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Note on a Simple Closed Curve Bounding a Pseudo-Projective Plane

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Introduction

A 3-*manifold* is a separable metric space each of whose points has a closed neighbourhood homeomorphic to a 3-cell. Hereafter we do not assume any conditions about connectedness, compactness, orientability and boundary for 3-manifolds except for Corollary 2 in Section 1. Any 3-manifold to be considered in the following is supposed without loss of generality to have a fixed triangulation. Moreover, every thing will be considered from the semi-linear point of view. For example, mappings are semi-linear, curves are polygonal and surfaces are polyhedral, and so on.

Let m be a positive integer and let P be a disc in euclidean plane E^2 which is mapped onto itself by each rotation r_a in E^2 defined by the equations:

$$x'=x\cos\left(\frac{a}{m}\cdot 2\pi\right)-y\sin\left(\frac{a}{m}\cdot 2\pi\right),$$
$$y'=x\sin\left(\frac{a}{m}\cdot 2\pi\right)+y\cos\left(\frac{a}{m}\cdot 2\pi\right) \qquad (a=1,\cdots,m).$$

Then if we identify each pair of points p, p' on [1] Bd P such that $r_a(p)=r_{a'}(p')$ for some a, a', we have from P a regular connected 2-dimensional polyhedron P_m, called a *pseudo-projective plane*, cf. [1] p. 266. In particular P_1 is a disc and P_2 is a projective plane. The above identification defines a mapping f of P onto P_m in a natural way. Then the set f (Bd P) is a simple closed curve L in P_m and is called a *boundary curve* of P_m.

In the previous paper [5], we gave a necessary and sufficient condition that a simple closed curve in a 3-manifold M bounds a disc in M. We now show in this note a property of a boundary curve L of P_m in M (Theorem 1) and give a condition that $L\subset M$ should be a boundary curve of $P_m\subset M$ (Theorem 2). We shall make use of the method of diagrams in proving these theorems. Furthermore at the end of this note, a remark for [5] will be added.

1. Pseudo-projective planes

First we indicate several definitions for diagrams. Let D be a singular

(1) Bd means boundary.

disc (i.e. a disc which may have self-intersections) in a 3-manifold M. Then by definition D is the image of a disc $\tilde{D}$ under a mapping $f: \tilde{D} \rightarrow M$ which takes Bd $\tilde{D}$ onto Bd D. The pair $(\tilde{D}, f)$ is called the *diagram* of the singular disc D. Now let a, b be points in M and let C, C' be oriented curves in M from a to b. If C is homotopic to C' in M, there exists a singular disc D such that $C^{-1} C'$ is the boundary of D. The diagram $(\tilde{D}, f)$ of D is called the *diagram* of this homotopy. Furthermore if C, C' are closed curves in M and if C is homotopic to C' in M, there exist an annulus $\tilde{A}$ and a mapping $f: \tilde{A} \rightarrow M$ such that $f(\tilde{C})=C$, $f(\tilde{C}')=C'$, where $\tilde{C}$, $\tilde{C}'$ are two components of Bd $\tilde{A}$. The pair $(\tilde{A}, f)$ is also called the *diagram* of the homotopy.

THEOREM 1. *Let P_m be a (polyhedral) pseudo-projective plane in a 3-manifold M and L a boundary curve of P_m. Then L is not homotopic in M to any closed curve disjoint P_m if $m \geqq 3$. This is also true for $m=2$ if a neighbourhood of L is orientable.*

PROOF. Suppose on the contrary that L is homotopic in M to a closed curve L' disjoint P_m. Deform P_m slightly to get a pseudo-projective plane P'_m, with L as its boundary curve, such that $(P'_m - L) \cdot \text{Bd } M = \emptyset$ and $P'_m \cdot L' = \emptyset$. In fact we show first that there exists a pseudo-projective plane P''_m, with L as its boundary curve, such that $(P''_m - L) \cdot \text{Bd } M$ is at most one dimensional polyhedron and $P''_m \cdot L' = \emptyset$. Suppose S is a 2-simplex in $P_m \cdot \text{Bd } M$. We can select a point e of [2] Int M near to S in such a way as the tetrahedron T constructed by e and S does not intersect L', $T \cdot \text{Bd } M = S$ and $(P_m + \text{Bd } T) - \text{Int } S$ is a pseudo-projective plane with boundary curve L. By using induction on the number of these 2-simplexes on $P_m \cdot \text{Bd } M$ we get such a P''_m. Next applying the similar way as above to each 1-simplex in the closure of $P''_m \cdot \text{Bd } M - L$ we have from P''_m a pseudo-projective plane P'''_m, with L as its boundary curve, such that $P'''_m \cdot L' = \emptyset$ and $P'''_m \cdot \text{Bd } M - L$ consists of a finite number of points. Finally if at each of these points we sink slightly P'''_m into Int M we have the desired P'_m.

Since $P'_m - L$ is an open disc, there exists a tubular neighbourhood V of L such that $L' \cdot \bar{V} = \emptyset$, $P'_m - V$ is a disc D and Bd D is a simple closed curve going around $\bar{V}$ longitudinally m times. We note here that $\bar{V}$ may be non-orientable for the case where M is non-orientable and hence Bd V is a Klein bottle. Therefore the following three cases are possible:

(a) Bd $V -$ Bd D, denoted by R, is an open annulus whose boundary consists of two copies C, C' of Bd D. Each closed curve J in R is homologous to a BdD in $\bar{R}$, for some integer a. If V is orientable, only this case may happen.

(b) R is an open Moebius band whose boundary is a simple closed curve covering twice Bd D. J in $\bar{R}$ is homologous to a Bd D for some integer a in $\bar{R}$.

(c) R consists of two open Moebius bands R_1, R_2 whose boundaries are

(2) Int means interior.

copies of Bd D. For J in R, [3] $J\sim a$ (Bd $D/2$) in $\bar{R}_i$ for some integer a and i (=1 or 2). This case is possible only for an even m.

Since L is homotopic to L' in M, $(L+L')\cdot D=\emptyset$ and D is a disc in Int M, [4]$L\simeq L'$ in $M-D$. Let $(\tilde{A},f)$ be the diagram of the homotopy and suppose that a component $\tilde{L}$ of the boundary of the annulus $\tilde{A}$ is taken homeomorphically onto L by f. Since f is semi-linear (cf. Introduction), $f^{-1}(\bar{V}\cdot f(\tilde{A}))$ consists of a finite number of simplexes of $\tilde{A}$. A slight adjustment of $f(\tilde{A})$ in the vicinity of Bd $V-D$ is enough to cause the adjusted $f(\tilde{A})$ to be such a polyhedron that each component of $f^{-1}(\bar{V}\cdot f(\tilde{A}))$ is a perforated disc (i.e. a set homeomorphic to a 2-sphere minus the sum of a finite number of mutually exclusive open discs). Let K be the component of $f^{-1}(\bar{V}\cdot f(\tilde{A}))$ containing $\tilde{L}$ and let $\tilde{C}_1,\cdots,\tilde{C}_k$ be the components of Bd K except for $\tilde{L}$. Then we can select orientations of $\tilde{L}$, $\tilde{C}_1,\cdots,\tilde{C}_k$ in such a way as

$$\tilde{L}\sim\tilde{C}_1+\cdots+\tilde{C}_k \text{ in } \tilde{A}. \tag{1}$$

Since $f(\tilde{C}_i)$ are closed curves in R (=Bd $V-$Bd D) and Bd $D\sim mL$ in $\bar{V}$ for each i there exists an integer a_i such that $f(\tilde{C}_i)\sim a_i m L$ in $\bar{V}$ (for Case (a) and (b)) or $f(\tilde{C}_i)\sim a_i(m/2)L$ in $\bar{V}$ (for Case (c)). On the other hand if we give the orientations induced by f from those of $\tilde{L}$, $\tilde{C}_1,\cdots,\tilde{C}_k$ to L, $f(\tilde{C}_1),\cdots,f(\tilde{C}_k)$, respectively, by (1)

$$L\sim f(\tilde{C}_1)+\cdots+f(\tilde{C}_k) \text{ in } \bar{V}.$$

Hence

$$L\sim(a_1+\cdots+a_k)mL \text{ in } \bar{V} \text{ (for Case (a) and (b))} \tag{2}$$

or

$$L\sim(a_1+\cdots+a_k)(m/2)L \text{ in } \bar{V} \text{ (for Case (c))}. \tag{3}$$

Since $m\geqq 2$ for (2) and $m\geqq 4$ for (3) by our assumption, and since L is the free generator of $H_1(\bar{V})$, (2) and (3) are impossible. This contradiction arose from the false assumption that $L\simeq L'$ in M.

Q. E. D.

COROLLARY 1. *Any 3-manifold M whose fundamental group $\pi_1(M)$ is* [5]*locally free does not contain pseudo-projective planes P_m ($m\geqq 2$).*

PROOF. Suppose P_m is embedded in such a 3-manifold M and let L be a boundary curve of P_m. Then L not $\simeq 0$ in M by the above theorem and hence the subgroup G of $\pi_1(M)$ generated by L is non-trivial. On the other hand

(3) $\sim$ means homologous to.
(4) $\simeq$ means homotopic to.
(5) A group G is said to be *locally free* provided that each finitely generated subgroup of G is free. We suppose the trivial group is locally free.

$L^m \simeq 0$ in M, because L is a boundary curve of P_m, and hence G is not free. This contradiction proves the corollary. Q. E. D.

REMARK. Corollary 1 is obtained by a result of Papakyriakopoulos, [4] p. 20, Theorem (31, 2), for the case where M is orientable.

Now we examine an application of Corollary 1 to the well-known result for a simply connected lens space (for alternate proofs cf. [2] p. 31, Theorem 3, [5] p. 213, Corollary 1). Let T be a solid torus. There exists a homeomorphic mapping h of the topological product $D \times C$ onto T, where D is the disc: $x^2+y^2 \leqq 1$ in E^2 and C is a simple closed curve. The simple closed curve $L=h((0, 0)\times C) \subset T$ is called a *loop of core* of T. Define a retraction r: $(D\times C) \times I \to D \times C$ by $r((x, y), z; t)=((tx, ty), z)$, where I is the unit interval $0 \leqq t \leqq 1$, $(x, y) \in D$ and $z \in C$. Then $hr(h^{-1}(T)\times I)=s(T\times I)$ is a retraction of T to L. If N is a subset of T, $s(N\times I)$ is called the *trace* of N by s.

COROLLARY 2. *A connected, simply connected*(6) *closed 3-manifold M is topologically S^3 if M is the sum of two tori T_1, T_2 such that $T_1 \cdot T_2 = \mathrm{Bd}\, T_1 = \mathrm{Bd}\, T_2$.*

PROOF. Let D be a meridian disc of T_2 and L a loop of core of T_1. Let R be the trace of Bd D under the retraction of T_1 to L defined above. If Bd D circles T_1 longitudinally m times, $D+R$ is a pseudo-projective plane P_m with boundary curve L. Therefore we conclude $m=1$ by virtue of Corollary 1. Now let Q be a 3-cell in T_2 such that Q contains D and $Q\cdot T_1$ is an annulus. Then both $Q+T_1$ and the closure of T_2-Q are 3-cells with common boundary and M is the sum of these 3-cells. Hence M is topologically S^3.

Q. E. D.

2. A Simple closed curve bounding P_m

We describe again the definition of ρ-homotopy defined in [5] p. 205, by using the term of diagram given in Section 1. Let N be a connected set in a 3-manifold M, a, b points on N and C a curve joining a to b such that $C-(a+b) \subset M-N$. Suppose that C is homotopic in M to a curve C' on N from a to b and let $(\tilde{D}, f)$ be the diagram of the homotopy. If $f(\mathrm{Int}\,\tilde{D})\cdot N=\emptyset$, it is called that C is *ρ-homotopic* to C' in M, with respect to N.

Moreover if for each of such a triple a, b, C there exists a curve C' as above, we denote this fact by $\pi_1(M-N, N)=1$.

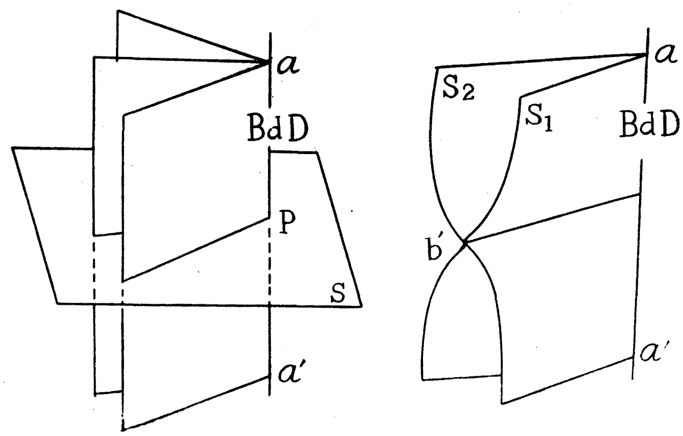
If N is a simple closed curve, we may suppose that C' has no turning point [5], p. 211. In the following, curves on a simple closed curve mean those without turning point.

The definitions used below of double curves, triple points, branch points

(6) Closed means compact and without boundary.

of the 1st kind and of the 2nd kind are due to [5] pp. 206–207. We shall define several new singularities of singular discs which will appear in the proof of Theorem 2.

Let D be a singular disc. If D has as one of its singularities the curve aa' on Bd D such that it is on k sheets and is a side of each sheet, we call aa' a *k-fold curve* (Fig. 1 (a)). Furthermore if a sheet s intersects a k-fold curve aa' at only one point p (Fig. 1 (a)), we call p a *k-fold piercing point* or simply a *piercing point.*



(a) (b)
Fig. 1

Let aa' be a 2-fold curve and let s_1, s_2 be two sheets containing aa' and actually crossing one another in the double curve bb' with $b \in aa'$ as an end point (Fig. 1 (b)). Then b is called a *branch point of the 3rd kind.* We note that a 1-fold piercing point is equal to "piercing point" defined in [5] p. 207.

THEOREM 2. *Let M be a 3-manifold, L a simple closed, polygonal curve in M. Then L is a boundary curve of a pseudo-projective plane if and only if there exists a neighbourhood U of L such that for an integer s $(\neq 0)$, $L^s \simeq 0$ in U and $\pi_1(U-L, L)=1$.*

PROOF. Necessity. Suppose L is a boundary curve of a pseudo-projective plane P_m in M. Let V be the tubular neighbourhood of L as defined in the proof of Theorem 1. Then we find a 3-cell V' such that V' contains the open disc $P_m - V$ and $V' \cdot \bar{V}$ is an annulus on Bd V containing Bd $(P_m - V)$. It will suffice to take $\bar{V} + V'$ as U.

Sufficiency. It is called that a singular disc is normal if the singularities consist of at most double curves, triple points, branch points of the 1st kind, of the 2nd kind, piercing points, k-fold curves and branch points of the 3rd kind. We enumerate the steps in the proof.

(i) A normal singular disc D_1. Let $(\tilde{D}_1, g)$ be the diagram of the homotopy $L^s \simeq 0$ in U. Then Bd $\tilde{D}_1$ covers L s times by g. By repeating a finite number of piecewise adjustments of the singular disc $g(\tilde{D}_1)$, we have a normal singular disc D_1 in U whose boundary circles L s times. (cf. [3] pp. 147–148, [4] p. 3, [5] p. 206–207).

(ii) Reduction of the number of piercing points. In this step we show that there exists a normal singular disc D_2, with boundary on L, containing at most one piercing point. We make use of induction on the number of these piercing points.

Suppose that D_1 has two or more piercing points and let $(\tilde{D}_1, f_1)$ be the diagram of D_1. We note that if p is a k-fold piercing point of D_1, $f_1^{-1}(p)$ consists of exactly $k+1$ points such that one of them lies in Int $\tilde{D}_1$ and the others are on Bd $\tilde{D}_1$. Hence the number of all the points in Int $\tilde{D}_1$, each of which is mapped on a piercing point of D_1, is equal to that of piercing points in D_1.

Let $\tilde{J}$ be a [7] cross cut of $\tilde{D}_1$ with end points $\tilde{a}$, $\tilde{b}$ such that each of components of $(\tilde{D}_1-\tilde{J})\cdot \text{Int}\,\tilde{D}_1$ intersects $f_1^{-1}(P)$, where P is the set of piercing points in D_1 and $\tilde{J}\cdot f_1^{-1}(P)=\emptyset$. Since $f_1(\tilde{J})$ is a curve in U such that $(f_1(\tilde{J})-(a+b))\cdot L=\emptyset$, where $a=f_1(\tilde{a})$ and $b=f_1(\tilde{b})$, and since $\pi_1(U, U-L)=1$, $f_1(\tilde{J})$ is ρ-homotopic to a curve with end points a, b in U, with respect to L. Let $(\tilde{D}_1', f_1')$ be the diagram of the homotopy. We may suppose without loss of generality that $\tilde{J}$ is a subarc of Bd $\tilde{D}_1'$. Then there exists a component $\tilde{D}_1''$ of $\tilde{D}_1-\tilde{J}$ as follows: If we denote the disc $\tilde{D}_1'+\tilde{D}_1''$ by $\tilde{D}_2$ and define the mapping $f_2\colon \tilde{D}_2\to U$ in such a way as $f_2=f_1'$ on $\tilde{D}_1'$ and $f_2=f_1$ on $\tilde{D}_1''$, $f_2(\text{Bd}\,\tilde{D}_2)$ is a closed curve without turning point. Thus we have a singular disc $f_2(\tilde{D}_2)$ with boundary on L, having fewer piercing points than D_1, because $L\cdot\text{Int}\,f_1'(\tilde{D}_1')=\emptyset$. Again apply the above step, which may destroy the normality, to $f_2(\tilde{D}_2)$ and $\tilde{D}_2$.

Hence we obtain a singular disc, not necessarily normal, with boundary on L and with at most one piercing point. By applying Step (i) to the resulted singular disc so that new piercing points do not appear, there results the desired D_2.

(iii) Removing of branch points of the 1st kind and of the 2nd kind. In this step, we obtain a normal singular disc D_3 with at most one piercing point and having neither branch point of the 1st kind nor of the 2nd kind.

First we remove branch points of the 2nd kind from D_2 (cf. [5] (ii) pp. 207–208) and reduce the multiplicity of each branch point of the 1st kind to 1 (cf. [4] p. 5, [5] (iii) pp. 208–209). The resulted singular disc is also denoted by D_2. Now let b be a branch point of the lst kind in D_2. If we pass along the double curve C issuing from b and go through triple points on our way, we reach (a) a branch point of the lst kind or (b) a piercing point or (c) a branch point b' of the 3rd kind. By cutting D_2 along C, b is removed from D_2 as

(7) A *cross cut* of a disc D is an arc in D joining two points on Bd D and whose interior is in Int D.

follows:

(a) cf. [5] (iv) (a), pp. 209–210.

(b) cf. loc. cit. (iv) (b), p. 210.

(c) $f_2^{-1}(C)$ consists of two arcs $\tilde{C}_1$, $\tilde{C}_2$ joining $f_2^{-1}(b)=\tilde{b}$ to $\tilde{b}'_1$, $\tilde{b}'_2$ respectively, where $f_2^{-1}(b')=\tilde{b}'_1+\tilde{b}'_2$ (Fig. 2). The arrows in the figure show two copies of the direction of C along which we have passed. Let us denote by $\tilde{D}''_2$ one of two discs in $\tilde{D}_2$ into which the cross cut $\tilde{C}_1+\tilde{C}_2$ of $\tilde{D}_2$ divides. Now match $\tilde{C}_1$ and $\tilde{C}_2$ in such a way as each pair of points $x_1 \in \tilde{C}_1, x_2 \in \tilde{C}_2$ such that $f_2(x_1)=f_2(x_2)$ is identified, and denote the disc thus obtained from $\tilde{D}''_2$ by $\tilde{D}'_2$. If we denote by f'_2 the mapping $\tilde{D}'_2 \to U$ induced from f_2 in a natural way, $f'_2(\tilde{D}'_2)$ is a singular disc, with boundary on L, whose branch points of the lst kind are fewer than D'_2.

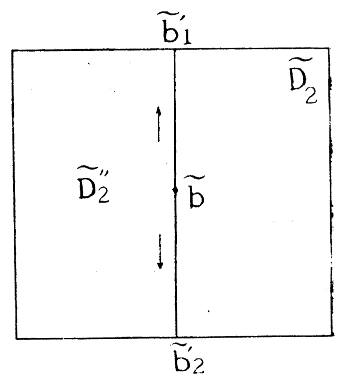


Fig. 2

Repeating a finite number of times the above process we finally obtain the desired D_3.

(iv) A normal singular disc D_4 with no piercing point. Suppose p is the k-fold piercing point of D_3 and let C be one of k double curves issuing from p. If we go along C and pass through triple points on our way, the following two cases may happen:

(a) We come again to the initial point p.

(b) C ends at a branch point of the 3rd kind.

In the diagram $(\tilde{D}_3, f_3)$ of D_3, $f_3^{-1}(C)$ consists of two curves $\tilde{C}_1$ and $\tilde{C}_2$. Let $\tilde{p}_1 \in \text{Int } \tilde{D}_3$, $\tilde{p}_i \in \text{Bd } \tilde{D}_3$ $(i=2,\cdots, k+1)$ be the inverse image of p under f_3^{-1}. Case (a) is divided into the following two cases (a_1) and (a_2).

Case (a_1): $\tilde{C}_1$ is a simple closed curve containing $\tilde{p}_1$ in Int $\tilde{D}_3$ and $\tilde{C}_2$ is a cross cut of $\tilde{D}_3$ such that $\tilde{C}_2 \cdot \tilde{C}_1 = \emptyset$ and Bd $\tilde{C}_2=\tilde{p}_2+\tilde{p}_3$, as shown in Fig. 3 (a_1). Let $\tilde{D}'_3$ be the disc in $\tilde{D}_3$ bounded by $\tilde{C}_1$ and $\tilde{D}''_3$ the disc in $\tilde{D}_3$ which is the

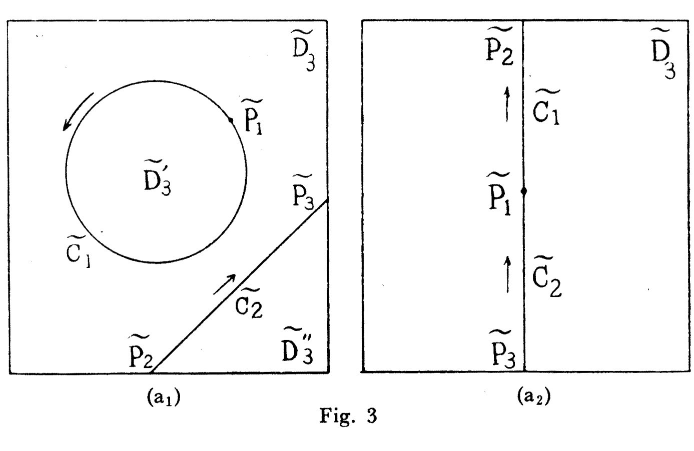


Fig. 3

closure of the component of $\tilde{D}_3-\tilde{C}_2$ not containing $\tilde{C}_1$. Now we construct a new disc $\tilde{D}_4$ gluing $\tilde{D}_3'$ and $\tilde{D}_3''$ along $\tilde{C}_1$ and $\tilde{C}_2$ in such a way as each pair of points $x_1 \in \tilde{C}_1$, $x_2 \in \tilde{C}_2$, such that $f_3(x_1)=f_3(x_2)$, is identified. Let f_4 be the mapping $\tilde{D}_4 \to U$ induced from f_3 in a natural way. Then $f_4(\tilde{D}_4)$ is a normal singular disc without piercing point.

Case (a_2): The diagram $(\tilde{D}_3, f_3)$ is illustrated in Fig. 3 (a_2). $\tilde{C}_1$ ($\tilde{C}_2$) has the end points $\tilde{p}_1$ and $\tilde{p}_2$ ($\tilde{p}_3$) and $\tilde{C}_1+\tilde{C}_2$ is a cross cut of $\tilde{D}_3$. The arrows in the figure show two copies of the direction of C along which we have passed. Since $f_3(\tilde{C}_1)$ is a closed curve in U such that $f_3(\tilde{C}_1)-p \subset U-L$ it is ρ-homotopic in U, with respect to L, to a closed curve on L. Let $(\tilde{D}_3', f_3')$ be the diagram of the homotopy and let $\tilde{C}$ be the arc on Bd $\tilde{D}_3'$ such that f_3' takes $\tilde{C}$ onto the closed curve $f_3(\tilde{C}_1)$. Then we match $\tilde{D}_3'$ with $\tilde{D}_3$ along $\tilde{C}_1$ in such a way that each pair of points $x_1 \in \tilde{C}_1$, $x_2 \in \tilde{C}$ such that $f_3(x_1)=f_3'(x_2)$ is identified. Denote the resulted set by F_1. Moreover we glue $\tilde{D}_3'$ to F_1 at $\tilde{C}_2$ so that the identification is compatible with f_3 and f_3' in $\tilde{C}_2$, and we denote thus obtained set by F_2. Then there exists a component $\tilde{D}_3''$ of $\tilde{D}_3-(\tilde{C}_1+\tilde{C}_2)$ such that the boundary of the disc $F_2-\tilde{D}_3''$ is mapped onto a closed curve on L without turning point by the mapping f_4 induced from f_3 and f_3' on $\tilde{D}_4$.

The singular disc $f_4(\tilde{D}_4)=D_4$ has no piercing point. It may be not normal. By applying Steps (i) and (iii) to D_4 so that no piercing point appears, we have a normal singular disc, denoted by the same notation D_4, without piercing point and having neither branch point of the lst kind nor of the 2nd kind.

Case (b): For this case the diagram $(\tilde{D}_3, f_3)$ is analogous to Fig. 3, and we can remove p from D_3 in the similar way as in Step (iii).

(v) A singular disc D_5, whose singularities on Bd D_5 consists of only k-fold curves, is obtained by this step. Let b be a branch point of the 3rd kind and let C be the double curve issuing from b. Then C will end at a branch point of the 3rd kind. Let $(\tilde{D}_4, f_4)$ be the diagram of D_4 and put $f_4^{-1}(b)=\tilde{b}_1+\tilde{b}_2$, $f_4^{-1}(b')=\tilde{b}_1'+\tilde{b}_2'$. Then $f_4^{-1}(C)$ consists of mutually exclusive cross cuts of $\tilde{D}_4$ whose end points are $\tilde{b}_1, \tilde{b}_1'$; $\tilde{b}_2, \tilde{b}_2'$, respectively (Fig. 4). Let $\tilde{D}_4'$ ($\tilde{D}_4''$) be the component of $\tilde{D}_4-\tilde{C}_1$ ($\tilde{D}_4-\tilde{C}_2$) not containing $\tilde{C}_2$ ($\tilde{C}_1$). If we glue $\tilde{D}_4'$ and $\tilde{D}_4''$ along $\tilde{C}_1$ and $\tilde{C}_2$ in such a way that the identification is compatible with f_4, the resulted disc and f_4 define a new singular disc whose branch points of the 3rd kind are fewer than D_4. By induction we have the desired singular disc D_5.

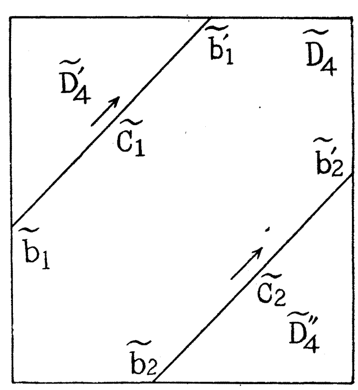


Fig. 4

(vi) Since the singularities of Bd D_5 consist of only m-fold curves, there exists a tubular neighborhood V of L, such that D_5-V is a Dehn disc and its boundary K circles $\bar{V}$ longitudinally m times.

Therefore we can find a disc D' in $U-V$. $D'+V\cdot D_5$ is a pseudo-projective plane P_m.

Q. E. D.

We give a relation of s and m, where s is the integer given in Theorem 2 and m is the suffix of P_m obtained at the end of the proof of the theorem. Suppose L, V, $\tilde{A}$, f, K, $\tilde{C}_1,\cdots,\tilde{C}_k$, $C_1,\cdots,C_k$ and a_i have the same meanings as in the proof of Theorem 1, then we have

$$\tilde{L}\sim\tilde{C}_1+\cdots+\tilde{C}_k \text{ in } \tilde{A},$$
$$sL\sim C_1+\cdots+C_k \text{ in } \bar{V},$$

and $f(\tilde{C}_i)\sim a_i mL$ in $\bar{V}$ (for Cases (a) and (b) in the proof of Theorem 1) or $f(\tilde{C}_i)\sim a_i(m/2)L$ in $\bar{V}$ (for Case (c), loc. cit.). Therefore

$$sL\sim(a_1+\cdots+a_k)mL \text{ in } \bar{V} \tag{1}$$

or

$$sL\sim(a_1+\cdots+a_k)(m/2)\,L \text{ in } \bar{V}. \tag{2}$$

Since L is the free generator of $H_1(\bar{V})$, $s=(a_1+\cdots+a_k)m$ or $s=(a_1+\cdots+a_k)(m/2)$. Therefore we conclude that $1\leqq m\leqq|s|$ (or $1\leqq m\leqq 2|s|$), and m (or $m/2$) is a divisor of $|s|$, because $|s|\geqq 1$ and a_i are integers.

Let us suppose M is orientable in the vicinity of L and L^t not$\simeq 0$ in U for $1\leqq|t|<|s|$. Then $|s|=m$. Furthermore if M is orientable in the vicinity of L, L not$\simeq 0$ in U and $|s|$ is primitive, we know that $|s|=m$.

The theorem in the previous paper [5] is the following

COROLLARY. *Let M be a 3-manifold and L a simple closed curve in M. L bounds a disc in M if and only if there exists a neighbourhood U of L such that $L\simeq 0$ in U and $\pi_1(U, U-L)=1$.*

REMARK. In [5] we did not explicitly describe the case where $L\cdot\text{Bd } M\neq\emptyset$. In this case the method of thickening in the proof of the theorem in [5] is modified as follows: First we show that there exists a simple closed curve L' in Int M near to L such that L' bounds a disc D'. Then we extend D' to a singular disc D with boundary L. By a finite number of piecewise deformations, D is adjusted to a singular disc whose singularities are at most double curves, a finite number of pairs of piercing points of the 1st kind. Again using the property $\pi_1(U-L, L)=1$, we can remove piercing points.

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