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A Remark to Existence and Uniqueness of Certain Stable Solutions of a Weakly Nonlinear System

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1. Introduction.

In this note, we consider the n-dimensional system

$$(1) \qquad x' = Ax + f(t, x) \qquad \left(' = \frac{d}{dt}\right)$$

such that

$1°$ the $k (k \leqq n)$ characteristic roots $\lambda_i (i = 1, 2, \dots, k)$ of the constant matrix A have negative real parts;

$2°$ the remaining $(n-k)$ characteristic roots λ_i $(i = k+1, \dots, n)$ have non-negative real parts;

$3°$ for a certain $\varDelta > 0$,

$$(2) \qquad f(t, x) = (|x|^{1+\varDelta})^{1)}$$

uniformly in t as $|x| \to 0$.

For such a system, in the book of E. A. Coddington and N. Levinson [1], there is stated a theorem[2] as follows:

When λ_i's $(i = 1, 2, \dots, k)$ are arranged so that $R\lambda_i \leqq R\lambda_{i+1}$ $(i = 1, 2, \dots, k-1)$, if $x = \varphi(t)$ is a solution of (1) *and it holds that*

$$\limsup_{t\to\infty} \frac{\log |\varphi(t)|}{t} = b < 0,$$

then there exist integers p and *q $(1 \leqq p \leqq q \leqq k)$ such that*

$$R\lambda_{p-1} < R\lambda_p = R\lambda_{p+1} = \dots = R\lambda_q = b < R\lambda_{q+1}$$

and there exists a $\delta > 0$ and a solution

$$(3) \qquad x = \psi(t) = \sum_{j=p}^{q} Q_j(t) e^{\lambda_j t}$$

of $x' = Ax$ such that

$$(4) \qquad \varphi(t) = \psi(t) + O(e^{(b-\delta)t})$$

as $t \to \infty$. Here $Q_j(t)$'s are column vectors each component of which is a polynomial in t.

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1) $|x|$ means $\sum_{i=1}^{n} |x^i|$, where x^i's are the components of the vector x. In the sequel, we use this convention.

2) In the sequel, we call this theorem the theorem of Coddington and Levinson.

Conversely, if, for any given $\varepsilon>0$, *there exist* $\xi>0$ *and* $T>0$ *such that*

$$|f(t,\tilde{x})-f(t,x)|\leqq\varepsilon|\tilde{x}-x| \tag{5}$$

for $t\geqq T$, $|\tilde{x}|\leqq\xi$ *and* $|x|\leqq\xi$, *then, corresponding to any solution* $x=\psi(t)$ *of the form* (3) *of* $x'=Ax$, *there exists a solution* $x=\varphi(t)$ *of* (1) *which satisfies* (4) *for sufficiently small* $\delta>0$. *Furthermore, if* $p=1$, *then the solution* $x=\varphi(t)$ *of* (1) *is uniquely determined by* $x=\psi(t)$.

In this note, we are concerned with the latter part of this theorem. Namely, in this note, dropping out the condition that $p=1$, we shall give a somewhat more general result on the converse case of the above theorem.

By the assumption, there exists a real matrix P such that

$$PAP^{-1}=B=\begin{pmatrix}B_1 & 0 & 0\\ 0 & B_2 & 0\\ 0 & 0 & B_3\end{pmatrix}, \tag{6}$$

where B_1 is a matrix of order $(p-1)$ of which the characteristic roots are all less than b in their real parts, B_2 is a matrix of order $(q-p+1)$ of which the characteristic roots are all equal to b in their real parts, and B_3 is a matrix of order $(n-q)$ of which the characteristic roots are all greater than b in their real parts. Using this P, let $z=Px$. Then (1) is transformed to the system of the same form as follows:

$$z'=Bz+g(t,z) \tag{7}$$

and the conditions (2) and (5) are changed respectively to the conditions as follows:

1° for certain $L>0$,

$$|g(t,z)|\leqq L|z|^{1+\Delta} \tag{8}$$

uniformly in t as $|z|\to 0$;

2° for any $\varepsilon>0$, there exist $\xi>0$ and $T>0$ such that

$$|g(t,\tilde{z})-g(t,z)|\leqq\varepsilon|\tilde{z}-z| \tag{9}$$

for $t\geqq T$, $|\tilde{z}|\leqq\xi$ and $|z|\leqq\xi$.

Then our result is as follows:

Theorem. *When any solution* $z=\psi(t)$ *of the form* (3) *of* $z'=Bz$ *and any* $(p-1)$*-dimensional vector* α *are given arbitrarily, there exists always one and only one solution* $z=\varphi(t)$ *of* (7) *which satisfies* (4) *and the initial condition*

$$\varphi^i(t_0)=\alpha^i\ (i=1,2,\cdots,p-1), \tag{10}$$

provided that $|\psi(t)|$ *and* $|\alpha|$ *are sufficiently small.*

When $p=1$, the initial condition (10) falls down, consequently the above theorem evidently implies the latter part of the theorem of Coddington and

Levinson.

Since the proof is not given explicitly for the latter part of the theorem of Coddington and Levinson in their book, we shall in this note give a proof in detail for our theorem.

2. Auxiliary theorem.

In order to prove our theorem, in this paragraph, let us prepare a theorem on the integral equation.

Corresponding to the partitions of the matrix B, we define

$$U_1(t)=\begin{pmatrix} e^{tB_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_2(t)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{tB_2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_3(t)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{tB_3} \end{pmatrix}.$$

Then, as is seen in the book of Coddington and Levinson, any solution of (7) which is of the form (4) satisfies the following integral equation

$$\varphi(t)=F[\varphi(t)]\overset{def}{=}U_1(t)c_1+U_2(t)c_2+J_1+J_2, \tag{11}$$

where c_1 and c_2 are respectively $(p-1)$- and $(q-p+1)$-dimensional constant vectors and

$$J_1=\int_{t_0}^{t}U_1(t-s)g\,(s,\varphi(s))ds \quad (t_0\geqq T),$$

$$J_2=-\int_{t}^{\infty}[U_2(t-s)+U_3(t-s)]g(s,\varphi(s))ds.$$

In this paragraph, for the integral equation (11), we prove an

Auxiliary Theorem. *For any* (c_1, c_2) *such that* $|c_1|+|c_2|$ *is sufficiently small, the integral equation* (11) *has always one and only one solution of the form* (4).

Proof. For any Δ_1 such that

$$0<\Delta_1<\Delta, \tag{12}$$

we take a δ so that

$$0<\delta<\frac{-b\Delta_1}{2+\Delta}, \tag{13}$$

and also take a σ so that

$$0<\sigma<\min\left(\delta,\ -b,\ \frac{-b\varDelta_1-\delta}{1+\varDelta_1}\right). \tag{14}$$

Then, if we put

$$\tilde{\sigma}=-b\varDelta_1-\sigma(1+\varDelta_1)-\delta, \tag{15}$$

we have:

$$\begin{aligned}\tilde{\sigma}+\sigma-\delta &= -b\varDelta_1-\sigma(1+\varDelta_1)-\delta+\sigma-\delta\\ &= -b\varDelta_1-\sigma\varDelta_1-2\delta\\ &> -b\varDelta_1-\delta\varDelta_1-2\delta\\ &= -b\varDelta_1-\delta(\varDelta_1+2)>0\end{aligned}$$

because of (13), namely we have

$$\tilde{\sigma}+\sigma>\delta, \tag{16}$$

from which, by (14), it is readily seen that

$$\tilde{\sigma}>0. \tag{17}$$

On the other hand, since

$$\sigma+\tilde{\sigma}=-b\varDelta_1-\sigma\varDelta_1-\delta<-b\varDelta_1,$$

we can choose $\varDelta_1$ so small that

$$\sigma+\tilde{\sigma}<b-R\lambda_{p-1}. \tag{18}$$

Then, for $\varDelta_1$, from (12), it is readily seen that the condition (8) is also valid even when $\varDelta$ is replaced by $\varDelta_1$. Also, for σ and $\tilde{\sigma}$, by (18) and (14), it is readily seen from the definitions of $U_j(t)$'s $(j=1, 2, 3)$ that

$$\begin{aligned}&|U_1(t)|\leqq Ke^{(b-\sigma-\tilde{\sigma})t} && (t\geqq 0),\\ &|U_2(t)|\leqq Ke^{(b+\sigma)t} && (t\geqq 0),\\ &|U_j(t)|\leqq Ke^{(b-\sigma)t} && (t\leqq 0) \qquad (j=2, 3)\end{aligned} \tag{19}$$

for a certain sufficiently large constant $K>0$.

We choose an $\varepsilon>0$ so that

$$r=\varepsilon K\left[\frac{1}{\sigma+\tilde{\sigma}-\delta}+\frac{2}{\delta-\sigma}\right]<1, \tag{20}$$

and, corresponding to this ε, we take $T>0$ and $\xi>0$ so that (9) may hold for $t\geqq T$, $|\tilde{z}|\leqq\xi$ and $|z|\leqq\xi$.

First, by successive approximations, we shall show that the equation (11) has a solution $x=\varphi(t, c_1, c_2)$ in $T\leqq t<\infty$ corresponding to any (c_1, c_2) such that

$$|c_1|+|c_2|=a<\min\left[\frac{\xi}{2K},\ \left(\frac{1}{6}\,\frac{\xi(\delta-\sigma)(1-r)}{LK^{2+\varDelta_1}}\right)^{\frac{1}{1+\varDelta_1}}\right]. \tag{21}$$

For (c_1, c_2) satisfying (21), if we put

$$m=\frac{3LK^{2+\varDelta_1}a^{1+\varDelta_1}}{\delta-\sigma}, \tag{22}$$

then evidently it holds that

$$\text{(23)} \qquad m(1+r+r^2+\cdots)=\frac{m}{1-r}<\frac{\xi}{2}.$$

Let $\varphi_0(t, c_1, c_2)=0$ and let $\varphi_1(t, c_1, c_2)=F[\varphi_0(t, c_1, c_2)]$. Then, from (8), (11) and (19), it follows that, for $t \geqq T$,

$$\text{(24)} \qquad |\varphi_1(t, c_1, c_2)| \leqq Ke^{(b-\sigma-\bar{\sigma})t}|c_1| + Ke^{(b+\sigma)t}|c_2| \leqq Kae^{(b+\sigma)t},$$

from which follows

$$\text{(25)} \qquad |\varphi_1(t, c_1, c_2)| < \frac{\xi}{2}e^{(b+\sigma)t} < \frac{\xi}{2}$$

because of (21) and (14). Then $\varphi_2(t, c_1, c_2)=F[\varphi_1(t, c_1, c_2)]$ is well defined and, from $\varphi_2-\varphi_1=F(\varphi_1)-F(\varphi_0)$, we can estimate $|\varphi_2(t, c_1, c_2)-\varphi_1(t, c_1, c_2)|$ as follows:

$$\text{(26)} \qquad |\varphi_2(t, c_1, c_2)-\varphi_1(t, c_1, c_2)|$$

$$\leqq \int_{t_0}^{t} |U_1(t-s)|\,|g(s, \varphi_1(s))|\,ds + \int_t^{\infty}\Big(|U_2(t-s)| + |U_3(t-s)|\Big)|g(s, \varphi_1(s))|\,ds \qquad \text{(by (8))}$$

$$\leqq LK^{2+\Delta_1}a^{1+\Delta_1}\left[\int_{t_0}^{t} e^{(b-\sigma-\bar{\sigma})(t-s)+(b+\sigma)(1+\Delta_1)s}\,ds + 2\int_t^{\infty} e^{(b-\sigma)(t-s)+(b+\sigma)(1+\Delta_1)s}ds\right] \qquad \text{(by (8), (19), (24))}$$

$$= LK^{2+\Delta_1}a^{1+\Delta_1}\left[e^{(b-\sigma-\bar{\sigma})t}\int_{t_0}^{t} e^{(\sigma-\delta)s}\,ds + 2e^{(b-\sigma)t}\int_t^{\infty} e^{[\sigma-(\bar{\sigma}+\delta)]s}\,ds\right] \qquad \text{(by (15))}$$

$$\leqq LK^{2+\Delta_1}a^{1+\Delta_1}\left[e^{(b-\delta)t}\int_{t_0}^{t} e^{(\sigma-\delta)s}\,ds + 2e^{(b-\sigma)t}\int_t^{\infty} e^{(\sigma-\delta)s}\,ds\right] \qquad \text{(by (17))}$$

$$\leqq \frac{3LK^{2+\Delta_1}a^{1+\Delta_1}}{\delta-\sigma}\,e^{(b-\delta)t}$$

$$= me^{(b-\delta)t} \qquad \text{(by (22))}$$

$$< \frac{\xi}{2}e^{(b-\delta)t} \qquad \text{(by (23)).}$$

Thus, from (25) and (26), we have

$$\begin{aligned}|\varphi_2| \leqq |\varphi_1| + |\varphi_2-\varphi_1| &< \frac{\xi}{2}e^{(b+\sigma)t} + \frac{\xi}{2}e^{(b-\delta)t}\\ &< \xi e^{(b+\sigma)t}\\ &\leqq \xi.\end{aligned}$$

Then, $\varphi_3(t, c_1, c_2)=F[\varphi_2(t, c_1, c_2)]$ is well defined and, by (9) and (26), we have:

$$|\varphi_3(t, c_1, c_2)-\varphi_2(t, c_1, c_2)|$$

$$\leqq mK\varepsilon\left[\int_{t_0}^{t} e^{(b-\sigma-\tilde{\sigma})(t-s)+(b-\delta)s}ds+2\int_{t}^{\infty} e^{(b-\sigma)(t-s)+(b-\delta)s}\,ds\right] \qquad \text{(by (19))}$$

$$\leqq mK\varepsilon\left[\frac{1}{\sigma+\tilde{\sigma}-\delta}+\frac{2}{\delta-\sigma}\right]e^{(b-\delta)t}$$

$$= mre^{(b-\delta)t} \qquad \text{(by (20))}.$$

Then, from (25) and (26), we have

$$\begin{aligned} |\varphi_3(t, c_1, c_2)| &\leqq |\varphi_1|+|\varphi_2-\varphi_1|+|\varphi_3-\varphi_2| \\ &\leqq \frac{\xi}{2}e^{(b+\sigma)t}+m(1+r)e^{(b-\delta)t} \\ &\leqq \frac{\xi}{2}e^{(b+\sigma)t}+\frac{\xi}{2}e^{(b-\delta)t} \qquad \text{(by (23))} \\ &< \xi e^{(b+\sigma)t} \\ &\leqq \xi. \end{aligned}$$

Continuing this process, by induction, we see that

$$\text{(27)} \qquad \begin{cases} |\varphi_k(t, c_1, c_2)-\varphi_{k-1}(t, c_1, c_2)| \leqq mr^{k-2}e^{(b-\delta)t}, \\ |\varphi_k(t, c_1, c_2)| < \xi e^{(b+\sigma)t} \leqq \xi \end{cases} \qquad (k=2, 3, \dots).$$

Then the series

$$\varphi_1(t, c_1, c_2)+\left\{\varphi_2(t, c_1, c_2)-\varphi_1(t, c_1\, c_2)\right\}+\cdots$$

$$+\left\{\varphi_k(t, c_1, c_2)-\varphi_{k-1}(t, c_1, c_2)\right\}+\cdots$$

converges uniformly in $(0<T\leqq)t_0\leqq t<\infty$ with respect to t and (c_1, c_2). Therefore there exists a limit function

$$\text{(28)} \qquad \varphi(t, c_1, c_2)=\lim_{k\to\infty}\varphi_k(t, c_1, c_2)$$

such that

$$|\varphi(t, c_1, c_2)|\leqq \xi e^{(b+\sigma)t}.$$

Then, from the proof of the appraisal of $|\varphi_2(t, c_1, c_2)-\varphi_1(t, c_1, c_2)|$, it is seen that there exists $F[\varphi(t, c_1, c_2)]$. Also, from the latter of (27), it is seen that the improper integral

$$\int_t^{\infty}[U_2(t-s)+U_3(t-s)]g(s, \varphi_k(s))\,ds$$

is equiconvergent with respect to k. Therefore it follows that

$$\lim_{k\to\infty}F[\varphi_k(t, c_1, c_2)]=F[\varphi(t, c_1, c_2)],$$

from which we see that $\varphi(t, c_1, c_2)$ is a solution of the integral equation (11), because

$$\varphi_{k+1}(t, c_1, c_2) = F[\varphi_k(t, c_1, c_2)] \qquad (k = 0, 1, 2, \cdots).$$

Now, since

$$\varphi_k(t, c_1, c_2) = \varphi_1 + (\varphi_2 - \varphi_1) + \cdots + (\varphi_k - \varphi_{k-1}),$$

from the former of (27) follows

$$|\varphi_k(t, c_1, c_2) - U_1(t)c_1 - U_2(t)c_2| \leqq \frac{m}{1-r} e^{(b-\delta)t}$$
$$< \frac{\xi}{2} e^{(b-\delta)t}.$$

Then, from (28), it is evident that

$$|\varphi(t, c_1, c_2) - U_1(t)c_1 - U_2(t)c_2| \leqq \frac{m}{1-r} e^{(b-\delta)t}$$
$$< \frac{\xi}{2} e^{(b-\delta)t}.$$

Then, since $U_1(t)c_1 = O(e^{(b-\delta)t})$ by (19) and $U_2(t)c_2$ can be written as (3), the solution $\varphi(t, c_1, c_2)$ of (11) obtained by the above successive approximations becomes the solution of the form (4). Thus, for the integral equation (11), the existence of a solution of the form (4) has been proved.

Lastly, we shall prove that, corresponding to any (c_1, c_2) satisfing (21), the solution of the form (4) of the integral equation (11) is determined uniquely. In fact, let $x = \varphi_1(t, c_1, c_2)$ and $x = \varphi_2(t, c_1, c_2)$ be such solutions, then, due to (19), from $\varphi_2(t, c_1, c_2) - \varphi_1(t, c_1, c_2) = F[\varphi_2(t, c_1, c_2)] - F[\varphi_1(t, c_1 . c_2)]$, we have:

$$|\varphi_2(t, c_1, c_2) - \varphi_1(t, c_1, c_2)| \leqq \varepsilon K\Big[\int_{t_0}^{t} e^{(b-\sigma-\bar{\sigma})(t-s)} |\varphi_2(s, c_1, c_2) - \varphi_1(s, c_1, c_2)|\, ds$$
$$+ 2\int_{t}^{\infty} e^{(b-\sigma)(t-s)} |\varphi_2(s, c_1, c_2) - \varphi_1(s, c_1, c_2)|\, ds\Big],$$

which can be written as follows:

(29) $$|\varphi_2(t, c_1, c_2) - \varphi_1(t, c_1, c_2)|\, e^{-(b-\delta)t}$$
$$\leqq \varepsilon K\Big[e^{(\delta-\sigma-\bar{\sigma})t} \int_{t_0}^{t} e^{(\sigma+\bar{\sigma}-\delta)s} |\varphi_2(s, c_1, c_2) - \varphi_1(s, c_1, c_2)|\, e^{-(b-\delta)s} ds$$
$$+ 2e^{(\delta-\sigma)t} \int_{t}^{\infty} e^{(\sigma-\delta)s} |\varphi_2(s, c_1, c_2) - \varphi_1(s, c_1, c_2)|\, e^{-(b-\delta)s} ds\Big].$$

Now, by the assumption,

$$\varphi_j(t, c_1, c_2) = O(e^{(b+\sigma)t}) \qquad (j = 1, 2),$$

therefore, from the proof of the appraisal of $|\varphi_2(t, c_1, c_2) - \varphi_1(t, c_1, c_2)|$, it is seen that

(30) $$\varphi_j(t, c_1, c_2) = U_1(t)c_1 + U_2(t)c_2 + O(e^{(b-\delta)t}) \qquad (j = 1, 2).$$

Then $|\varphi_2(t, c_1, c_2)-\varphi_1(t, c_1, c_2)|e^{-(b-\delta)t}$ becomes bounded in $t_0 \leqq t < \infty$, therefore we put

$$\max_{t_0 \leqq t < \infty} \left|\varphi_2(t, c_1, c_2)-\varphi_1(t, c_1, c_2)\right| e^{-(b-\delta)t} = M.$$

Then, from (29), it follows that

$$M \leqq rM$$

which, on account of (20), implies $M=0$, namely $\varphi_2(t, c_1, c_2)=\varphi_1(t, c_1, c_2)$. This proves the uniqueness of the solutions of the form (4) of the integral equation (11).

Thus our auxiliary theorem has been proved.

3. Proof of theorem.

As is stated at the outset of §2, any solution $x=\varphi(t)$ of the form (4) of the equation (7) satisfies the integral equation (11). Consequently for certain c_1 and c_2, it holds that

$$\varphi(t) = \psi(t) + O(e^{(b-\delta)t}) = U_1(t)c_1 + U_2(t)c_2 + J_1 + J_2. \tag{31}$$

Now, as is seen from (30), for any solution $x=\varphi(t)$ of the form (4), it is valid that

$$J_1, J_2 = O(e^{(b-\delta)t}).$$

Therefore, by (19), from (31), it is seen that

$$\psi(t) = U_2(t)c_2. \tag{32}$$

Also, from (31), it follows that

$$\varphi^i(t_0) = \Big[U_1(t_0)c_1\Big]^i \qquad (i=1, 2, \cdots, p-1). \tag{33}$$

Therefore (c_1, c_2) is uniquely determined when $\psi(t)$ and the initial values $\varphi^i(t_0)=\alpha^i$ $(i=1, 2, \cdots, p-1)$ are appointed beforehand. Now, by our auxiliary theorem, the solution of the form (4) of the integral equation (11) is uniquely determined for any given (c_1, c_2), therefore, we see that the solution $x=\varphi(t)$ is uniquely determined when $\psi(t)$ and its initial values $\varphi^i(t_0)=\alpha^i (i=1, 2, \cdots, p-1)$ are appointed beforehand.

Now, as is readily seen, the solution of the form (4) of the integral equation (11) satisfies the initial differential equation (7). Therefore, the solution $x=\varphi(t, c_1, c_2)$ of the form (4) of the integral equation (11) corresponding to (c_1, c_2) determined by (32) and (33) (the existence of such a solution follows from our auxiliary theorem), becomes the desired solution of the initial differential equation (7). This proves the existence of the desired solution of the differential equation (7).

Thus the proof of our theorem is completed.

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Reference

[1] E. A. CODDINGTON and N. LEVINSON, Theory of Ordinary Differential Equations, New York (1955).

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