \times 10^{-3} = 3.2 \times 10^{-3}$, which is evidently less than δ.

This example is equivalent to solving the real simultaneous equations

$$\begin{cases} x = -\dfrac{4}{9}x - \dfrac{4\sqrt{3}}{9}y, \\ y = \dfrac{4\sqrt{3}}{9}x - \dfrac{4}{9}y, \end{cases}$$

assuming that the norm of the vector (x, y) is $\sqrt{x^2+y^2}$.

2° *The case where the relation* (3.1) *occurs.* In this case, the same conclusion as in the case 1° holds and the sequence $\{x_n^*\}$ oscillates taking m values.

When the functional equation (1.1) is to be solved in the set $F'(\subset F)$ consisting of the certain definite interpolation formulas — for example, polynomials of the definite degree or Fourier series consisting of only a definite finite number of terms —, $U' = U \cap F'$ becomes totally bounded because $U' \subset U$ is bounded. In such a case, since $x_n^* \in U'$, the same discussions as in the case 1° prevail and we see that the relation (3.1) surely occurs for a certain n and $m > 0$. Thus such a case falls under the present case.

3° *The case where the relation* (3.1) *never occurs for any n and $m > 0$.* In this case, from the discussion in the case 1°, it is evident that the sequence $\{x_n^*\}$ cannot oscillate nor converge, in other words the sequence $\{x_n^*\}$ diverges — contrary to the convergence of the true sequence $\{x_n\}$. Then, can we determine the solution from this divergent sequence?

Subtracting $\bar{x} = T\bar{x}$ from (2.6), we have:

$$\| x^*_{n+1} - \bar{x} \| \leqq K \| x^*_n - \bar{x} \| + \varepsilon.$$

Putting $n = N, N+1, \cdots, N+m-1$, from the above formula, we have:

$$\begin{aligned} \| x^*_{N+m} - \bar{x} \| &\leqq K \| x^*_{N+m-1} - \bar{x} \| + \varepsilon \\ &\leqq K^2 \| x^*_{N+m-2} - \bar{x} \| + \varepsilon (1 + K) \\ &\quad \cdots\cdots \\ &\leqq K^m \| x^*_N - \bar{x} \| + \varepsilon (1 + K + \cdots + K^{m-1}) \\ &\leqq K^m \| x^*_{N+m} - x^*_N \| + K^m \| x^*_{N+m} - \bar{x} \| + \varepsilon \cdot \frac{1-K^m}{1-K}. \end{aligned}$$

Consequently it follows that

$$\| x^*_{N+m} - \bar{x} \| \leqq \frac{K^m}{1-K^m} \| x^*_{N+m} - x^*_N \| + \delta. \tag{3.5}$$

Put

$$E_m = \frac{K^m}{1-K^m} \| x^*_{N+m} - x^*_N \| + \delta,$$

then

$$E_m \geqq \delta.$$

This expresses that δ is the least value of the bounds of errors for all m, in other words, the approximate solution corresponding to $E_m = \delta$ is best of the approximate solutions obtained by the method of iteration. This is also seen from (2.5) because $\lim_{n\to\infty} x_n = \bar{x}$.

Now, in the present case, $\|x^*_{N+m} - x^*_N\| = 0$ never occurs, consequently E_m can not become δ. But, if we take m sufficiently large, we can make $E_m - \delta$ as small as we desire. For, from (2.9) follows

$$E_m < \frac{K^m}{1-K^m} 2m\delta_1 + \delta$$

and $mK^m \to 0$ as $m \to \infty$. Then, for any preassigned small positive number e, we can determine M such that, for $m \geqq M$, $E_m < e + \delta = \delta_1$, namely we can determine the solution within the error $\delta_1 = \delta + e$ taking any one of x^*_{N+m}'s ($m = M, M+1, \cdots$) as the solution.

Thus we see that, although the sequence $\{x^*_n\}$ diverges, all x^*_{N+m}'s ($m = M, M+1, \cdots$)

lies in the δ_1-neighborhood of the true solution and that, for determination of the solution within the error δ_1, it is only necessary to compute x_n^* till x_{N+M}^* and the further computation of x_n^* is unnecessary. From this feature, we call the state of the sequence $\{x_n^*\}$ after x_{N+M}^* also *the state of numerical convergence* within the error δ_1.

Summarizing the above three cases, we have the conclusion:

The numerical iteration process necessarily attains the state of numerical convergence after finite times of repetition of the process. The approximate solutions are offered by any of x_n^*'s in the state of numerical convergence. The bound of errors of these approximate solutions is $\delta = \varepsilon/(1-K)$ or $\delta_1 = \delta + e$ according as the sequence $\{x_n^*\}$ oscillates or diverges, where e is any preassigned small positive number.

§ 4. The case where R is a real line

In the present case, the equation (1.1) is written as

$$x = \varphi(x). \tag{4.1}$$

Here we assume that $\varphi(x)$ is continuous with its derivative of the first order in the closed interval F. By §3, x_n^* in the state of numerical convergence lies in the interval I: $|x-\bar{x}|\leqq\delta$ where $\bar{x}\in F$ is a root of (4.1). On computation, we assume that the computation of $\varphi(x)$ is carried on so accurately that the bound ε of the error of the obtained value may be equal to that of rounding errors. Then, if the computation is carried on so minutely that ε may be sufficiently small, then the state of numerical convergence becomes an oscillation where x_n^* takes only two values at most. In the sequel, we shall prove this.

Let a and b be the rounded numbers nearest to $\bar{x}$ in such a way that $a\leqq \bar{x} < b$.

1° *The case where* $\varphi'(\bar{x}) = 0$. Since $\varphi'(x)$ is continuous, there exists a closed neighborhood F' of $\bar{x}$ such that $|\varphi'(x)| < 1/2$ in F'. If the computation is carried on so minutely that ε may be small and the interval $I[\bar{x}-\frac{\varepsilon}{1-K}, \bar{x}+\frac{\varepsilon}{1-K}]$ may be contained in F', then x_n^* in the state of numerical convergence lies in F'. In F', we may suppose that $K<1/2$, consequently, from (3.4) follows

$$|x_n^* - \bar{x}| \leqq \frac{\varepsilon}{1-K} < 2\varepsilon.$$

From this, it must be that $x_n^* = a$ or b, namely *the state of numerical convergence becomes an oscillation where x_n^* takes only two values at most.*

2° *The case where* $\varphi'(\bar{x}) > 0$. Since $\varphi'(x)$ is continuous, there exists a closed neighborhood F' of $\bar{x}$ such that $\varphi'(x) > 0$ in F'. If the computation is carried on so minutely that ε may be small and the interval $I[\bar{x} - \frac{\varepsilon}{1-K}, \bar{x} + \frac{\varepsilon}{1-K}]$ may be contained in F', then x_n^* in the state of numerical convergence lies in F'. Now, from mean value theorem follows

$$\varphi(x_n^*) - \bar{x} = \varphi'(\xi)(x_n^* - \bar{x})$$

where $\xi \in F'$. Also it is evident that

$$|\varphi(x_n^*) - \bar{x}| \leqq K|x_n^* - \bar{x}| < |x_n^* - \bar{x}|.$$

Consequently $\varphi(x_n^*)$ lies between x_n^* and $\bar{x}$. Then it must be that

$$x_n^* \leqq x_{n+1}^* \leqq b \quad \text{or} \quad a \leqq x_{n+1}^* \leqq x_n^* \tag{4.2}$$

according as $x_n^* \leqq a$ or $x_n^* \geqq b$.

Let us consider the case where $x_n^* \leqq a$. If $x_{n+1}^* = x_n^*$, then the sequence $\{x_n^*\}$ converges. Otherwise $x_n^* < x_{n+1}^* \leqq b$. Then, repeating this process, we see that the sequence $\{x_n^*\}$ converges or certain x_n^* becomes b after finite times of repetition. Now certain x_n^* can become b only when $a + \varepsilon < \bar{x} \leqq b$. Then, since $\bar{x} < \varphi(b) < b$, the rounded number of $\varphi(b)$ must be b, consequently the sequence $\{x_n^*\}$ must converge to b. Thus it is seen that the sequence $\{x_n^*\}$ always converges. Then, since initial $x_n^* \leqq a$ is assumed to be in the state of numerical convergence, the limiting value of the convergent sequence $\{x_n^*\}$ must be $x_n^* \leqq a$ itself, consequently it never happens that certain x_n^* becomes b.

The case where $x_n^* \geqq b$ is treated in the same way and the same conclusion is deduced.

Thus we see that, in the present case, *the numerical sequence $\{x_n^*\}$ always converges.*

3° *The case where* $\varphi'(\bar{x}) < 0$. As in the case 2°, if the computation is carried on sufficiently minutely, we may suppose that $\varphi'(x) < 0$ in the interval $I[\bar{x} - \frac{\varepsilon}{1-K}, \bar{x} + \frac{\varepsilon}{1-K}]$.

Suppose that x_n^* is in the state of numerical convergence. Let us consider the case where $x_n^* \leqq a$. From mean value theorem follows

$$\varphi(x_n^*) - \bar{x} = \varphi'(\xi)(x_n^* - \bar{x})$$

where $\varphi'(\xi) < 0$. Consequently $\varphi(x_n^*) \geqq \bar{x}$. Then, since

$$|\varphi(x_n^*) - \bar{x}| \leqq K|x_n^* - \bar{x}| < |x_n^* - \bar{x}|,$$

it is valid that

$$0 \leqq \varphi(x_n^*) - \bar{x} < \bar{x} - x_n^*. \tag{4.3}$$

Put

$$\eta = b - \bar{x} > 0$$

and

$$a - x_n^* = 2N\varepsilon,$$

then, from (4.3) follows

$$\varphi(x_n^*) - b < \bar{x} - x_n^* - \eta = 2(N+1)\varepsilon - 2\eta.$$

Consequently we have:

(i) when $\eta \leqq \varepsilon/2$, $\quad b \leqq x_{n+1}^* \leqq b + 2(N+1)\varepsilon$;

(ii) when $\varepsilon/2 < \eta \leqq \varepsilon$, $\quad b \leqq x_{n+1}^* \leqq b + 2N\varepsilon$;

(iii) when $\varepsilon < \eta \leqq \frac{3}{2}\varepsilon$, $\quad a \leqq x_{n+1}^* \leqq b + 2N\varepsilon$;

(iv) when $\frac{3}{2}\varepsilon < \eta \leqq 2\varepsilon$, $\quad a \leqq x_{n+1}^* \leqq b + 2(N-1)\varepsilon$.

When $x_n^* \geqq b$, the analogous results are obtained. Consequently, when $x_n^* \leqq a$, repeating the above process, we see that,

$$\begin{cases} \text{in the case (i), (ii) and (iii),} & x_n^* \leqq x_{n+2}^* \leqq b; \\ \text{in the case (iv),} & x_n^* \leqq x_{n+2}^* \leqq a. \end{cases} \tag{4.4}$$

If $x_{n+2}^* = x_n^*$, then the sequence $\{x_n^*\}$ oscillates taking at most two values. Otherwise $x_n^* < x_{n+2}^* \leqq b$. Consequently, after finite times of repetition of iteration, *the sequence $\{x_n^*\}$ oscillates taking at most two values or certain x_n^* becomes b.*

In the same way, it is shown that, *when $x_n^* \geqq b$, the sequence $\{x_n^*\}$ oscillates taking at most two values or certain x_n^* becomes a.*

Let us consider the case where certain x_n^* becomes a.

In the case (i), $\varphi^*(a) = b$ [1] or $b + 2\varepsilon$. If $\varphi^*(a) = b$, then, since $\varphi^*(b) = b$, the sequence $\{x_n^*\}$ must converge to b. Now, since $x_n^* = a$ is assumed to be in

1) $\varphi^* = T^*$, namely $\varphi^*(x_n^*) = x_{n+1}^*$

the state of numerical convergence, the limiting value of the convergent sequence $\{x_n^*\}$ must be a. This is a contradiction. Thus it must be that $\varphi^*(a) = b + 2\varepsilon$. Now $\varphi^*(b+2\varepsilon) = a$ or b. But, if $\varphi^*(b+2\varepsilon) = b$, then, since $\varphi^*(b) = b$, the sequence $\{x_n^*\}$ must converge to b. This is a contradiction as is pointed out just now. Therefore it must be that $\varphi^*(b+2\varepsilon) = a$. Thus we see that *the sequence* $\{x_n^*\}$ *oscillates taking two values* a *and* $b + 2\varepsilon$.

In the case (ii), $\varphi^*(a) = b$ and $\varphi^*(b) = b$ or a. But, if $\varphi^*(b) = b$, the sequence $\{x_n^*\}$ must converge to b. This is a contradiction as is pointed out in the above. Thus it must be that $\varphi^*(a) = b$ and $\varphi^*(b) = a$. This expresses that *the sequence* $\{x_n^*\}$ *oscillates taking two values* a *and* b.

In the case (iii), $\varphi^*(a) = a$ or b. When $\varphi^*(a) = a$, *the sequence* $\{x_n^*\}$ *converges.* When $\varphi^*(a) = b$, since $\varphi^*(b) = a$, *the sequence* $\{x_n^*\}$ *oscillates taking two values* a *and* b.

In the case (iv), $\varphi^*(a) = a$, consequently *the sequence* $\{x_n^*\}$ *converges.*

Thus, *when certain* x_n^* *becomes* a, *the sequence* $\{x_n^*\}$ *oscillates taking at most two values.* Likewise, *when certain* $\{x_n^*\}$ *becomes* b, *the sequence* $\{x_n^*\}$ *oscillates taking at most two values.*

Thus, in all cases, it is seen that *the sequence* $\{x_n^*\}$ *oscillates taking at most two values.*

Summarizing the above three cases, we have the conclusion :

When the computation is carried on so minutely that the bound ε *of errors may be sufficiently small and* $\varphi(x)$ *is computed so accurately that the error may not exceed* ε, *after finite times of repetition of iteration, the numerical sequence* $\{x_n^*\}$ *necessarily attains the state of numerical convergence where* x_n^* *takes at most two values.*

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