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Dimension Functions on Certain General Lattices

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Introduction

There have been developed in the literature two kinds of the theory of dimensionality in lattices, that is, one is the theory in the continuous geometries (von Neumann [13], Halperin [5], Iwamura [7]) and the other is the theory in the operator rings (Murray and von Neumann [12], Segal [15]) and AW*-algebras (Kaplansky [8], Sasaki [14]). The main purpose of the present paper is to investigate a dimensionality in certain general (not necessarily modular) lattices so that all the above cases may be treated from the uniform standpoint.

Before describing the outline of this paper, it is convenient to explain the above theories in some more details.

(I) In the theory of continuous geometries, von Neumann [13, I] introduced dimensionality by perspectivity. He constructed numerical dimension functions in the irreducible case, and in the reducible case Iwamura [6] introduced dimension functions whose ranges are sets of continuous functions on the Boolean space which represents the center Z of the lattice.

In continuous geometries, generalizing the idea of Halperin [5], Iwamura [7] introduced the concept of the p-relation which means a relation $a \sim b$, satisfying the following conditions:

(1) $a \sim b$ is an equivalence relation;

(2) if a and b are perspective, then $a \sim b$;

(3) (complete additivity) if $a_\alpha \sim b_\alpha$ for every $\alpha \in I$, then $\oplus a_\alpha \sim \oplus b_\alpha$, where $\oplus$ means the l. u. b. of an independent system in von Neumann's sense [13, I, p. 9];

(4) if $a \sim \oplus b_\alpha$, then there exists a decomposition $a = \oplus a_\alpha$ with $a_\alpha \sim b_\alpha$;

(5) (finiteness) there exists no pair of elements a, b such that $a \sim b < a$.

He defined the relative center with respect to the given p-relation as the set of all z such that $a \sim z$ implies $a = z$. A dimensionality is induced by a p-relation, and the relative center plays the rôle of the center Z.

(II) In the theory of operator rings, the dimensionality was introduced in the lattice of projections in an operator ring **M** with the unit operator as follows: two projections P, Q in **M** are said to have the same relative dimension if there exists a partially isometric operator in **M** whose initial and final projections are P and Q respectively.

Murray and von Neumann [12] constructed the theory of dimensionality in the case where **M** is a factor. In general case, Segal [15, p. 405] gave axioms of dimension functions and proved that every operator ring has a dimension function in his sense.

On the other hand Dixmier [4] proved the existence of a pseudo-application $\natural$ which is normal, faithful and essential in a semi-finite ring. The restriction of this application to projections is essentially a dimension function in Segal's sense, and its range is a set of continuous functions on the space Ω, the spectre of the center $\mathbf{M}^{\natural}$ of **M**.

The projections in an AW*-algebra form a complete lattice and the dimensionality was introduced there by the same way as in an operator ring (see kaplansky [8]).

In the case (I) the lattices are modular, but in the case (II) they are not modular in general, and in the later case there exists the concept of orthogonality, and the complete additivity of the relation which induces the dimensionality holds for orthogonal families, while in the former case it holds for independent systems.

In §1, we postulate a relation $a \perp b$ in a complete lattice L which may be interpreted as $a \frown b \doteq 0$ in a continuous geometry and as $ab = 0$ in the lattice of projections in a operator ring. Using this relation we define in L independent systems by the similar way as in the continuous geometries. We show that the center Z of L is a complete Boolean sublattice.

In §2, modifying the axioms of Iwamura's p-relation, we postulate a relation $a \sim b$ in L so that the dimensionality induced by this relation may be available to both of the cases (I), (II).

Here the relative center Z_0 is defined as the set of $z \in Z$ such that $a \sim b \leqq z$ implies $a \leqq z$, and we show that Z_0 is a complete Boolean sublattice of L. We develope the theory of dimensionality in L along the similar lines as in the cases (I), (II). First, in this section we give the reduction theorem.

In §3, we give the comparability theorem and the decomposition theorem which play a fundamental rôle in the sequel. §4 is devoted to the preliminaries to the rest of the paper.

In §5, we give the axioms of dimension functions on L. Let Ω be the Boolean space which represents Z_0 and let $\boldsymbol{Z}$ be the set of non-negative continuous functions on Ω. We denote by $\phi(z)$ the characteristic function of the subset of Ω which corresponds to $z \in Z_0$. A function d on L to $\boldsymbol{Z}$ will be termed a dimension function on L if the following axioms are satisfied:

(1°) if $a \sim b$, then $d(a) = d(b)$;

(2°) if $a \perp b$, then $d(a \cup b) = d(a) + d(b)$;

(3°) if $z \in Z_0$, then $d(z \frown a) = \phi(z)\, d(a)$;

(4°) if $a > 0$, then $d(a) > 0$;

(5°) if a is finite, then $d(a)$ is finite valued except on a set of the first category.

This definition is obtained by modifying that of Segal [15, p. 405]. We show the existence of such a dimension function which is unique in a certain sense (Theorem 5.1, 5.3). This is the main result of this paper. We discuss the further properties of the dimension functions: the complete additivity and the properties of their ranges.

In §6, to make clear the relation of our dimension functions to those of Segal and Dixmier (explained in (II)), we consider a function d^* satisfying the first three axioms (1°), (2°) and (3°), and show that (4°) implies the complete additivity of d^* and that (5°) corresponds to the essentiality in Dixmier's sense.

§ 1. Independent systems in a complete lattice

We assume that in a complete lattice there is a binary relation "$\perp$" which has the following properties:

(1, α) *$a \perp a$ implies $a = 0$;*

(1, β) *$a \perp b$ implies $b \perp a$;*

(1, γ) *$a \perp b$, $a_1 \leqq a$ imply $a_1 \perp b$;*

(1, δ) *$a \perp b$, $a \cup b \perp c$ imply $a \perp b \cup c$;*

(1, ε) *if $a_\delta \uparrow a$ and $a_\delta \perp b$ for all δ, then $a \perp b$.* ($a_\delta \uparrow a$ means that $\{a_\delta\}$ is an ascending directed set with the l. u. b. a.)

Then a subset S of the lattice is said to be an *independent system* (defined by the given relation "$\perp$"), in notation $(a\,;\, a \in S) \perp$, if $\bigvee(a\,;\, a \in S_1) \perp \bigvee(a\,;\, a \in S_2)$ for every pair of disjoint subsets S_1, S_2 of S. $a \cup b$ will be denoted by $a \oplus b$ if $a \perp b$, and $\bigvee(a\,;\, a \in S)$ will be denoted by $\oplus(a\,;\, a \in S)$ if S is an independent system.

Clearly, if S is an independent system, then every subset of S is also an independent system. Next we shall show some properties of the independent systems. (Theorem 1.1 and 1.2 can be proved without (1, α).)

THEOREM 1.1. *If every finite subset of S is an independent system, then so is S.*

PROOF. Let S_1, S_2 be two disjoint subsets of S. If F_i is a finite subset of $S_i (i=1, 2)$, then we have $\bigvee(a\,;a\in F_1)\perp\bigvee(a\,;a\in F_2)$ by our assumption. Therefore, applying (1, ε), we get $\bigvee(a\,;a\in S_1)\perp\bigvee(a\,;a\in S_2)$, which means that S is an independent system.

THEOREM 1.2. *If S_α is an independent system for every $\alpha\in I$ and $(\bigoplus(a\,;a\in S_\alpha)\,;\alpha\in I)\perp$, then $\bigcup(S_\alpha\,;\alpha\in I)$ is also an independent system.*

PROOF. We first prove the theorem in the case $I=\{1, 2\}$. Let T_1, T_2 be two disjoint subsets of $S_1\cup S_2$. Setting $a_{ij}=\bigvee(a\,;a\in T_i\cap S_j)$ $(i, j=1, 2)$, then by the assumption and (1, γ) we have $a_{1j}\perp a_{2j}$ $(j=1, 2)$ and $a_{11}\cup a_{21}\perp a_{12}\cup a_{22}$. It follows from (1, δ) that $a_{11}\perp a_{21}$, $a_{11}\cup a_{21}\perp a_{22}$ imply $a_{11}\perp a_{21}\cup a_{22}$ and that $a_{12}\perp a_{22}$, $a_{12}\cup a_{22}\perp a_{11}\cup a_{21}$ imply $a_{12}\perp a_{11}\cup a_{21}\cup a_{22}$. Hence, by (1, δ) again, we have $a_{11}\cup a_{12}\perp a_{21}\cup a_{22}$, i. e., $\bigvee(a\,;a\in T_1)\perp\bigvee(a\,;a\in T_2)$.

Next we treat the general case. Let F be arbitrary finite subset of $\bigcup(S_\alpha\,;\alpha\in I)$. Then there exists a finite subset $\{\alpha_i\}$ of I such that $F\subset\bigcup_i S_{\alpha_i}$. By the first paragraph of this proof $\bigcup_i S_{\alpha_i}$ is an independent system, then so is F. By Theorem 1.1, $\bigcup_\alpha S_\alpha$ is an independent system, completing the proof.

In a continuous geometry (the case (I) in the introduction) we define $a\perp b$ if $a\cap b=0$. Then (1, α)—(1, γ) are evidently true, and (1, δ) and (1, ε) are true since the lattice is modular (see [13, I], Theorem 1.2) and upper-continuous. In this case the concept of the independent system coincides with that of the independent system in von Neumann's sense [13, I, p. 9].

In a complete lattice formed by all projections in an AW*-algebra (the case (II) in the introduction) we define $a\perp b$ if a and b are orthogonal. Then (1, α)—(1, γ) are evidently true and (1, δ), (1, ε) are true since every element has a unique orthocomplement. And then the concept of the independent system coincides with that of the orthogonal family.

Moreover, in both cases, the following condition is satisfied:

(1, ζ) *if $a\leqq b$, then there exists $c\in L$ such that $a\perp c$, $a\cup c=b$, i. e. $a\oplus c=b$.*

From now on, if nothing in particular is said, L is always a complete lattice having a binary relation "$\perp$" with the six properties (1, α)—(1, ζ).

We know that the center of a lattice with 0, 1 is a Boolean sublattice (see

Birkhoff [1, pp. 27–29]). Now we shall prove that the center Z of L is a complete sublattice of L.

We shall write $(a, b)M$ if $(c \cup a) \cap b = c \cup (a \cap b)$ for every $c \leqq b$.

LEMMA 1.1. *$a \perp b$ implies $a \cap b = 0$ and $(a, b)M$.*

PROOF. By $(1, \gamma)$, $(1, \beta)$ it follows from $a \perp b$ that $a \cap b \perp a \cap b$. Thus $a \cap b = 0$ by $(1, \alpha)$. Next let $c \leqq b$. Since $(c \cup a) \cap b \geqq c \cup (a \cap b) = c$, we may write $(c \cup a) \cap b = c \oplus d$ by $(1, \zeta)$. Since $c \cup d \leqq b \perp a$ we have $d \perp c \cup a$ by $(1, \delta)$. But $d \leqq c \cup a$, therefore $d = 0$ which shows $(a, b)M$.

Next we shall prove the following result, which is known both in the continuous geometries (von Neumann [13, I], Theorem 5.3) and in the operator rings (Maeda [9], Theorem 1).

LEMMA 1.2. *$z \in L$ is in Z if and only if z has a unique complement.*

PROOF. The "only if" part is trivial. To prove the converse, assume that z has a unique complement z'. It suffices to show that the correspondence $x \to [z \cup x, z' \cap x]$ is an isomorphism between L and the product of the sublattices $L(0, z) = \{x \in L ; x \leqq z\}$ and $L(0, z')$. We shall first prove that $a \cup b$ corresponds to $[a, b]$ for every $a \leqq z$, $b \leqq z'$. Since the complement of z is unique, necessarily $z \perp z'$ holds by $(1, \zeta)$. Hence we have $b \perp z$, therefore by Lemma 1.1 $(a \cup b) \cap z = a$. Similarly $(a \cup b) \cap z' = b$.

Next we shall prove the correspondence is one-to-one, that is, $x = (z \cap x) \cup (z' \cap x)$. Put $x = (z \cap x) \oplus a$. Then $z \cap a = z \cap x \cap a = 0$. To prove that $a \leqq z'$, we put $(z \cup a) \oplus b = 1$. Then $(a \cup b) \cap (z \cup a) = a$ by Lemma 1.1, so $z \cap (a \cup b) = z \cap (z \cup a) \cap (a \cup b) = z \cap a = 0$. Thus $a \cup b$ is a complement of z, so that $a \cup b = z'$, therefore $a \leqq z'$. Hence $x = (z \cap x) \cup a \leqq (z \cap x) \cup (z' \cap x) \leqq x$. Since the correspondence is clearly order-preserving, it is an isomorphism. This completes the proof.

THEOREM 1.3. *Z is a complete sublattice of L.*

PROOF. As $z \in Z$ has a unique complement, we denote it by $1 - z$; clearly $1 - z \in Z$. Let $z_\delta \uparrow a$, $z_\delta \in Z$ and let a' be a complement of a. Since $a' \cap z_\delta = 0$ implies $a' \leqq 1 - z_\delta$ for every δ, we put $a' \oplus b = \bigwedge_\delta (1 - z_\delta)$. But $z_{\delta'} \cap \bigwedge_\delta (1 - z_\delta) = 0$ implies $z_{\delta'} \perp \bigwedge_\delta (1 - z_\delta)$, and hence by $(1, \varepsilon)$ we have $a \perp \bigwedge_\delta (1 - z_\delta)$. By $(1, \delta)$ we have $b \perp a \cup a' = 1$, so $b = 0$. Thus $a' = \bigwedge_\delta (1 - z_\delta)$ which shows that a' is uniquely determined, therefore $a \in Z$ by Lemma 1.2.

If $z_\delta \downarrow a$, $z_\delta \in Z$, then $\{1 - z_\delta\}$ is an ascending set. Then $z = \bigvee_\delta (1 - z_\delta)$ is in Z and the complement $1 - z$ is equal to $\bigwedge_\delta z_\delta$. Hence $a = \bigwedge_\delta z_\delta = 1 - z \in Z$. Thus the theorem is proved.

For later use we prove the following lemma:

LEMMA 1.3. *Let* $a_\delta \uparrow a$. *If* $b \in Z$ *or* $a_\delta \in Z$ *for all* δ, *then* $a_\delta \cap b \uparrow a \cap b$.

PROOF. Let $a \cap b = \bigvee_\delta (a_\delta \cap b) \oplus c$ and $b \oplus b' = 1$. We have $c \oplus (a_\delta \cap b) \perp a_\delta \cap b'$ since $b \perp b'$, and then $c \perp (a_\delta \cap b) \oplus (a_\delta \cap b') = a_\delta$. By $(1, \varepsilon)$ we have $c \perp a$, so $c = 0$.

REMARK 1.1. (i) *L is relatively complemented.* Proof: let $a \leqq x \leqq b$ and $x \oplus y = b$. $(a \cup y) \cup x = b$ holds obviously, and $a \leqq x \perp y$ imply $(a \cup y) \cap x = a$ by Lemma 1.1. Therefore $a \cup y$ is a relative complement of x in the closed interval $[a, b]$.

(ii) *If S is an independent system,* $\{\bigvee(a\,;\, a \in M)\,;\, M \subset S\}$ *form a Boolean sublattice of L.* Proof: put $a(M) = \bigvee(a\,;\, a \in M)$. For arbitrary M_1, $M_2 \subset S$, $a(M_1) \cup a(M_2) = a(M_1 \cup M_2)$ holds obviously. We have $a(M_1 - M_2) \perp a(M_2)$, whence $a(M_1) \cap a(M_2) = \{a(M_1 \cap M_2) \cup a(M_1 - M_2)\} \cap a(M_2) = a(M_1 \cap M_2)$ by Lemma 1.1. Therefore the set $\{\bigvee(a\,;\, a \in M)\,;\, M \subset S\}$ is a sublattice of L and is lattice-isomorphic to $\{M\,;\, M \subset S\}$, which is a Boolean lattice.

REMARK 1.2. In a lattice with 0,1 (not necessarily complete) we assume that there is a binary relation "$\perp$" having the properties $(1, \alpha)$—$(1, \zeta)$ except $(1, \varepsilon)$. Then we define an independent system for a *finite* subset of the lattice by the same way as before. In this case Lemma 1.1, 1.2 and Remark 1.1 are also valid.

For example, in an orthocomplemented lattice we define $a \perp b$ by $a \leqq b^\perp$ where $b^\perp$ is the orthocomplement of b. Then it is easy to show that $(1, \alpha)$—$(1, \delta)$ hold. $(1, \zeta)$ holds if and only if $(a, a^\perp)M$ for all a. Proof: the "only if" part is true by Lemma 1.1. Conversely if $(a, a^\perp)M$ and $a \leqq b$, then $b^\perp \leqq a^\perp$, so $(b^\perp \cup a) \cap a^\perp = b^\perp \cup (a \cap a^\perp) = b^\perp$. Hence setting $c = b \cap a^\perp$ we have $a \oplus c = b$.

§ 2. Relation "$\sim$" in L

Let L be a complete lattice having a relation "$\perp$" with the properties $(1, \alpha)$—$(1, \zeta)$, and Z be the center of L. By Theorem 1.3, Z is a complete sublattice of L.

LEMMA 2.1. *Let $a \equiv b$ be an equivalence relation in L (i. e., it is reflexive, symmetric and transitive) and have the following property: if $0 \neq a_1 \leqq a \equiv b$, then there exists $b_1 \neq 0$ such that $a_1 \equiv b_1 \leqq b$. Then the set $Z' = \{z \in Z\,;\, a \equiv b \leqq z$ implies $a \leqq z\}$ is a Boolean lattice with 0,1 of L, and is a complete sublattice of L.*

PROOF. It is obvious that $1 \in Z'$ and that if $z_\alpha \in Z'$ for all α then $\bigwedge_\alpha z_\alpha \in Z'$. To complete the proof, it suffices to show that if $z \in Z'$ then $1 - z \in Z'$. Let

$a \equiv b \leqq 1-z$. If $z \cap a \neq 0$ then there exists $b_1 \neq 0$ such that $z \cap a \equiv b_1 \leqq b \leqq 1-z$. But $z \in Z'$ implies $b_1 \leqq z$. Hence $b_1 = 0$, a contradiction. So we must have $z \cap a = 0$, whence $a \leqq 1-z$. Therefore $1-z \in Z'$, and the lemma is proved.

Now we shall postulate a relation which induces a dimensionality in L.

We assume that in L there is a binary ralation "$\sim$" which has the following six properties $(2, \alpha)$—$(2, \zeta)$.

$(2, \alpha)$ *$a \sim b$ is an equivalence relation;*

$(2, \beta)$ *$a \sim 0$ implies $a = 0$;*

$(2, \gamma)$ *if $a \sim b_1 \oplus b_2$, then there exists a decomposition $a = a_1 \oplus a_2$ with $a_i \sim b_i$ $(i = 1, 2)$;*

$(2, \delta)$ *if $(a_\alpha; \alpha \in I) \perp$, $(b_\alpha; \alpha \in I) \perp$ and $a_\alpha \sim b_\alpha$ for all $\alpha \in I$, then $\oplus (a_\alpha; \alpha \in I) \sim \oplus (b_\alpha; \alpha \in I)$.*

Before describing the property $(2, \varepsilon)$, we shall define the relative center with respect to the given relation "$\sim$". Let Z_0 be the set of $z \in Z$ such that $a \sim b \leqq z$ implies $a \leqq z$. Then using the first three properties $(2, \alpha)$—$(2, \gamma)$, it follows from Lemma 2.1 that Z_0 is a Boolean lattice with 0,1 of L and that it is a complete sublattice of L. Z_0 will be called the *relative center* with respect to the given relation "$\sim$". Since Z_0 is complete, for any $a \in L$ there is the smallest element $z \in Z_0$ such that $a \leqq z$. We shall denote it by $e(a)$.

$(2, \varepsilon)$ *If $e(a) \cap e(b) \neq 0$, then there exist a_1, b_1 such that $0 \neq a_1 \leqq a$, $0 \neq b_1 \leqq b$, $a_1 \sim b_1$;*

$(2, \zeta)$ *if $a_1 \oplus a_2 = b_1 \oplus b_2$, $a_1 \sim a_2$, $b_1 \sim b_2$, then $a_1 \sim b_1$.*

Using these postulates $(2, \alpha)$—$(2, \zeta)$, we shall develope the theory of dimensionality in L along the similar lines as in the continuous geometries and in the operator rings.

EXAMPLE 1. Let L be a continuous geometry. In §1 we defined $a \perp b$ if $a \cap b = 0$ and showed that $(1, \alpha)$—$(1, \zeta)$ are valid. We assume that in L there is a relation $a \sim b$ which has the following properties:

(1) *$a \sim b$ is an equivalence relation;*

(2) *if a and b are perspective, then $a \sim b$;*

(3) *if $a_\alpha \sim b_\alpha$ for every $\alpha \in I$, then $\oplus a_\alpha \sim \oplus b_\alpha$;*

(4) *if $a \sim \oplus b_\alpha$, then there exists a decomposition $a = \oplus a_\alpha$ with $a_\alpha \sim b_\alpha$;*

(5) *$a \sim 0$ implies $a = 0$.*

If we add the finiteness, this relation coincides with Iwamura's p-relation (see (I) in the introduction), and it follows from [7], Lemma 4.1, 4.2 that Z_0 coincides with the relative center in his sense.

This relation $a \sim b$ includes also Halperin's relation "$\equiv$" which is defined as follows: $a \equiv b$ if there exist decompositions $a = \oplus(a_\alpha ; \alpha \in I)$, $b = \oplus(b_\alpha ; \alpha \in I)$ such that, for every $\alpha \in I$, b_α is perspective to $T_\alpha a_\alpha$ for some $T_\alpha \in G$, where G is a group of lattice-automorphisms of L. For, if we define $a \sim b$ by $a \equiv b$, then (2), (5) are clearly hold, and by [5], Theorem 4.1, 4.2 and [10], Lemma 1.9 (1), (3), (4) hold also. And it is easy to show that

$$Z_0 = \{z \in Z ;\ Tz = z \text{ for all } T \in G\}.$$

Now we shall show that this relation $a \sim b$ has all the properties (2, α)—(2, ζ). (2, α)—(2, δ) clearly hold. We shall show that (2, ε) holds. For arbitrary $a \in L$, let $M(a) = \{x \in L ; a_1 \leqq a,\ b_1 \leqq x,\ a_1 \sim b_1 \text{ imply } a_1 = b_1 = 0\}$ and $a^* = \bigvee\{x ;\ x \in M(a)\}$. Then by the same method as in the proof of [5], Lemma 4.1 we can show that $a^* \in M(a)$, and the details are omitted. Let a', a'' be complements of a^*. Put $a'' = (a' \cap a'') \oplus c_1$, $c_2 = (a' \cup a'') \cap a^*$. Then $c_1 \cap a' = 0 = c_2 \cap a'$, $c_2 \cup a' = (a' \cup a'') \cap (a^* \cup a') = a' \cup a'' = c_1 \cup a'$. By (2) $c_1 \sim c_2 \leqq a^* \in M(a)$, hence we have $c_1 \in M(a)$, so $c_1 \leqq a^*$. But $c_1 \leqq a''$, so $c_1 = 0$. Therefore $a' = a''$. By Lemma 1.2 we have $a^* \in Z$. It is easily seen that $a^* \in Z_0$. Since $a^* \cap a = 0$, we have $a \leqq 1 - a^*$. If $b \in M(a)$, then $b \leqq a^*$ and hence $e(a) \cap e(b) = 0$. This shows that (2, ε) holds. We shall examine (2, ζ) later (in § 3).

EXAMPLE 2. Let L be a complete lattice formed by all projections in an AW*-algebra A. In §1 we defined $a \perp b$ if $ab = 0$ and showed that (1, α)—(1, ζ) are valid. We define $a \sim b$ if there is $x \in A$ such that $xx^* = a$, $x^*x = b$ (see (II) in the introduction). This relation $a \sim b$ satisfies (2, α)—(2, γ) clearly and (2, δ) by Kaplansky [8], Theorem 5.5. The center Z of L is the set of all projections in the center of A. Hence if $z \in Z$ and $xx^*z = xx^*$, then $x^*xz = x^*x$, so $Z_0 = Z$. Next it is easy to show that $e(a) \cap e(b) \neq 0$ if and only if $aAb \neq 0$. Hence (2, ε) is satisfied by [8], Lemma 3.3. We shall examine (2, ζ) later.

We note that in both examples the relations $a \sim b$ have moreover the following properties: *if* $a \sim \oplus b_\alpha$, *then there exists a decomposition* $a = \oplus a_\alpha$ *with* $a_\alpha \sim b_\alpha$ *(*$\{\alpha\}$ *is infinite)*; *if* a *and* b *are perspective then* $a \sim b$ *(equivalently if* $a \cup b = a \oplus c$, $b = (a \cap b) \oplus d$ *then* $c \sim d$*)*.

In the remainder of this section and in § 3 we assume that the relation $a \sim b$ has the first five properties (2, α)—(2, ε), and now deduce the reduction theorem.

LEMMA 2.2. (i) $a\sim b$ *implies* $e(a)=e(b)$.

(ii) $e(\bigvee_\alpha a_\alpha)=\bigvee_\alpha e(a_\alpha)$.

(iii) *If* $z\in Z_0$, *then* $e(z\cap a)=z\cap e(a)$.

PROOF. (i). This is obvious by the definition of $e(a)$.

(ii). We have $e(a_\alpha)\leqq e(\bigvee_\alpha a_\alpha)$, so $\bigvee_\alpha e(a_\alpha)\leqq e(\bigvee_\alpha a_\alpha)$. On the other hand $\bigvee_\alpha a_\alpha\leqq\bigvee_\alpha e(a_\alpha)\in Z_0$, so $e(\bigvee_\alpha a_\alpha)\leqq\bigvee_\alpha e(a_\alpha)$.

(iii). Using (ii), $e(z\cap a)\cup e((1-z)\cap a)=e(a)=z\cap e(a)\oplus(1-z)\cap e(a)$. Clearly $e(z\cap a)\leqq z\cap e(a)$, $e((1-z)\cap a)\leqq(1-z)\cap e(a)$, therefore $e(z\cap a)=z\cap e(a)$.

As in the continuous geometries, we shall write $a\prec b$ if there exists b_1 such that $a\sim b_1<b$, and write $a\ll b$ if for any $z\in Z_0$ either $z\cap a\prec z\cap b$ or $z\cap a=z\cap b=0$.

Clearly $a\ll b$ implies either $a\prec b$ or $a=b=0$, and $a\prec b$ implies $e(a)\leqq e(b)$ from Lemma 2.2 (i). Moreover, we obtain:

LEMMA 2.3. (i) *If* $z\in Z_0$ *and* $a\sim b$, *then* $z\cap a\sim z\cap b$. *From this* $a\prec b$ *implies* $z\cap a\precsim z\cap b$, $a\ll b$ *implies* $z\cap a\ll z\cap b$.

(ii) $a\sim b$, $b\prec c$, $c\sim d$ *imply* $a\prec d$; $a\prec b$, $b\prec c$ *imply* $a\prec c$; $a\precsim b$, $b\ll c$, $c\precsim d$ *imply* $a\ll d$.

(iii) $a\succsim b$, $a\precsim b$ *imply* $a\sim b$.

(iv) *if* $a\oplus b=c$ *and* $e(b)=e(c)$, *then* $a\ll c$.

PROOF. (i). By (2, γ) $a=z\cap a\oplus(1-z)\cap a$ implies that there exist b_1, b_2 such that $b=b_1\oplus b_2$, $b_1\sim z\cap a$, $b_2\sim(1-z)\cap a$. As $z\in Z_0$ we have $b_1\leqq z\cap b$, $b_2\leqq(1-z)\cap b$. But $z\cap b\oplus(1-z)\cap b=b$, so $b_1=z\cap b$, hence $z\cap a\sim z\cap b$.

(ii). The first and second statements are obvious by (2, γ), (2, β), and hence, using (i), the third one is easily proved.

(iii). Using (2, γ), (2, δ), the statement (iii) can be proved by the same method as in the proof of Murray and von Neumann [12], Lemma 6.1.3.

(iv). Let $z\in Z_0$. In case $z\cap b\neq 0$, we have $z\cap a<z\cap c$. In case $z\cap b=0$, we have $z\cap e(c)=z\cap e(b)=e(z\cap b)=0$, whence $z\cap a=z\cap c=0$.

DEFINITION 2.1. $a\in L$ is called *infinite* if $a\prec a$ holds, and otherwise *finite*.

By Lemma 2,3 (ii) and (2, δ), if a is finite and $a\succsim b$, then b is finite.

LEMMA 2.4. *Let* $z_\alpha\in Z_0\,(\alpha\in I)$. *If* $z_\alpha\cap a$ *is finite for every* $\alpha\in I$, *then so is* $\bigvee(z_\alpha;\,\alpha\in I)\cap a$.

PROOF. Let $a_0=\bigvee_\alpha z_\alpha\cap a$, $a_0\sim b\leqq a_0$. By Lemma 2.3 (i) we have $z_\alpha\cap a_0\sim z_\alpha\cap b\leqq z_\alpha\cap a_0$. As $z_\alpha\cap a_0=z_\alpha\cap a$ is finite, this gives $z_\alpha\cap a_0=z_\alpha\cap b$. Thus by Lemma 1.3 $a_0=\bigvee_\alpha z_\alpha\cap a_0=\bigvee_\alpha(z_\alpha\cap a_0)=\bigvee_\alpha(z_\alpha\cap b)=b$, whence a_0 is finite.

DEFINITION 2.2. $a\in L$ is called *properly infinite* if $a\neq 0$ and $a\ll a$ holds, that is, for any $z \in Z_0$, $z \cap a$ is infinite or zero.

THEOREM 2.1. *For any* $a \in L$ *there exist* $e^f(a)$, $e^i(a) \in Z_0$ *which have the following properties:*

(1) $e^f(a) \oplus e^i(a) = e(a)$;

(2) $e^f(a) \cap a$ *is finite;*

(3) *if* $e^i(a) \neq 0$ *then* $e^i(a) \cap a$ *is properly infinite.*

Then $e^f(a)$, $e^i(a)$ *are uniquely determined.*

PROOF. Let $e^f(a) = \bigvee(z \in Z_0 ; z \leqq e(a),\ z \cap a$ is finite), $e^i(a) = e(a) - e^f(a)$. By Lemma 2.4 it is clear that $e^f(a) \cap a$ is finite. If $e^i(a) \neq 0$, then $e(e^i(a) \cap a) = e^i(a)$ implies $e^i(a) \cap a \neq 0$. Assume that $z \cap e_i(a) \cap a$ is finite for $z \in Z_0$. Then $z \cap e^i(a) \leqq e^f(a)$, so $z \cap e^i(a) = 0$. Hence $e^i(a) \cap a$ is properly infinite. Next let e_1^f, $e_1^i \in Z_0$ have also the properties (1), (2) and (3). Since $e_1^f \cap e^i \cap a$ is finite by (2), we have $e_1^f \cap e^i \cap a = 0$ by (3), whence $e_1^f \cap e^i = e(e_1^f \cap e^i \cap a) = 0$. Hence $e_1^f \leqq e^f$. Similarly $e^f \leqq e_1^f$, so $e^f = e_1^f$ and $e^i = e_1^i$.

The similar result of this theorem is known both in the continuous geometries (Maeda [10], Lemma 1.13) and in the operator rings (Dixmier [3], Lemma 1.2).

As in the continuous geometies (von Neumann [13, III], Definition 3.1) we define :

DEFINITION 2.3. $a \in L$ is called *minimal* if $a \neq 0$ and $x \ll a$ implies $x = 0$.

By Lemma 2.3 (ii) it is clear that if a is minimal and $a \gtrsim b \neq 0$, then b is also minimal.

LEMMA 2.5. *Let* $0 \neq a \in L$. a *is minimal if and only if* $a_1 \oplus a_2 \leqq a$, $a_1 \sim a_2$ *imply* $a_1 = a_2 = 0$.

PROOF. Let a be minimal and $a_1 \oplus a_2 \leqq a$, $a_1 \sim a_2$. $e(a_1) = e(a_2) = e(a_1 \oplus a_2)$ implies $a_1 \ll a_1 \oplus a_2 \leqq a$ by Lemma 2.3 (iv). Hence $a_1 = 0$. We shall prove the converse. If $b \ll a$, then there exists b_1 such that $b \sim b_1 \leqq a$. Setting $a = b_1 \oplus c$, we have $e(b_1) \cap e(c) = 0$ by our assumption and (2, $\mathcal{E}$). Hence $e(b_1) \cap a = e(b_1) \cap b_1$. But we have $b_1 \ll a$, therefore $b_1 = e(b_1) \cap b_1 = 0$, so $b = 0$. This completes the proof.

This lemma shows that the concept of "minimal" is equivalent to that of "irreducible" in Dixmier [2], [3]. Furthermore it is easily seen that this is equivalent to the concept of "non-zero abelian" in Kaplansky [8].

LEMMA 2.6. (i) *Every minimal element is finite.*

(ii) *If* a_α *is minimal for every* $\alpha \in I$ *and* $(e(a_\alpha) ; \alpha \in I) \perp$, *then* $a = \oplus(a_\alpha ; \alpha \in I)$ *is also minimal.*

PROOF. (i). Let a be minimal. By Theorem 2.1 we have $e^i(a)\cap a \ll e^i(a)\cap a \leqq a$, whence $e^i(a)\cap a=0$. Therefore $a=e^f(a)\cap a$ is finite.

(ii). Let $b\ll a$. By Lemma 2.3 (i) we have $e(a_\alpha)\cap b \ll e(a_\alpha)\cap a = a_\alpha$. This gives $e(a_\alpha)\cap b=0$ since a_α is minimal. Therefore $b=e(a)\cap b=\oplus_\alpha(e(a_\alpha)\cap b) =0$.

DEFINITION 2.4. L is said to be of *type I* if it has a minimal element a such that $e(a)=1$; *type II* if it has no minimal elements and has a finite element b such that $e(b)=1$; *type III* if all non-zero elements are infinite. L is said to be of *type* $I_{(1)}$ (resp. $II_{(1)}$) if it is of type I (resp. II) and finite (i. e. 1 is finite); *type* $I_{(\infty)}$ (resp. $II_{(\infty)}$) if it is of type I (resp. II) and properly infinite.

Then the reduction theorem is obtained as follows:

THEOREM 2.2. (*Reduction theorem*). *There exists a unique decomposition* $1=z_{\mathrm{I}}\oplus z_{\mathrm{II}} \oplus z_{\mathrm{III}}$, *where* z_{I}, z_{II}, $z_{\mathrm{III}}\in Z_0$, *such that the sublattice* $L(0, z_{\mathrm{I}})$ (*resp.* $L(0, z_{\mathrm{II}})$, $L(0, z_{\mathrm{III}})$) *is of type I* (*resp. type II, type III*).

PROOF. Let $z_{\mathrm{I}}=\bigvee(e(a)\,;\,a$ is minimal$)$. Since Z_0 is a complete Boolean lattice, we may write $z_{\mathrm{I}}=\oplus(z_\alpha\,;\,\alpha\in I)$ where $z_\alpha=e(a_\alpha)$ and a_α is minimal. We see that $a_0=\oplus(a_\alpha\,;\,\alpha\in I)$ is minimal by Lemma 2.6 (ii), and that $e(a_0)=z_{\mathrm{I}}$. Next let $z^*=\bigvee(e(b)\,;\,b$ is finite$)$. As above, using Lemma 2.4, we have $z^*=e(b_0)$, where b_0 is finite. By Lemma 2.6 (i) we have $z^*\geqq z_{\mathrm{I}}$, so we put $z_{\mathrm{II}}=z^*-z_{\mathrm{I}}$, $z_{\mathrm{III}}=1-z^*$. Then evidently $L(0, z_{\mathrm{I}})$ (resp. $L(0, z_{\mathrm{II}})$, $L(0, z_{\mathrm{III}})$) is of type I (resp. type II, type III). The uniqueness of the decomposition is trivial.

From Theorem 2.1 we have $e^f(1)\oplus e^i(1)=1$ and clearly $z_{\mathrm{III}}\cap e^f(1)=0$. Hence we have a decomposition $1=z_{\mathrm{I}}^{(1)}\oplus z_{\mathrm{I}}^{(\infty)}\oplus z_{\mathrm{II}}^{(1)}\oplus z_{\mathrm{II}}^{(\infty)}\oplus z_{\mathrm{III}}$, where $z_{\mathrm{I}}^{(1)}= z_{\mathrm{I}}\cap e^f(1)$, $z_{\mathrm{I}}^{(\infty)}=z_{\mathrm{I}}\cap e^i(1)$, $z_{\mathrm{II}}^{(1)}=z_{\mathrm{II}}\cap e^f(1)$, $z_{\mathrm{II}}^{(\infty)}=z_{\mathrm{II}}\cap e^i(1)$. Thus L *is uniquely expressible as a product of five lattices, respectively of type* $I_{(1)}$, $I_{(\infty)}$, $II_{(1)}$. $II_{(\infty)}$ *and III.*

LEMMA 2.7. *If* $0\neq a\leqq z_{\mathrm{II}}$, *then for any natural number* n *there exists a decomposition* $a=\oplus(a_i\,;\,1\leqq i\leqq n)$ *such that* $a_i\sim a_1$ *for all* i.

PROOF. Consider the set $\mathfrak{S}$ of independent systems $[b_1, \cdots, b_n]$ such that $0\neq b_i\leqq a$, $b_i\sim b_1$. There exists a maximal subset $\{[b_1^{(\alpha)}, \ldots, b_n^{(\alpha)}]\,;\,\alpha\in I\}$ of $\mathfrak{S}$ such that $(\oplus(b_i^{(\alpha)}\,;\,1\leqq i\leqq n)\,;\,\alpha\in I)\perp$, by Zorn's lemma. If $b=\oplus(b_i^{(\alpha)};\,1\leqq i\leqq n, \alpha\in I)<a$, we put $a=b\oplus c$. Then since z_{II} includes no minimal element, applying Lemma 2.5 repeatedly, we get an independent system $[c_1, \cdots, c_n]$ such that $0\neq c_i\leqq c$, $c_i\sim c_1$, which is a contradiction. Hence $b=a$. Setting $a_i=\oplus(b_i^{(\alpha)}\,;\,\alpha\in I)$, we have $\oplus(a_i\,;\,1\leqq i\leqq n)=a$ and $a_i\sim a_1$ by (2, δ).

§ 3. Relation "$\sim$" in L, continued

In this § we shall prove the comparability theorem and the decomposition theorem which are known in the continuous geometries (von Neumann [13, III], Theorem 2.1, 2.7, Maeda [11, Ch. 4], Thorem 1.1, 1.2), and in the operator rings (Maeda [9], Theorem III, Dixmier [2], Theorem 6).

THEOREM 3.1. (*Comparability theorem*). *Let a and b be arbitrary elements in L. There exist decompositions $a=a'\oplus a''$, $b=b'\oplus b''$ such that $a'\sim b'$, $e(a'')\frown e(b'')=0$; then $e(a')=e(b')=e(a)\frown e(b)$.*

PROOF. Consider pairs of independent systems $\{a_\alpha\}$, $\{b_\alpha\}$ such that $a_\alpha\leqq a$, $b_\alpha\leqq b$, $a_\alpha\sim b_\alpha$ for all α. Among these there exists a maximal pair $\{a_\alpha;\ \alpha\in I\}$, $\{b_\alpha;\ \alpha\in I\}$ by Zorn's lemma, and we put $a'=\oplus(a_\alpha;\ \alpha\in I)$, $b'=\oplus(b_\alpha;\ \alpha\in I)$, $a=a'\oplus a''$, $b=b'\oplus b''$. Then $a'\sim b'$ by (2, δ). If $e(a'')\frown e(b'')\neq 0$, then by (2, ε) the pair can be enlarged, a contradiction. So $e(a'')\frown e(b'')=0$. Clearly $e(a')=e(b')$, whence $e(a)\frown e(b)=e(a')\smile(e(a'')\frown e(b''))=e(a')$.

THEOREM 3.2. (*Decomposition theorem*). *Let a and b be arbitrary elements in L. There exist z_1, z_2, $z_3\in Z_0$ such that $z_1\oplus z_2\oplus z_3=1$ and $z_1\frown a\gg z_1\frown b$, $z_2\frown a\ll z_2\frown b$, $z_3\frown a\sim z_3\frown b$. More simply, there exists $z\in Z_0$ such that $z\frown a\gtrsim z\frown b$, $(1-z)\frown a\lesssim(1-z)\frown b$.*

PROOF. From Theorem 3.1 we get decompositions $a=a'\oplus a''$, $b=b'\oplus b''$, such that $a'\sim b'$, $e(a'')\frown e(b'')=0$. Put $z_1=e(a'')$, $z_2=e(b'')$, $z_3=1-(z_1\oplus z_2)$. We have $z_3\frown a''=z_3\frown z_1\frown a''=0$, whence $z_3\frown a=z_3\frown a'$. Similarly $z_3\frown b=z_3\frown b'$. Hence $a'\sim b'$ implies $z_3\frown a\sim z_3\frown b$. Next $z_1\frown a=z_1\frown a'\oplus a''$, $z_1\frown b=z_1\frown b'$. Hence for any $z\in Z_0$, $z\frown a''\neq 0$ implies $z\frown z_1\frown a\succ z\frown z_1\frown b$, and $z\frown a''=0$ implies $z\frown z_1=e(z\frown a'')=0$. Therefore $z_1\frown a\gg z_1\frown b$. Similarly $z_2\frown a\ll z_2\frown b$. This completes the proof.

The following lemmas are consequences of these theorems and are useful in the following sections.

LEMMA 3.1. *Let $a_1\oplus a_2\sim b_1\oplus b_2$.* If *$a_1\oplus a_2$ is finite and $a_1\sim b_1$, then $a_2\sim b_2$.*

PROOF. By (2, γ) we may suppose that $a_1\oplus a_2=b_1\oplus b_2=a$. From Theorem 3.2 there is $z\in Z_0$ such that $z\frown a_2\gtrsim z\frown b_2$, $(1-z)\frown a_2\lesssim(1-z)\frown b_2$. If $z\frown a_2\succ z\frown b_2$, then since $z\frown a_1\sim z\frown b_1$ we have $z\frown a\succ z\frown a$, contradicting that a is finite. Hence $z\frown a_2\sim z\frown b_2$. Similarly $(1-z)\frown a_2\sim(1-z)\frown b_2$, so $a_2\sim b_2$.

LEMMA 3.2. *Let b be finite. If $a_\delta\uparrow a$ and $a_\delta\lesssim b$ for all δ, then $a\lesssim b$.*

PROOF. Now the lemma is trivial if $D=\{\delta\}$ is finite. Let D be infinite, and assume that the lemma is true for every directed set D' such that $\bar{\bar{D}}'<\bar{\bar{D}}$ ($\bar{\bar{D}}$ denotes the cardinal of D). Since D is infinite, there exists a well ordered ascending set of directed subsets $\{D_\rho;\ \rho<\alpha\}$ of D such that $\bar{\bar{D}}_\rho<\bar{\bar{D}}$ and $\bigcup(D_\rho;\ \rho<\alpha)=D$. Setting $a_\rho=\bigvee(a_\delta;\ \delta\in D_\rho)$, we have $a_\rho\lesssim b$ by our assumption, and $a_\rho\uparrow a$. Then by the same method as in the proof of Kaplansky [8] Lemma 6.4, we can prove that $a\lesssim b$, so that the lemma is true for D. The details are omitted.

REMARK. Kaplansky [8, pp. 246–247] showed that the projections in a finite AW*-algebra form a continuous geometry and the relation $a\sim b$ coincides with the perspectivity (cf. Example 2 in § 2).

Analogously we obtain the following results.

(i) *Let the relation $a\sim b$ in L has the properties* $(2,\alpha)$—$(2,\varepsilon)$ *and has the following property: if a and b are perspective, then $a\sim b$. If L is finite, then L is an upper-continuous complemented modular lattice.* For, let $a\leqq c$ and $x=(a\cup b)\cap c$, $y=a\cup(b\cap c)$. Then x and y are perspective, so $x\sim y$. Since $x\geqq y$ and x is finite, we have $x=y$, which shows that L is modular. Next using Lemma 3.1, 3.2 we can prove that L is upper-continuous by the same method as in the proof of [8], Theorem 6.5, and the details are omitted.

(ii) *If moreover $Z_0=Z$, then $a\sim b$ if and only if a and b are perspective. Thus the perspectivity is transitive.* For, by the decomposition theorem in the continuous geometries (see von Neumann [13, III], Theorem 2.7, Maeda [11, Ch. 4], Theorem 1.2), for any a, $b\in L$ we have $z\in Z=Z_0$ such that $z\cap a\gtrsim^* z\cap b$, $(1-z)\cap a\lesssim^* (1-z)\cap b$, where "$\sim^*$" is the perspectivity. Let $a\sim b$. If $z\cap a\succ^* z\cap b$ then $z\cap a\succ z\cap b\sim z\cap a$, a contradiction. Hence $z\cap a\sim^* z\cap b$. Similarly $(1-z)\cap a\sim^* (1-z)\cap b$, so $a\sim^* b$.

LEMMA 3.3. *If a is properly infinite, then for any natural number n, there exists a decomposition $a=\oplus(a_i;\ 1\leqq i\leqq n)$ such that $a_i\sim a$ for all i.*

PROOF. Since a is infinite, using Theorem 3.1, we can get $z\in Z_0$ such that $0\neq z\cap a\sim\oplus(b_\alpha;\ \alpha\in I)$ where $b_\alpha\sim b_\beta$ for all α, $\beta\in I$ and I is infinite. This can be proved by the same way as in the proof of [3], Lemma 1.3, and the details are omitted. Thus we have $c_1,\cdots,c_n$ such that $z\cap a\sim\oplus(c_i;\ 1\leqq i\leqq n)\sim c_i$. By $(2,\gamma)$ we have $z\cap a=\oplus(d_i;\ 1\leqq i\leqq n)\sim d_i$. Let

$$z_0=\bigvee(z\in Z_0;\ z\cap a=\oplus(d_i;\ 1\leqq i\leqq n)\sim d_i).$$

Then it is easy to show that there exist $a_1,\cdots,a_n$ such that $z_0\cap a=\oplus(a_i;\ 1\leqq i\leqq n)\sim a_i$. If $(1-z_0)\cap a\neq 0$, then $(1-z_0)\cap a$ is infinite and then, as above, there exists $z\in Z_0$ such that $0\neq(1-z_0)\cap z\cap a=\oplus(d_i;\ 1\leqq i\leqq n)\sim d_i$, a contradiction. Therefore $(1-z_0)\cap a=0$, so $a=z_0\cap a=\oplus(a_i;\ 1\leqq i\leqq n)\sim a_i$.

We remark that if the relation $a \sim b$ has the property that $a \sim \oplus b_\alpha$ implies $a = \oplus a_\alpha$ with $a_\alpha \sim b_\alpha$, then for any properly infinite element a there exists a decomposition $a = \oplus (a_i;\ 1 \leqq i < \infty)$ such that $a_i \sim a$.

Now let the relation $a \sim b$ satisfies the first five conditions $(2, \alpha)$—$(2, \varepsilon)$, and we shall examine the sixth condition $(2, \zeta)$. First we shall show that $(2, \zeta)$ holds in the finite case.

LEMMA 3.4. *If a is finite and $a = a_1 \oplus a_2 = b_1 \oplus b_2$, $a_1 \sim a_2$, $b_1 \sim b_2$, then $a_1 \sim b_1$.*

PROOF. Applying Theorem 3.2, we get $z \in Z_0$ such that $z \cap a_1 \gtrsim z \cap b_1$, $(1-z) \cap a_1 \lesssim (1-z) \cap b_1$. If $z \cap a_1 \succ z \cap b_1$, then $z \cap a_2 \succ z \cap b_2$, whence $z \cap a \succ z \cap a$, a contradiction. Hence $z \cap a_1 \sim z \cap b_1$. Similarly $(1-z) \cap a_1 \sim (1-z) \cap b_1$, so $a_1 \sim b_1$.

In general we obtain :

LEMMA. 3.5. *$(2, \zeta)$ is equivalent to the following condition: if a_1, a_2 are finite and $a_1 \perp a_2$, $a_1 \sim a_2$, then $a_1 \oplus a_2$ is also finite. Furthermore this condition implies that if a, b are finite and $a \perp b$, then $a \oplus b$ is also finite.*

PROOF. Let $a = a_1 \oplus a_2$, $a_1 \sim a_2$ and a_1 be finite. By Theorem 2.1 and Lemma 3.3 there are b_1, b_2 such that $e^i(a) \cap a = b_1 \oplus b_2 \sim b_1 \sim b_2$. If $(2, \zeta)$ holds, then $e^i(a) \cap a_1 \oplus e^i(a) \cap a_2 = b_1 \oplus b_2$ implies $e^i(a) \cap a_1 \sim b_1 \sim e^i(a) \cap a$. Since $e^i(a) \cap a_1$ is finite, $e^i(a) \cap a$ is also finite, so $e^i(a) \cap a = 0$. Therefore a is finite. To prove the converse, let $a = a_1 \oplus a_2 = b_1 \oplus b_2$, $a_1 \sim a_2$, $b_1 \sim b_2$. By Theorem 2.1 and Lemma 3.4, it is sufficient to prove the relation $a_1 \sim b_1$ when a is properly infinite. If $z \cap a_1$ is non-zero, finite for some $z \in Z_0$, then by the assumption $z \cap a_1 \oplus z \cap a_2 = z \cap a$ is again non-zero, finite, that is a contradiction. So a_1 is also properly infinite, whence there are a', a'' such that $a_1 = a' \oplus a'' \sim a' \sim a''$. Hence we have $a = a_1 \oplus a_2 \sim a' \oplus a'' = a_1$. Similarly $a \sim b_1$, so $a_1 \sim b_1$.

Next let $a \perp b$ and a, b be finite. From Theorem 3.2 we get $z \in Z_0$ such that $z \cap a \gtrsim z \cap b$, $(1-z) \cap a \lesssim (1-z) \cap b$. Put $z' = e^i(a \oplus b) \cap z$. There are b_1, b_2 such that $z' \cap (a \oplus b) = b_1 \oplus b_2 \sim b_1 \sim b_2$. Then there are a_1, a_2 such that $z' \cap a \sim a_1 \sim a_2$, $a_1 \leqq b_1$, $a_2 \leqq b_2$. Since $z' \cap a \gtrsim z' \cap b$, we have $z' \cap (a \oplus b) \lesssim a_1 \oplus a_2$. But $a_1 \oplus a_2$ is finite since a_1, a_2 are finite, hence so is $z' \cap (a \oplus b)$. Thus we have $z' \cap (a \oplus b) = 0$, so $z' = 0$. Similarly $e^i(a \oplus b) \cap (1-z) = 0$. Therefore $e^i(a \oplus b) = 0$, then $a \oplus b$ is finite. This completes the proof.

Assume moreover that the relation $a \sim b$ has the property: if a and b are perspective, then $a \sim b$. Then it is easily seen that if a and b are finite, then so is $a \cup b$.

We shall examine that both in Examples 1 and 2 (see § 2) $(2, \zeta)$ holds. Let $a_1 \oplus a_2 = b_1 \oplus b_2$. We shall show that there is $z \in Z_0$ such that $z \cap a_1 \lesssim z \cap b_1$,

$(1-z)\cap a_2 \lesssim (1-z)\cap b_2$. This is obvious in Example 2 from [8], Lemma 6.1. In Example 1, it follows from [5], Theorem 3.1 that there are decompositions $a_i = a_i' \oplus a_i''$, $b_i = b_i' \oplus b_i''$ $(i=1, 2)$ such that $a_1' \sim b_1'$, $a_1'' \sim b_2'$, $a_2' \sim b_1''$, $a_2'' \sim b_2''$. By Theorem 3.2 there is $z \in Z_0$ such that $z\cap a_1'' \lesssim z\cap a_2'$, $(1-z)\cap a_1'' \gtrsim (1-z)\cap a_2'$. Then $z\cap a_1 = z\cap a_1' \oplus z\cap a_1'' \lesssim z\cap b_1' \oplus z\cap b_1'' = z\cap b_1$. Similarly $(1-z)\cap a_2 \lesssim (1-z)\cap b_2$.

Assume $a_1 \oplus a_2 = b_1 \oplus b_2$, $a_1 \sim a_2$, $b_1 \sim b_2$. From the above result we get $z \in Z_0$ such that $z\cap a_1 \lesssim z\cap b_1$, $(1-z)\cap a_2 \lesssim (1-z)\cap b_2$ and we get $z' \in Z_0$ such that $z'\cap a_1 \gtrsim z'\cap b_1$, $(1-z')\cap a_2 \gtrsim (1-z')\cap b_2$. Then we have $(1-z)\cap a_1 \lesssim (1-z)\cap b_1$, $(1-z')\cap a_1 \gtrsim (1-z')\cap b_1$ by the assumption. Hence $a_1 \lesssim b_1$ and $a_1 \gtrsim b_1$, whence $a_1 \sim b_1$ by Lemma 2.3 (iii).

§ 4. Preliminary theorems

In the remainder of this paper we assume that the relation $a \sim b$ has the six properties $(2, \alpha)$—$(2, \zeta)$.

As in the continuous geometries (von Neumann [13, I], Chap. VI) it is convenient to give the following notations.

DEFINITION 4.1. Let $[a]$ denote the class of all elements x such that $x \sim a$, and $[L]$ denote the set of all $[a]$, $a \in L$. We write $[a] < [b]$ if $a \prec b$, $[a] \ll [b]$ if $a \ll b$.

If there exist $a_1 \in [a]$, $b_1 \in [b]$ such that $a_1 \perp b_1$, we define $[a]+[b] = [a_1 \oplus b_1]$.

Clearly, if $[a] \leqq [b]$, there is $[c]$ such that $[a]+[c]=[b]$.

If $([a]+[b])+[c]$ exists, it is easily shown that $([a]+[b])+[c] = [a]+([b]+[c])$. Define $0[a]=[0]$. For any natural number n, if $(n-1)[a]$ has been defined and $(n-1)[a]+[a]$ exists, then we put $n[a]=(n-1)[a]+[a]$.

Lemma 3.3 shows that, if a is properly infinite, $[a]=n[a]$ for every n.

REMARK. We shall show the following properties of $[L]$.

(i) $[L]$ *is a lattice with the order* "$\leqq$". Proof: it is clear from Lemma 2.3 (ii), (iii) that $[L]$ is a partially ordered set. For arbitrary a, $b \in L$, from Theorem 3.2 there is $z \in Z_0$ with $z\cap a \lesssim z\cap b$, $(1-z)\cap a \gtrsim (1-z)\cap b$. Put $c = z\cap a \oplus (1-z)\cap b$. Then clealy $c \lesssim a$, b and if $x \lesssim a$, b, then $x \lesssim c$. Therefore $[c]=[a]\cap[b]$. Similarly $[z\cap b \oplus (1-z)\cap a] = [a]\cup[b]$.

(ii) $[L]$ *is totally ordered if and only if* $Z_0 = \{0, 1\}$.

LEMMA 4.1. *If* $n[a] \leqq [b]$ *for every natural number* n *and* b *is finite, then* $a=0$.

PROOF. Suppose that there exists an independent system $\{a_i;\ 1 \leqq i \leqq n\}$ such that $a_i \leqq b$, $a_i \in [a]$. Put $b = \oplus(a_i\,;\ 1 \leqq i \leqq n) \oplus d$. Since $n[a] \leqq (n+1)[a] \leqq [b]$, there are $c_1 \in n[a]$ and $c_2 \in (n+1)[a]$ with $c_1 \leqq c_2 \leqq b$. Put $c_1 \oplus d_1 = c_2$, $c_2 \oplus d_2 = b$. Then it follows from Lemma 3.1 that $a \sim d_1 \leqq d_1 \oplus d_2 \sim d$. Hence there is a_{n+1} such that $a \sim a_{n+1} \leqq d$. We see $(a_i\,;\ 1 \leqq i \leqq n+1)\perp$, $a_i \leqq b$, $a_i \in [a]$. Thus if a_1 is any element in $[a]$, we have an infinite sequence $\{a_i\}$ such that $(a_i\,;\ 1 \leqq i < \infty)\perp$, $a_i \leqq b$, $a_i \in [a]$. If $a \neq 0$, $\oplus(a_i\,;\ 1 \leqq i < \infty)$ is infinite. Therefore by the finiteness of b we have $a = 0$.

THEOREM 4.1. *Let c be finite. For arbitrary $a \in L$ and $n = 0, 1, 2, \dots$ there exists a unique element $q_n(c, a) \in Z_0$ which has the following property:*

(1) *for any $z \in Z_0$, $z \leqq q_n(c, a)$ if and only if $n[z \cap c]$ exists and $n[z \cap c] \leqq [z \cap a]$.*
Then setting $r_n(c, a) = q_n(c, a) - q_{n+1}(c, a)$, the following relations hold.

(2) $[r_n(c, a) \cap a] = n[r_n(c, a) \cap c] + [p]$ *with* $[p] \ll [r_n(c, a) \cap c]$.

(3) $1 = q_0(c, a) = \oplus(r_n(c, a)\,;\ 0 \leqq n < \infty) \oplus \bigwedge(q_n(c, a)\,;\ 0 \leqq n < \infty)$,

$$\bigwedge(q_n(c, a)\,;\ 0 \leqq n < \infty) = e^i(a) \cup (1 - e(c)).$$

PROOF. Let $q_n(c, a) = \bigvee(z \in Z_0\,;\ n[z \cap c] \leqq [z \cap a])$. Then we have $q_n(c, a) = \oplus(z_\alpha \in Z_0\,;\ \alpha \in I)$ with $n[z_\alpha \cap c] \leqq [z_\alpha \cap a]$. Hence $n[q_n(c, a) \cap c] \leqq [q_n(c, a) \cap a]$. Therefore (1) holds. The uniqueness of $q_n(c, a)$ is obvious.

Next we shall prove (2). Since $r_n \leqq q_n$ it follows from (1) that $[r_n \cap a] = n[r_n \cap c] + [p]$ for some $p \leqq r_n$. From Theorem 3.2, there is $z \in Z_0$ such that $[z \cap r_n \cap c] \leqq [z \cap p]$, $[(1-z) \cap r_n \cap c] \gg [(1-z) \cap p]$. We see $[z \cap r_n \cap a] = n[z \cap r_n \cap c] + [z \cap p] \geqq (n+1)[z \cap r_n \cap c]$, whence $z \cap r_n \leqq q_{n+1}$ by (1). Hence $z \cap r_n = 0$, so $r_n \leqq 1 - z$. Therefore $[r_n \cap c] \gg [p]$.

We shall prove (3). $1 = q_0 = \oplus_n r_n \oplus \bigwedge_n q_n$ is obvious. Now put $z = \bigwedge_n q_n \cap (1 - e^i(a))$. Then $n[z \cap c] \leqq [z \cap a]$ for all n and $z \cap a$ is finite. By Lemma 4.1 we have $z \cap c = 0$, so $z \leqq 1 - e(c)$. Hence $\bigwedge_n q_n \leqq e^i(a) \cup z \leqq e^i(a) \cup (1 - e(c))$. On the other hand $n[(1 - e(c)) \cap c] = [0]$ implies $1 - e(c) \leqq q_n$ for every n. Since $r_n \cap c$ is finite, so is $r_n \cap a$ by (2) and Lemma 3.5. Then $r_n \cap e^i(a) \cap a = 0$ since $e^i(a) \cap a$ is properly infinite. So $r_n \cap e^i(a) = 0$ for every n. Therefore $e^i(a) \leqq \bigwedge_n q_n$. This completes the proof.

The similar result of this theorem is known in the continuous geometries ([13, III], Theorem 2.14, 2.15, 2.16).

We remark that if $c \sim c'$, $a \sim a'$ then $q_n(c, a) = q_n(c', a')$ for all n.

By Theorem 2.2 there exists a minimal element h such that $e(h) = z_{\mathrm{I}}$. If h_1, h_2 are minimal and $e(h_1) = e(h_2) = z_{\mathrm{I}}$, then by Theorem 3.1 we have $h_i = h_i' \oplus h_i''$ $(i = 1, 2)$ with $h_1' \sim h_2'$, $e(h_1') = e(h_2') = z_{\mathrm{I}}$. By Lemma 2.3 (iv) $h_i'' \ll h_i$ and so $h_i'' = 0$.

Therefore $h_1 \sim h_2$. This means that $[h]$ is uniquely determined. Hereafter we suppose that h is such a element.

THEOREM 4.2. (1) *For any* $z \leqq z_{\mathrm{I}}$, $z \leqq r_n(h, a)$ *if and only if* $n[z \frown h] = [z \frown a]$.

(2) $r_0(h, a) = z_{\mathrm{I}} \frown (1 - e(a))$.

PROOF. Let $z \leqq r_n(h, a)$. Applying Theorem 4.1, we have $[z \frown a] = n[z \frown h] + [z \frown p]$, $[z \frown p] \ll [z \frown h]$. Since $z \frown h$ is minimal, we have $z \frown p = 0$. To prove the converse, let $z \leqq z_{\mathrm{I}}$, $[z \frown a] = n[z \frown h]$. Clearly $z \leqq q_n(h, a)$. Setting $z' = z \frown q_{n+1}(h, a)$, we have $(n+1)[z' \frown h] \leqq [z' \frown a] = n[z' \frown h]$, whence $z' \frown h = 0$. But $z' \leqq e(h)$, so $z' = 0$. Therefore $z \leqq r_n(h, a)$.

We shall prove (2). $[z_{\mathrm{I}} \frown (1 - e(a)) \frown a] = [0]$ implies $z_{\mathrm{I}} \frown (1 - e(a)) \leqq r_0$. On the other hand, $r_0 \leqq z_{\mathrm{I}}$ is obvious and $[r_0 \frown a] = [0]$ implies $r_0 \leqq 1 - e(a)$. So the proof is complete.

From this theorem we get the following decomposition of $z_{\mathrm{I}}^{(1)}$.

COROLLARY. *Let* $e_{(n)} = r_n(h, 1)$. *Then*

(1) $e_{(0)} = 0$, $\oplus(e_{(n)}\,;\ 1 \leqq n < \infty) = z_{\mathrm{I}}^{(1)}$.

(2) *For any* $z \leqq z_{\mathrm{I}}$, $z \leqq e_{(n)}$ *if and only if* $n[z \frown h] = [z]$.

PROOF. By Theorem 4.1 (3) we have $\oplus_n e_{(n)} = 1 - \wedge_n q_n(h, 1) = (1 - e^i(1)) \frown e(h) = e^f(1) \frown z_{\mathrm{I}} = z_{\mathrm{I}}^{(1)}$. The other statements follow from Theorem 4.2.

Furthermore we can prove the following properties.

LEMMA 4.2. (i) $q_m(c, a) \frown q_n(c, b) \leqq q_{m+n}(c, a \oplus b)$.

(ii) $(1 - q_m(c, a)) \frown (1 - q_n(c, b)) \leqq 1 - q_{m+n}(c, a \oplus b)$.

Especially $(1 - q_{m+1}(h, a)) \frown (1 - q_{n+1}(h, b)) \leqq 1 - q_{m+n+1}(h, a \oplus b)$.

(iii) $q_n(c, z \frown a) \frown e(c) = z \frown q_n(c, a) \frown e(c)$ *if* $n \geqq 1$.

PROOF. (i). Put $z = q_m(c, a) \frown q_n(c, b)$. By Theorem 4.1 we have $m[z \frown c] \leqq [z \frown a]$, $n[z \frown c] \leqq [z \frown b]$ and then $(m+n)[z \frown c] \leqq [z \frown (a \oplus b)]$. This implies $z \leqq q_{m+n}(c, a \oplus b)$.

(ii). Let $m' < m$, $n' < n$ and put $z = q_{m+n}(c, a \oplus b) \frown r_{m'}(c, a) \frown r_{n'}(c, b)$. If $z \frown c \neq 0$, we have $(m+n)[z \frown c] \leqq [z \frown (a \oplus b)]$ and $[z \frown a] < m[z \frown c]$, $[z \frown b] < n[z \frown c]$. Hence $(m+n)[z \frown c] < (m+n)[z \frown c]$, a contradiction. So $z \frown c = 0$. But $z \leqq e(c)$, hence $z = 0$. This implies $r_{m'}(c, a) \frown r_{n'}(c, b) \leqq 1 - q_{m+n}(c, a \oplus b)$ for any $m' < m$, $n' < n$. Since $\oplus(r_{m'}\,;\ m' < m) = 1 - q_m$, we have $(1 - q_m(c, a)) \frown (1 - q_n(c, a)) \leqq 1 - q_{m+n}(c, a \oplus b)$.

Next let $m' \leqq m$, $n' \leqq n$ and put $z = q_{m+n+1}(h, a \oplus b) \frown r_{m'}(h, a) \frown r_{n'}(h, b)$. By Theorem 4.2 we have $[z \frown (a \oplus b)] \geqq (m+n+1)[z \frown h]$, $[z \frown a] = m'[z \frown h]$, $[z \frown b] = n'[z \frown h]$. But since $m + n + 1 > m' + n'$, we have $z \frown h = 0$, so $z = 0$. Therefore $(1 - q_{m+1}(h, a)) \frown (1 - q_{n+1}(h, a)) \leqq 1 - q_{m+n+1}(h, a \oplus b)$.

(iii). Since $n[z\frown q_n(c, a)\frown c]\leqq[z\frown q_n(c, a)\frown a]$, we have $z\frown q_n(c, a)\leqq q_n(c, z\frown a)$. On the other hand $q_n(c, z\frown a)\leqq q_n(c, a)$ is obvious. Now it suffices to show that $q_n(c, z\frown a)\frown e(c)\leqq z$. Put $z'=q_n(c, z\frown a)\frown e(c)\frown(1-z)$. We see that $n[z'\frown c]\leqq[z'\frown z\frown a]=[0]$. But $n\geqq 1$, so $z'\frown c=0$. Since $z'\leqq e(c)$, we have $z'=0$.

§ 5. Dimension functions

DEFINITION 5.1. The complete Boolean lattice Z_0 is isomorphic to the lattice of all compact open subsets of a totally disconnected compact Hausdorff space Ω in which the closure of any open set is open. The compact open subset of Ω which corresponds to $z\in Z_0$ will be denoted by $E(z)$. Then $\{E(z)\,;\, z\in Z_0\}$ is a basis of Ω.

Let $\boldsymbol{Z}$ be the set of all non-negative (finite or infinite) continuous functions on Ω. $\boldsymbol{Z}$ is a complete lattice by the usual order. After Dixmier [4, p. 25] we define $f+g$, ξf, fg, $f^{-1}\in\boldsymbol{Z}$ for any f, $g\in\boldsymbol{Z}$ and $0\leqq\xi\leqq\infty$. The infinite sum $\sum(f_\alpha\,;\,\alpha\in I)$ is defined by the l.u.b. of all finite sums $f_{\alpha_1}+\cdots+f_{\alpha_n}$.

DEFINITION 5.2. A function d on L to $\boldsymbol{Z}$ is called a *dimension function* on L if d has the following properties:

(1°) *if* $a\sim b$, *then* $d(a)=d(b)$;

(2°) *if* $a\perp b$, *then* $d(a\smile b)=d(a)+d(b)$;

(3°) *if* $z\in Z_0$, *then* $d(z\frown a)=\phi(z)d(a)$, *where* $\phi(z)$ *is a characteristic function of* $E(z)$;

(4°) *if* $a>0$, *then* $d(a)>0$;

(5°) *if* a *is finite, then* $d(a)$ *is finite valued a. e., where a. e. means "except on a set of the first category".*

For arbitrary $\omega\in\Omega$ we shall write $d(a, \omega)=(d(a))(\omega)$.

Now we shall show the existence of a dimension function on L which is normalized in a certain sense.

THEOREM 5.1. (*Theorem of existence*). *There exists a dimension function* d *of* L *such that* $d(h)=\phi(z_{\mathrm{I}})$, $d(z_{\mathrm{II}}^{(1)})=\phi(z_{\mathrm{II}}^{(1)})$, *where* h *is a minimal element with* $e(h)=z_{\mathrm{I}}$.

PROOF. By the property (3°) in Definition 5.2 and the reduction theory (Theorem 2.2) it suffices to show that the theorem is true in any of the following cases: (i) L is of type I, (ii) type II, (iii) type III.

(i). Let L be of type I. For arbitrary $a\in L$, we set $U(n, a)=E(q_n(h, a))$ $(n=0, 1, 2, \cdots)$. Clearly $U(0, a)=\Omega$, $U(n, a)\supset U(n+1, a)$, so we define

$$d(a, \omega)=\text{l. u. b. }\{n\,;\,\omega\in U(n, a)\}.$$

For all $n\geqq 0$, $\{\omega\in\Omega\,;\, d(a, \omega)\geqq n\}=U(n, a)$ and $\{\omega\in\Omega\,;\, d(a, \omega)\leqq n\}=\Omega-U(n+1, a)$

are closed. Hence $d(a)$ is continuous and $d(a)\in \boldsymbol{Z}$. It is easy to show that $U(1, h)=\Omega$, $U(2, h)=0$. Hence $d(h)=1$. Now we shall prove that d has the properties (1°)—(5°).

If $a\sim b$, then $q_n(h, a)=q_n(h, b)$ for all n, and then $d(a)=d(b)$.

Let $a\perp b$. By Lemma 4.2 (i), (ii), we have

$$U(m, a)\cap U(n, b)\subset U(m+n, a\oplus b).$$

and $$(\Omega-U(m+1, a))\cap(\Omega-U(n+1, b))\subset\Omega-U(m+n+1, a\oplus b).$$

Then it follows that $d(a, \omega)\geqq m$, $d(b, \omega)\geqq n$ imply $d(a\oplus b, \omega)\geqq m+n$ and that $d(a, \omega)\leqq m$, $d(b, \omega)\leqq n$ imply $d(a\oplus b, \omega)\leqq m+n$. Hence if $d(a, \omega)$, $d(b, \omega)<\infty$, then clearly $d(a, \omega)+d(b, \omega)=d(a\oplus b, \omega)$. If either $d(a, \omega)=\infty$ or $d(b, \omega)=\infty$, then $d(a\oplus b, \omega)=\infty=d(a, \omega)+d(b, \omega)$. Therefore $d(a)+d(b)=d(a\oplus b)$.

Let $z\in Z_0$. By Lemma 4.2 (iii) we have $U(n, z\cap a)=E(z)\cap U(n, a)$ for all $n\geqq 1$. Hence $d(z\cap a, \omega)\equiv d(a, \omega)$ on $E(z)$ and $d(z\cap a, \omega)\equiv 0$ on $\Omega-E(z)$. Therefore $d(z\cap a)=\phi(z)d(a)$.

Assume that $d(a)=0$. Then $U(1, a)=0$, so $q_1(h, a)=0$. By Theorem 4.2 (2), we have $1-e(a)=r_0(h, a)=1$, whence $a=0$. Therefore if $a>0$, then $d(a)>0$.

Let a be finite. Then by Theorem 4.1 (3) $\bigwedge_n q_n(h, a)=e^i(a)=0$. Therefore $\{\omega\in\Omega\,;\, d(a, \omega)=\infty\}=\bigcap_n U(n, a)$ is a non-dense set, that is, $d(a, \omega)<\infty$ a.e.

(ii). Let L be of type II. There is a finite element c_0 with $e(c_0)=1$. We may suppose that $z_{\rm II}^{(1)}\leqq c_0$. By Lemma 2.7 there is a sequence $\{c_k\}$ such that $2^k[c_k]=[c_0]$. Put $\Lambda=\left\{\frac{n}{2^k}\,;\, k=1, 2, \dots\,;\, n=0, 1, 2, \dots\right\}$. For arbitrary $a\in L$ we set $U\left(\frac{n}{2^k}, a\right)=E(q_n(c_k, a))$. For $\lambda\in\Lambda$, $U(\lambda, a)$ is uniquely determined since $q_{2n}(c_{k+1}, a)=q_n(c_k, a)$. Clearly $U(0, a)=\Omega$, and if $\lambda_1\leqq\lambda_2$ then $U(\lambda_1, a)\supset U(\lambda_2, a)$, so we define

$$d(a, \omega)=\text{l.u.b. }\{\lambda\in\Lambda\,;\,\omega\in U(\lambda, a)\}.$$

For all $\xi\geqq 0$ (ξ: real number), $\{\omega\in\Omega\,;\, d(a, \omega)\geqq\xi\}=\bigcap(U(\lambda, a)\,;\,\lambda<\xi)$ and $\{\omega\in\Omega\,;\, d(a, \omega)\leqq\xi\}=\bigcap(\Omega-U(\lambda, a)\,;\,\lambda>\xi)$ are closed. Hence $d(a)$ is continuous and $d(a)\in\boldsymbol{Z}$. It is easy to show that $U(1, c_0)=\Omega$ and $U(\lambda, c_0)=0$ if $\lambda>1$. Hence $d(c_0)=1$ and $d(z_{\rm II}^{(1)})=\phi(z_{\rm II}^{(1)})$. Now we shall prove that d has the properties (1°)—(5°).

If $a\sim b$, then $q_n(c_k, a)=q_n(c_k, b)$ for all k, n, whence $d(a)=d(b)$.

Let $a\perp b$, and ξ, η be non-negative real numbers. If $\lambda\in\Lambda$, $\lambda<\xi+\eta$, there are k, m, n such that $\lambda=\frac{m+n}{2^k}$, $\frac{m}{2^k}<\xi$, $\frac{n}{2^k}<\eta$. So using Lemma 4.2 we have

$$\bigwedge(U(\lambda, a)\,;\,\lambda<\xi)\cap\bigwedge(U(\lambda, b)\,;\,\lambda<\eta)\subset\bigwedge(U(\lambda, a\oplus b)\,;\,\lambda<\xi+\eta).$$

Similarly we have

$$\bigwedge(\Omega-U(\lambda, a)\,;\,\lambda>\xi)\cap\bigwedge(\Omega-U(\lambda, b)\,;\,\lambda>\eta)\subset\bigwedge(\Omega-U(\lambda, a\oplus b)\,;\,\lambda>\xi+\eta).$$

Then it follows that $d(a, \omega)\geqq\xi$ (resp. $\leqq\xi$), $d(b, \omega)\geqq\eta$ (resp. $\leqq\eta$) imply $d(a\oplus b, \omega)\geqq\xi+\eta$ (resp. $\leqq\xi+\eta$). Therefore we have $d(a)+d(b)=d(a\oplus b)$.

Let $z\in Z_0$. By Lemma 4.2 (iii) $U(\lambda, z\cap a)=E(z)\cap U(\lambda, a)$ for all $\lambda>0$. Therefore $d(z\cap a)=\phi(z)d(a)$.

Assume that $d(a)=0$. Then $E(q_1(c_k, a))=U\left(\frac{1}{2^k}, a\right)=0$, so $r_0(c_k, a)=1$ for all k. By Theorem 4.1, $[a]\ll[c_k]$, whence $2^k[a]\leqq[c_0]$ for all k. Since c_0 is finite, we have $a=0$ by Lemma 4.1.

Let a be finite. We have $\bigwedge_n q_n(c_0, a)=0$, whence $\{\omega\in\Omega\,;\,d(a, \omega)=\infty\}=\bigwedge_n U(n, a)$ is a non-dense set.

(iii). Let L be of type III. Set $d(a)=\infty\phi(e(a))$. Then clearly $d(a)\in\boldsymbol{Z}$ and d has the properties (1°)—(5°) from the following properties: $a\sim b$ implies $e(a)=e(b)$; $e(a\oplus b)=e(a)\cup e(b)$; $e(z\cap a)=z\cap e(a)$; $a>0$ implies $e(a)>0$; 0 is the only finite element. This completes the proof.

The dimension functions necessarily have the following further properties.

THEOREM 5.2. *Let d be a dimension function on L.*

(i) $d(a, \omega)\equiv 0$ *on* $\Omega-E(e(a))$;

$0<d(a, \omega)<\infty$ *a. e. on* $E(e^f(a))$;

$d(a, \omega)\equiv\infty$ *on* $E(e^i(a))$.

Especially $a>0$ if and only if $d(a)>0$; a is finite if and only if $d(a, \omega)<\infty$ a. e.

(ii) *$a\gtrsim b$ implies $d(a)\geqq d(b)$ and the converse holds if b is finite. From this if a (or b) is finite, then $a>b$ (resp. $\sim$, $<$) is equivalent to $d(a)>d(b)$ (resp. $=$, $<$).*

PROOF. (i). Since $0\in Z_0$, it follows from (3°) that $d(0)=0$. Thus $z\leqq 1-e(a)\rightleftarrows z\cap a=0\rightleftarrows\phi(z)d(a)=0\rightleftarrows d(a, \omega)\equiv 0$ on $E(z)$. Hence $d(a, \omega)\equiv 0$ on $\Omega-E(e(a))$ and $d(a, \omega)>0$ a. e. on $E(e(a))$. Since $e^f(a)\cap a$ is finite, it follows from (3°), (5°) that $d(a, \omega)<\infty$ a. e. on $E(e^f(a))$. Next by Lemma 3.4 $[e^i(a)\cap a]=2[e^i(a)\cap a]$. Hence by (1°), (2°) we have $d(e^i(a)\cap a)=2d(e^i(a)\cap a)$, whence $d(e^i(a)\cap a, \omega)=0$ or ∞. Since $d(e^i(a)\cap a, \omega)>0$ a. e. on $E(e^i(a))$, $d(a, \omega)\equiv\infty$ on $E(e^i(a))$.

(ii). It follows from (1°), (2°) that $a\gtrsim b$ implies $d(a)\geqq d(b)$. Let b be finite and $d(a)\geqq d(b)$. From Theorem 3.1 there are decompositions $a=a'\oplus a''$, $b=b'\oplus b''$ with $a'\sim b'$, $e(a'')\cap e(b'')=0$. $d(a')=d(b')$ is finite valued a. e. by (5°), and hence $d(a')+d(a'')\geqq d(b')+d(b'')$ implies $d(a'')\geqq d(b'')$. Hence $0=d(e(b'')\cap a'')$ $=\phi(e(b''))d(a'')\geqq\phi(e(b''))d(b'')=d(b'')$. Therefore $b''=0$, whence $a\gtrsim b'=b$.

REMARK. We assume that relation $a\sim b$ has moreover the following property: if a and b are perspective, then $a\sim b$. Then for arbitrary a, $b\in L$ we have $d(a\cup b)+d(a\cap b)=d(a)+d(b)$. Proof: let $a\cup b=a\oplus c_1$, $b=a\cap b\oplus c_2$. Since $c_1\sim c_2$ by the assumption, $d(a\cup b)+d(a\cap b)=d(a)+d(c_1)+d(a\cap b)=$ $d(a)+d(c_2)+d(a\cap b)=d(a)+d(b)$.

Next we prove the uniqueness of the dimension function in a certain sense.

THEOREM 5.3. *If d_1, d_2 are dimension functions on L, then there exists a function $f\in \boldsymbol{Z}$, $0<f(\omega)<\infty$ a. e. such that $d_1(a)=f\cdot d_2(a)$ for all $a\in L$.*

PROOF. Let L be of type I. By Theorem 5.2 (i) we have $0<d_i(h,\omega)<\infty$ a. e. $(i=1, 2)$. Therefore $d_i(h)^{-1}$ are also non-zero, finite valued a. e. Put $d_i'(a)$ $=d_i(h)^{-1}d_i(a)$ for all $a\in L$. Then d_1', d_2' are also dimension functions on L and $d_1'(h)=d_2'(h)=1$. To prove the theorem, it suffices to show that both d_1' and d_2' coincide with the dimension function d which appeared in Theorem 5.1. Since $n[q_n(h,a)\cap h]\leqq[q_n(h,a)\cap a]$ and $n[r_n(h,a)\cap h]=[r_n(h,a)\cap a]$, we have

$$d_i'(a,\omega)\equiv\infty \text{ on } \bigcap\nolimits_n(U(n,a))$$

and

$$d_i'(a,\omega)\equiv n \text{ on } U(n,a)-U(n+1,a).$$

Hence $d_i'(a)=d(a)$ for all $a\in L$.

Let L be of type II. By the same reason as above it suffices to show that if d' is a dimension function and $d'(c_0)=1$, then $d'=d$. Since $d'(c_k)=1/2^k$, $d'(a,\omega)\geqq\lambda$ on $U(\lambda,a)$ and $d'(a,\omega)\leqq\lambda$ on $\Omega-U(\lambda,a)$. Hence

$$d'(a,\omega)\geqq\xi \text{ on } \bigcap(U(\lambda,a)\,;\,\lambda<\xi)$$

and

$$d'(a,\omega)\leqq\xi \text{ on } \bigcap(\Omega-U(\lambda,a)\,;\,\lambda>\xi).$$

Therefore $d'(a)=d(a)$ for all $a\in L$.

Let L be of type III. By Theorem 5.2 (i) we have $d_1(a)=d_2(a)=\infty\phi(e(a))$, i. e., in this case the dimension function is uniquely determined.

COROLLARY. *Let $z_{\text{III}}=0$ and let d_0 be a dimension function on L. There exists a one-to-one correspondence between the dimension functions d on L and the functions $f\in\boldsymbol{Z}$ with $0<f(\omega)<\infty$ a. e., where the correspondence is given by the equation $d(a)=f\cdot d_0(a)$, $a\in L$.*

PROOF. Let $f\in\boldsymbol{Z}$ and $0<f(\omega)<\infty$ a. e. Then $f\cdot d_0$ is clearly a dimension function on L. Conversely, if d is any dimension function on L, then there is $f\in\boldsymbol{Z}$, $0<f(\omega)<\infty$ a. e. such that $d=f\cdot d_0$ by Theorem 5.3. Since $z_{\text{III}}=0$, there is a finite element a_0 with $e(a_0)=1$. Clearly $0<d_0(a_0,\omega)<\infty$ a. e. If $f_1\cdot d_0=$

$f_2 \cdot d_0$, we have $f_1 \cdot d_0(a_0) = f_2 \cdot d_0(a_0)$, whence $f_1 = f_2$. Therefore the correspondence is one-to-one.

Furthermore we shall examine some properties of the dimension functions.

DEFINITION 5.3. A dimension function d on L will be called a *normalized dimension function* if $d(h) = \phi(z_{\mathrm{I}})$, $d(z_{\mathrm{II}}^{(1)}) = \phi(z_{\mathrm{II}}^{(1)})$, where h is a minimal element with $e(h) = z_{\mathrm{I}}$.

It is clear from the proof of Theorem 5.3 that the normalized dimension function on L is uniquely determined if $z_{\mathrm{II}}^{(\infty)} = 0$.

For L, $\boldsymbol{Z}_L$ will denote the set of $f \in \boldsymbol{Z}$ which have the following properties:

$$\begin{aligned} f(\omega) &= 0, 1, \cdots, n && \text{on} \quad E(e_{(n)}); \quad (n: \text{natural number}) \\ f(\omega) &= 0, 1, \cdots, \infty && \text{on} \quad E(z_{\mathrm{I}}^{(\infty)}); \\ f(\omega) &\leqq 1 && \text{on} \quad E(z_{\mathrm{II}}^{(1)}); \\ f(\omega) &= 0, \infty && \text{on} \quad E(z_{\mathrm{III}}). \end{aligned}$$

It is easily shown that $\boldsymbol{Z}_L$ is a complete sublattice of $\boldsymbol{Z}$.

THEOREM 5.4. *If d is a normalized dimension function on L, then* $\{d(a)\,;\, a \in L\} = \boldsymbol{Z}_L$.

PROOF Let d be the normalized dimension function which appeared in Theorem 5.1 and d' be arbitrary normalized dimension function on L. By Theorem 5.3 we have $d' = f \cdot d$, where $f(\omega) \equiv 1$ on $\Omega - E(z_{\mathrm{II}}^{(\infty)})$, $0 < f(\omega) < \infty$ a. e. on $E(z_{\mathrm{II}}^{(\infty)})$. It is obvious that $f \cdot \boldsymbol{Z}_L = \boldsymbol{Z}_L$, thus to prove $\{d'(a)\,;\, a \in L\} = \boldsymbol{Z}_L$ it suffices to show that $\{d(a)\,;\, a \in L\} = \boldsymbol{Z}_L$.

First we shall prove $d(a) \in \boldsymbol{Z}_L$ for all $a \in L$. If L is of type I, then it is obvious that $d(a, \omega) = 0, 1, \cdots, \infty$. Since $U(n, 1) - U(n+1, 1) = E(e_{(n)})$, we have $d(1, \omega) \equiv n$ on $E(e_{(n)})$, whence $d(a, \omega) \leqq n$ on $E(e_{(n)})$. Now $d(z_{\mathrm{II}}^{(1)}) = \phi(z_{\mathrm{II}}^{(1)})$ implies $d(a, \omega) \leqq 1$ on $E(z_{\mathrm{II}}^{(1)})$. It is clear that $d(a, \omega) = 0, \infty$ on $E(z_{\mathrm{III}})$.

Next we shall show that for any $f \in \boldsymbol{Z}_L$ there exists $a \in L$ such that $d(a) = f$. Let L be of type I. Since $n[q_n(h, 1) \frown h]$ exists for every n, we get an independent system $\{h_n\,;\; 1 \leqq n < \infty\}$ with $h_n \in [q_n(h, 1) \frown h]$. Since $f(\omega) = 0, 1, \cdots, \infty$, $A_n = \{\omega \in \Omega\,;\, f(\omega) \geqq n\}$ is open and closed, then there is $z_n \in Z_0$ such that $E(z_n) = A_n$. We see that $f(\omega) \leqq i$ on $E(e_{(i)})$, so $z_n \leqq 1 - \oplus(e_{(i)}\,;\, i < n) = q_n(h, 1) = e(h_n)$. Hence $d(z_n \frown h_n) = \phi(z_n)$. Set $a = \oplus_n (z_n \frown h_n)$. We have

$$d(a) \geqq n \quad \text{on} \quad E(z_n) = \{\omega \in \Omega\,;\, f(\omega) \geqq n\}$$

and
$$d(a) \leqq n \quad \text{on} \quad \Omega - E(z_{n+1}) = \{\omega \in \Omega\,;\, f(\omega) \leqq n\}.$$

Therefore $d(a) = f$.

Let L be of type II. For every $k \geqq 1$, we get an independent system

$\{c_k^n\,;\ 1\leqq n<\infty\}$ such that $c_k^n\in[c_k]$ if $n\leqq 2^k$, $c_k^n\in[z_{\mathrm{II}}^{(\infty)}\cap c_k]$ if $n>2^k$. We may suppose that $c_k^n=c_{k+1}^{2n-1}\oplus c_{k+1}^{2n}$. The interior of the closed set $A_k^n=\{\omega\in\Omega\,;\,f(\omega)\geqq n/2^k\}$ is open and closed. Hence we get $z_k^n\in Z_0$ such that $E(z_k^n)=\mathrm{int}\,(A_k^n)$. Set $a_k=\bigoplus_n(z_k^n\cap c_k^n)$, $b_k=\bigoplus_n(z_k^{n-1}\cap c_k^n)$. Then $a_k\leqq a_{k+1}\leqq b_{k+1}\leqq b_k$ since

$$z_k^n\cap c_k^n\leqq(z_{k+1}^{2n-1}\cap c_{k+1}^{2n-1})\oplus(z_{k+1}^{2n}\cap c_{k+1}^{2n})\leqq(z_{k+1}^{2n-2}\cap c_{k+1}^{2n-1})\oplus(z_{k+1}^{2n-1}\cap c_{k+1}^{2n})\leqq z_k^{n-1}\cap c_k^n.$$

Set $a=\bigvee_k a_k$, and we shall show $d(a)=f$. Since $f(\omega)\leqq 1$ on $E(z_{\mathrm{II}}^{(1)})$, $z_k^n\leqq z_{\mathrm{II}}^{(\infty)}=e(c_k{}^n)$ if $n>2^k$. Hence $d(z_k^n\cap c_k^n)=(1/2^k)\phi(z_k^n)$. We have $d(a,\omega)\geqq d(a_k,\omega)\geqq n/2^k$ on $E(z_k^n)$, so that

$$d(a,\omega)\geqq\xi\ \text{ on }\ \bigwedge(E(z_k^n)\,;\ \frac{n}{2^k}<\xi)\subset\{\omega\in\Omega\,;\,f(\omega)\geqq\xi\}$$

for any non-negative real number ξ. On the other hand $d(a,w)\leqq d(b_k,\omega)\leqq n/2^k$ on $\Omega-E(z_k^n)$, so that

$$d(a,\omega)\leqq\xi\ \text{ on }\ \bigwedge\{\Omega-E(z_k^n)\,;\ \frac{n}{2^k}>\xi\}\subset\{\omega\in\Omega\,;\,f(\omega)\leqq\xi\}.$$

Therefore $d(a)=f$.

Let L be of type III. Since $A=\{\omega\,;\,f(\omega)=\infty\}$ is open and closed, we get $z\in Z_0$ with $E(z)=A$. Clearly $d(z)=\infty\phi(z)=f$. This completes the proof.

COROLLARY. *$\{[a]\,;\ a$ is finite$\}$ is lattice-isomorphic to $\{f\in \boldsymbol{Z}_L\,;\,f(\omega)<\infty$ a. e.$\}$. Especially if L is finite, then $[L]$ is lattice-isomorphic to $\boldsymbol{Z}_L$ and then it is a complete lattice.*

PROOF. Let d be a normalized dimension function on L. By Theorem 5.2 (ii) and Theorem 5.4, the correspondence $[a]\to d(a)$ is a lattice-isomorphism of $\{[a]\,;\ a$ is finite$\}$ with $\{f\in\boldsymbol{Z}_L\,;\,f(\omega)<\infty$ a. e$\}$.

THEOREM 5.5. *Let d be a dimension function on L. If $a_\delta\uparrow a$, then $d(a_\delta)\uparrow d(a)$. Especially d is completely additive, that is, if $a=\bigoplus(a_\alpha\,;\,\alpha\in I)$ then $d(a)=\sum(d(a_\alpha)\,;\,\alpha\in I)$.*

PROOF. By Theorem 5.3 we may suppose that d is a normalized dimension function. Let $a_\delta\uparrow a$. By Theorem 5.4 there is $b\in L$ such that $d(b)=\bigvee_\delta d(a_\delta)$. Clearly $d(b)\leqq d(a)$. By Theorem 5.2 (i) $d(b,\omega)\equiv d(a,\omega)\equiv\infty$ on $E(e^i(b))$. Put $z=1-e^i(b)$. Since $d(z\cap a_\delta)\leqq d(z\cap b)$ for all δ and $z\cap b$ is finite, we have $d(z\cap a)\leqq d(z\cap b)$ by Theorem 5.2 (ii) and Lemma 3.2. Hence $d(a,\omega)\equiv d(b,\omega)$ on $E(z)=\Omega-E(e^i(b))$. Therefore $d(a)=d(b)$, this gives $d(a_\delta)\uparrow d(a)$. The rest of the statements is obvious.

§ 6. Remarks on the axioms of dimension functions

In this section we shall make clear the relation of our dimension functions defined in §5 to those of Segal and Dixmier (see (II) in the introduction). Let d^* be a function on L to $\boldsymbol{Z}$ satisfying the first three axioms (1°)—(3°) in Definition 5.2. As in the proof of Theorem 5.2 (i) it is easily shown that $d^*(a, \omega)\equiv 0$ on $\Omega - E(e(a))$ and that if a is properly infinite, then $d^*(a, \omega)=0$ or ∞.

To make clear the relation of our dimension functions to those of Segal, we prove the following theorem :

THEOREM 6.1. *For any* d^*, *there exists a decomposition* $z_1 \oplus z_2 \oplus z_3 \oplus z_4 = 1$, $z_i \in Z_0$ *such that*

(1) d^* *is a dimension function on* $L(0, z_1)$;

(2) *if* $a \leqq z_2$, *then* $d^*(a) = \infty\phi(e(a))$; (*where* $z_2 \frown z_{\mathrm{III}} = 0$)

(3) *if* $a \leqq z_3$, *then* $d^*(a) = 0$;

(4) $d^*(z_4) = \infty\phi(z_4)$ *and there exists* a_0 *such that* $e(a_0) = z_4$, $d^*(a_0) = 0$.

PROOF. Let L be of type I. Put $A_1 = \{\omega \in \Omega ; d^*(h, \omega) < \infty\}$, $A_2 = \{\omega \in \Omega ; d^*(h, \omega) > 0\}$. Since $\bar{A}_1$, $\bar{A}_2$ are open and closed, there are z_1, z_2, $\bar{z} \in Z_0$ such that $\bar{A}_1 \frown \bar{A}_2 = E(z_1)$, $\Omega - \bar{A}_1 = E(z_2)$. $\Omega - \bar{A}_2 = E(\bar{z})$. Clearly $z_1 \oplus z_2 \oplus \bar{z} = 1$. Next put $A_3 = \{\omega \in E(\bar{z}) ; d^*(\bar{z}, \omega) > 0\}$ and $\bar{A}_3 = E(z_4)$, $\bar{z} - z_4 = z_3$.

We have $0 < d^*(h, \omega) < \infty$ a. e. on $E(z_1)$. Hence as in the proof of Theorem 5.3, it is easy to show that d^* is a dimension function on $L(0, z_1)$.

We have $d^*(h, \omega) \equiv \infty$ on $E(z_2)$ and hence if $a \leqq z_2$, then $d^*(a, \omega) \equiv \infty$ on $E(q_1(h, a))$. But $q_1(h, a) = 1 - r_0(h, a) = e(a)$. Therefore $d^*(a) = \infty\phi(e(a))$.

We have $d^*(z_3) = 0$, and hence if $a \leqq z_3$, then $d^*(a) = 0$.

We have $d^*(h, \omega) \equiv 0$ on $E(z_4)$. So setting $a_0 = z_4 \frown h$, we have $e(a_0) = z_4$, $d^*(a_0) = 0$. If $a \leqq z_4$, then $d^*(a, \omega) = 0$ on $\Omega - E(q_n(h, a))$ for all n. Hence if a is finite, $d^*(a) = 0$. But if $z_4 \neq 0$, then $d^*(z_4, \omega) > 0$ a. e. on $E(z_4)$. Hence z_4 is properly infinite and then $d^*(z_4, \omega) = 0$ or ∞. Therefore $d^*(z_4) = \infty\phi(z_4)$.

Let L be of type II. We use c_0 instead of h (see the proof of Theorem 5.1), and get z_i as above. Then it is clear that (1), (3), (4) is true. Regarding (2) we have $d^*(c_0, \omega) \equiv \infty$ on $E(z_2)$, so $d^*(c_k, \omega) \equiv \infty$ on $E(z_2)$. Hence if $a \leqq z_2$, then $d^*(a, \omega) \equiv \infty$ on $E(q_1(c_k, a))$. Put $z = \bigwedge_k r_0(c_k, a)$. Since $2^k[z \frown a] \leqq [z \frown c_0]$ for all k, we have $z \frown a = 0$ and then $e(a) \leqq 1 - z = \bigvee_k q_1(c_k, a)$. Therefore $d^*(a) = \infty\phi(e(a))$.

Let L be of type III. Set $\bar{z} = \bigvee(e(a) ; d^*(a) = 0)$. We may write $\bar{z} = \oplus(e(a_\alpha) ; \alpha \in I)$ with $d^*(a_\alpha) = 0$. Put $\bar{a} = \oplus(a_\alpha ; \alpha \in I)$. Then $e(\bar{a}) = \bar{z}$, and clearly $d^*(\bar{a}) = 0$. Put $z_1 = 1 - \bar{z}$. If $a \leqq z_1$ and $d^*(a) = 0$, then $e(a) \leqq \bar{z}$, so $a = 0$. Thus (4°) in Definition 5.2 is true for $a \leqq z_1$, whence d^* is a dimension function on $L(0, z_1)$. .

We get a decomposition $\bar{z} = z_3 \oplus z_4$ as above. Clearly (3) is true, and $e(z_4 \cap \bar{a}) = z_4$, $d^*(z_4 \cap \bar{a}) = 0$. Since z_4 is zero or properly infinite, we have $d^*(z_4) = \infty \phi(z_4)$. This completes the proof.

COROLLARY 1. *d^* is completely additive if and only if the following condition is satisfied:*

(6°) *if $a_\delta \uparrow a$, then $d^*(a_\delta) \uparrow d^*(a)$.*

PROOF. The "if" part is obvious since d^* is finitely additive. To prove the converse, let d^* be completely additive and let $z_1 \oplus z_2 \oplus z_3 \oplus z_4 = 1$ be the decomposition for d^* in Theorem 6.1. If $z_4 \neq 0$, then by Theorem 6.1 (4) there is $a_0 \neq 0$ such that $e(a_0) = z_4$, $d^*(a_0) = 0$. Using $(2, \mathcal{E})$ and Zorn's lemma we have an independent system $\{a_\alpha ; \alpha \in I\}$ such that $a_\alpha \lesssim z_4$, $a_\alpha \lesssim a_0$ for all $\alpha \in I$ and $\oplus_\alpha a_\alpha = z_4$. Since $d^*(a_\alpha) = 0$ for all $\alpha \in I$, we have $d^*(z_4) = 0$, a contradiction. Therefore $z_4 = 0$. d^* satisfies (6°) on $L(0, z_1)$ by Theorem 5.5, and it is easy to show that d^* satisfies (6°) on $L(0, z_2)$ and $L(0, z_3)$. Therefore d^* satisfies (6°) on L. This completes the proof.

From this proof it is clear that d^* is completely additive if and only if $z_4 = 0$. We can see that there exists d^* such that $z_4 \neq 0$, so that d^* is not completely additive. For example, let L be of type $\mathrm{I}_{(\infty)}$ (or $\mathrm{II}_{(\infty)}$) and $d^*(a) = \infty \phi(e^i(a))$. It is easy to show that d^* has the properties (1°)—(3°) and $z_4 = 1$.

COROLLARY 2. *d^* satisfies the condition (4°) in Definition 5.2 if and only if d^* satisfies the condition (6°) and the following condition:*

(7°) *if $z \in Z_0$, $z > 0$, then $d^*(z) > 0$.*

PROOF. By the Theorem 6.1 d^* satisfies (4°) if and only if $z_3 = z_4 = 0$, and d^* satisfies (7°) if and only if $z_3 = 0$. It follows from the proof of Corollary 1 that d^* satisfies (6°) if and only if $z_4 = 0$. This completes the proof.

This corollary shows that if d^* satisfies (4°) then d^* is completely additive.

The axioms of Segal's dimension functions ([15, p. 405], (1)—(5)) correspond to (1°), (2°), (3°), (6°), (7°), (5°) and the converse of (5°). As above, (6°), (7°) may be replaced by (4°), and by Theorem 5.2 (i) the converse of (5°) may be omitted. Thus the Segal's axioms are equivalent to (1°)—(5°) in Definition 5.2.

Moreover using Theorem 6.1 we otain:

COROLLARY 3. *Suppose $z_{\mathrm{III}} = 0$, and let d_0 be a dimension function on L. There exists a one-to-one correspondence between d^* satisfying (6°) and $f \in \boldsymbol{Z}$, where the correspondence is given by the equation $d^*(a) = f \cdot d_0(a)$, $a \in L$. d^* satisfies (4°) (resp. (5°)) if and only if $f(\omega) > 0$ a. e. (resp. $f(\omega) < \infty$ a. e.).*

PROOF. For $f \in \boldsymbol{Z}$ clearly $f \cdot d_0$ satisfies (1°)—(3°) and (6°). Conversely if d^* satisfies (6°), then $z_4 = 0$. Since d^* is a dimension function on $L(0, z_1)$, by Theorem 5.3 there is a continuous function f_1 on $E(z_1)$ such that $0 < f_1(\omega) < \infty$ a. e. and $f_1(\omega) d_0(a, \omega) \equiv d^*(a, \omega)$ on $E(z_1)$. Set

$$f(\omega) \equiv f_1(\omega) \text{ on } E(z_1)\,;\; f(\omega) \equiv \infty \text{ on } E(z_2)\,;\; f(\omega) \equiv 0 \text{ on } E(z_3).$$

Then $f \in \boldsymbol{Z}$ and $f \cdot d_0 = d^*$ by Theorem 6.1 (2), (3).

Since $z_{\mathrm{III}} = 0$, there is a finite element b with $e(b) = 1$. Clearly $0 < d_0(b, \omega) < \infty$ a. e. If $f_1 \cdot d_0 = f_2 \cdot d_0$, then $f_1 \cdot d_0(b) = f_2 \cdot d_0(b)$, whence $f_1 = f_2$. Therefore the correspondence is one-to-one.

d^* satisfies (4°) (resp. (5°)) if and only if $z_3 = 0$ (resp. $z_2 = 0$) which is equivalent to $f(\omega) > 0$ a. e. (resp. $f(\omega) < \infty$ a. e.).

Let φ be a pseudo-application ♮ on an operator ring (see Dixmier [4, p. 25]). Then it is clear that the restriction of φ to the projections satisfies the axioms (1°), (2°) and (3°). The conditions "normal", "faithful" correspond to the conditions (6°), (4°) respectively and the following theorem shows that the condition "essential in semi-finite part" corresponds to the condition (5°).

THEOREM 6.2 *d^* satisfies (5°) in Definition 5.2 if and only if the following condition is satisfied: if $0 \neq a \leqq 1 - z_{\mathrm{III}}$, then there exists $a_1 \neq 0$, $a_1 \leqq a$ such that $d^*(a_1)$ is bounded valued.*

PROOF. Let d^* satisfy (5°). By Theorem 2.2 there is a finite element b such that $e(b) = 1 - z_{\mathrm{III}}$. If $0 \neq a \leqq 1 - z_{\mathrm{III}}$, then $e(a) \cap e(b) = e(a) \neq 0$. So there exist $a' \neq 0$ such that $a \geqq a' \lesssim b$. Since a' is finite, $d^*(a')$ is finite valued a. e. Then for sufficiently large number K, $A = \{\omega \in E(e(a'))\,;\, d^*(a', \omega) < K\} \neq 0$. Since $\bar{A}$ is open and closed, there is $z \in Z_0$ such that $E(z) = \bar{A}$. Since $0 \neq z \leqq e(a')$, we have $0 \neq z \cap a' \leqq a$ and clearly $d^*(z \cap a')$ is bounded valued.

To prove the converse, let $a \neq 0$ be finite. Then $a \leqq 1 - z_{\mathrm{III}}$. If there exists $0 \neq z \in Z_0$ such that $d^*(a, \omega) \equiv \infty$ on $E(z)$, then $z \cap a \neq 0$ and then there is $c \neq 0$, $c \leqq z \cap a$ such that $d^*(c) \leqq K$ for some K. By Theorem 4.1 $\bigwedge_n q_n(c, a) = e^i(a) \cup (1 - e(c)) = 1 - e(c) \neq 1$, whence $q_n(c, a) \neq 1$ for some n. Let $\omega \in \Omega - E(q_n(c, a))$. Then we have $d^*(a, \omega) \leqq n d^*(c, \omega) \leqq nK$. But since $1 - q_n(c, a) \leqq e(c) \leqq z$, we have $d^*(a, \omega) = \infty$, a contradiction. Therefore $d^*(a, \omega) < \infty$ a. e. If $a = 0$, then $d^*(a) = 0$. Thus d^* satisfies (5°).

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