J. Sci. Hiroshima Univ. Ser. A
Vol. 23, No. 1, April, 1959

On the Theory of Multiplicities in Finite Modules over Semi-Local Rings

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(Received January 20, 1959)

The theory of multiplicities in semi-local rings was originated by Chevalley (in geometric local rings) and continued mainly by Samuel [8] and Nagata [6]. Because of the fact that they used high techniques of the theory of local rings, especially the structure theorem of complete local rings, it is desirable to construct the theory in a more elementary and simpler way.

Godement has shown in [2] that the notion of Hilbert characteristic functions in semi-local rings can be naturally extended to that in finite modules over these rings and Serre continued the study by the homological method [14]. And recently Lech has succeeded in proving the associative formula for multiplicities without using the structure theorem [5]. We develop here systematically, following closely [14] and [5], the full theory of multiplicities in finite modules over semi-local rings and simplify their proofs in some points.

In §1 we generalize the intersection theorem of Zariski to that in finite modules. Some remarks on finite modules are stated in §2. Then, in §3, we shall generalize the theorem of Lech concerning an expression for the multiplicity of a primary ideal, which is generated by a system of parameters, in a local ring to that of a defining ideal in a finite module over a semi-local ring. In the main §4, we shall give fundamental theorems on multiplicities. Finally in §5, we treat the complete tensor product of modules.

Conventions and terminology

i) Unless stated otherwise, we assume throughout this paper that all rings are commutative Noetherian rings with identity elements and that all modules are unitary and finitely generated, therefore Noetherian.

ii) Let A be a ring. We denote by rank A the maximal number n such that there exists a chain of prime ideals of A

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

where each inclusion is strict. For an ideal $\mathfrak{a}$ in A, we define corank $\mathfrak{a} =$ rank $A/\mathfrak{a}$.

iii) Let E be a finite A-module. For an ideal $\mathfrak{a}$ of A and a submodule F

of E, we denote by $\mathfrak{a}F$ the submodule of E generated by elements ax where $a \in \mathfrak{a}$ and $x \in F$. An element a in A (resp. an ideal $\mathfrak{a}$ of A) called zero divisor in E/F if there exists $x \in E$ such that $x \notin F$ and $ax \in F$ (resp. $\mathfrak{a}x \subset F$). Zero submodule of E shall be denoted by 0 and zero ideal of A by (0). $F: \mathfrak{a}$ (resp. $F:E$) means as usual the submodule of E consisting of $x \in E$ such that $\mathfrak{a}x \subset F$ (resp. the ideal of A consisting of $a \in A$ such that $aE \subset F$). As for the notions of prime and primary submodules, we refer the reader to [4]. Then, let $F = \bigcap_{i \in I} Q_i$ be an irredundant primary representation of F in E. $Q_i : E$ is a primary ideal belonging to a prime ideal $\mathfrak{p}_i$ (say). We call $\mathfrak{p}_i$ $(i \in I)$ the associated prime divisors of F in E. The set of $a \in A$ which are zero divisors in E/F is equal to $\bigcup_{i \in I} \mathfrak{p}_i$.

§ 1. Zariski's theorem.

The following lemma is well-known but we state it here because our theory much depends on it.

LEMMA 1. (ARTIN-REES). *Let E be a finite module over a Noetherian ring A and let M and N be submodules of E. Then, for any ideal $\mathfrak{a}$ of A, there exists an integer r such that*

$$\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-r}(\mathfrak{a}^r M \cap N)$$

for any $n \geqq r$.

For the proof we refer the reader to [2, Exposé 2].

COROLLARY 1. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring A and let E be a finite module over A. Put $N = \bigcap_{n=1}^{\infty} \mathfrak{a}^n E$. Then we have $\mathfrak{a}N = N$.*

In fact, by Lemma 1, we have

$$N = \mathfrak{a}^n E \cap N \supseteq \mathfrak{a}(\mathfrak{a}^{n-1}E \cap N) \supseteq \cdots \supseteqq \mathfrak{a}^{n-r}(\mathfrak{a}^r E \cap N) = \mathfrak{a}^n E \cap N = N.$$

Hence
$$N = \mathfrak{a}^n E \cap N = \mathfrak{a}(\mathfrak{a}^{n-1}E \cap N) = \mathfrak{a}N.$$

COROLLARY 2. (*With the same notations and assumptions.*) *The following conditions are equivalent:*

i) $\bigcap_{n=1}^{\infty} \mathfrak{a}^n E = 0.$

ii) *Any element $\alpha \in A$ such that $\alpha \equiv 1$ $(\mathfrak{a})$ is a not zero divisor in E.*

iii) *$\mathfrak{p} + \mathfrak{a} = A$ for any prime divisor $\mathfrak{p}$ of 0 in E.*

PROOF. i)$\rightarrow$ii). Assume there exists an element $\alpha \in A$ such that $\alpha \equiv 1 (\mathfrak{a})$ and $\alpha x = 0$ for some $x \neq 0$, $x \in E$. Then $(1-\alpha)^n x = (1 - n\alpha + n(n-1)\alpha^2 - \cdots + (-1)^n \alpha^n)x = x$, hence $x \in \mathfrak{a}^n E$. Therefore $0 \neq x \in \bigcap \mathfrak{a}^n E$, which is a contradiction.

ii)$\rightarrow$i). Put $N = \bigcap \mathfrak{a}^n E$. Then, by Corollary 1, we have $\mathfrak{a}N = N$. Take a system of generators $x_1, \cdots, x_s$ of N and we can write $x_i (i=1, \cdots, s)$ as

$$x_i = \alpha_{i1} x_1 + \cdots + \alpha_{is} x_s, \quad \text{with } \alpha_{ij} \in \mathfrak{a}.$$

Then $\delta = \det (\delta_{ij} - \alpha_{ij})$ satisfies $\delta x_i = 0$ for $i = 1, \cdots, s$, hence $\delta N = 0$. Since $\delta - 1 \in \mathfrak{a}$, we conclude that $N = 0$.

ii) $\leftrightarrow$ iii). This equivalence follows at once from the fact that $\bigcup_i \mathfrak{p}_i$ ($\mathfrak{p}_i$ runs over the prime divisors of 0) is the set of zero divisors in E.

By passing to the residue module we have the following

COROLLARY 3. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring A. And let E be a finite A-module and N a submodule of E. Denote by $\mathfrak{p}_i (i=1, \cdots, r)$ prime divisors of N. Then the following* i) *and* ii) *are equvalent*:

i) $\bigcap_{n=1}^{\infty} N + \mathfrak{a}^n E = N.$

ii) $\mathfrak{a} + \mathfrak{p}_i \neq A \ (i=1, \cdots, r).$

PROOF. Let $N = Q_1 \cap \cdots \cap Q_r$ be an irredundant primary representation of N. Put $\mathfrak{b} = N : E$ and $\mathfrak{q}_i = Q_i : E \ (i=1, \cdots, r)$. Then $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary and E/N may be considered as a finite $A/\mathfrak{b}$-module. Moreover we see easily that $\bigcup_i (\mathfrak{p}_i/\mathfrak{b})$ is the set of zero divisors in E/N. Therefore, by applying Corollary 2 to the $A/\mathfrak{b}$-module E/N, we see that the following conditions are equivalent:

i′) $\bigcap_n (\mathfrak{a} + \mathfrak{b}/\mathfrak{b})^n (E/N) = 0$, ii′′) $(\mathfrak{p}_i/\mathfrak{b}) + (\mathfrak{a} + \mathfrak{b}/\mathfrak{b}) \neq A/\mathfrak{b} \ (i=1, \cdots, r)$.

From this we see immediately that our conditions i) and ii) are equivalent.

THEOREM 1. (ZARISKI). *Let $\mathfrak{a}$ be an ideal of a Noetherian ring A. And let E be a finite A-module and N a submodule of E. Let $N = Q_1 \cap \cdots \cap Q_r$ be an irredundant primary representation of N in E. If $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ denote the prime divisors of N and if*

$$\mathfrak{p}_i + \mathfrak{a} \neq A \ (i=1, \cdots, s) \quad \textit{and} \quad \mathfrak{p}_i + \mathfrak{a} = A \ (j=s+1, \cdots, r),$$

then we have

$$\bigcap_{n=1}^{\infty} N + \mathfrak{a}^n E = Q_1 \cap \cdots \cap Q_s. \quad [8, \text{p. } 6]$$

PROOF. In the case when $r=s$, the theorem is true by Corollary 3. Therefore we may assume $s<r$. Put $\mathfrak{q}_i = Q_i : E$. Then $(\mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_r) + \mathfrak{a}^n = A \ (n = 1, 2, \cdots\cdots)$. Therefore

$$\begin{aligned} Q_1 \cap \cdots \cap Q_s &= ((\mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_r) + \mathfrak{a}^n)(Q_1 \cap \cdots \cap Q_s) \\ &= ((Q_{s+1} : E) \cap \cdots \cap (Q_r : E) + \mathfrak{a}^n)(Q_1 \cap \cdots \cap Q_s) \\ &= ((Q_{s+1} \cap \cdots \cap Q_r) : E)(Q_1 \cap \cdots \cap Q_s) + \mathfrak{a}^n (Q_1 \cap \cdots \cap Q_s) \\ &\subseteq (Q_{s+1} \cap \cdots \cap Q_r) \cap (Q_1 \cap \cdots \cap Q_s) + \mathfrak{a}^n E = N + \mathfrak{a}^n E. \end{aligned}$$

Conversely, $Q_1 \cap \cdots \cap Q_s = \bigcap_n (Q_1 \cap \cdots \cap Q_s) + \mathfrak{a}^n E \supseteq \bigcap_n N + \mathfrak{a}^n E.$

§ 2. Some remarks on finite modules.

LEMMA 2. *Let A and A' be any two commutative rings with identity elements. Let T be a covariant additive exact functor on the category of A-modules with values in the category of A'-modules and suppose T satisfies the following conditions*:

α) $T(A)=A'$.

β) *If* $F=Ax$ $(x\in F)$, *then* $T(F)=A'x'$ *for some* $x'\in T(F)$.

We fix an A-module E and an A'-module $E'=T(E)$ and confine our attention to submodules of E and E'. Then, for submodules F and G of E, we have

i) $T(F+G)=T(F)+T(G)$.

ii) $T(F/G)\approx T(F)/T(G)$ *if* $G\subseteq F$.

iii) $T(F\cap G)=T(F)\cap T(G)$.

iv) $T(F:G)=T(F):T(G)$ *when G is finitely generated over A.*

PROOF. We first remark that $T(F)$ may be considered a submodule of $T(E)$ provided F is a submodule of E. Therefore ii) holds obviously from the exactness of T. As for i), we see clearly $T(F)+T(G)\subseteq T(F+G)$. Conversely, from $0\to F\to F+G\to (F+G)/F\to 0$ (exact) and $(F+G)/F\approx G/(F\cap G)$, we have $0\to T(F)\to T(F+G)\to T(G/F\cap G)\to 0$ (exact). Therefore $0\to T(F)\to T(F+G)\to T(G)/T(F\cap G)\to 0$ (exact) by ii). Hence $T(F+G)\subseteq T(F)+T(G)$.

iii) From the relations $0\to T(F)\to T(F+G)\to T(F+G)/T(F)\to 0$ (exact) and $T(F+G)/T(F)=(T(F)+T(G))/T(F)\approx T(G)/(T(F)\cap T(G))$, we have $0\to T(F)\to T(F+G)\to T(G)/(T(F)\cap T(G))\to 0$ (exact). Therefore $T(F\cap G)=T(F)\cap T(G)$.

iv) We first show that it is sufficient to prove iv) in the case when G is generated by one element. In fact, let $G=Ax_1+\cdots+Ax_r$. Then

$$\begin{aligned}T(F:G)&=T(F:(Ax_1+\cdots+Ax_r))=T((F:Ax_1)\cap\cdots\cap(F:Ax_r))\\&=T(F:Ax_1)\cap\cdots\cap T(F:Ax_r)=(T(F):T(Ax_1))\cap\cdots\cap(T(F):T(Ax_r))\\&=T(F):(T(Ax_1)+\cdots+T(Ax_r))=T(F):T(Ax_1+\cdots+Ax_r)=T(F):T(G).\end{aligned}$$

Now, let $G=Ax$. Then we see easily that $Ax/(Ax\cap F)\approx A/(F:Ax)$ as A-modules. Therefore $T(G/F\cap G)\approx T(A)/T(F:G)=A'/T(F:G)$ and $T(G/F\cap G)\approx T(G)/T(F\cap G)\approx T(G)/(T(F)\cap T(G))=A'x'/(T(F)\cap A'x')\approx A'/(T(F):A'x')=A'/(T(F):T(G))$ by β). Hence $T(F:G)=T(F):T(G)$.

To show the generality of Lemma 2, we shall give some examples of covariant additive exact functors which satisfy our conditions:

i) Let A be a commutative ring with identity and A' its quotient ring with respect to a multiplicatively closed subset S of A which does not contain zero (usually denoted by A_S). As for the functor T, we set $T(E)=E\otimes_A A_S$[1)] where E is an A-module [11, Lemma 1, p. 241].

ii) Let A be an $\mathfrak{m}$-adic Zariski ring and A' its completion (usually denoted by $\hat{A}$). As for the functor T, we set $T(E)=E\otimes_A\hat{A}$[1)] where E is an A-

1) From now on we shall omit A if any confusion does not occur.

module.

On this occasion we recall the following definition: Given a (finite) module E over a commutative ring A and an ideal $\mathfrak{m}$ of A such that $\cap \mathfrak{m}^n E = 0$, we may topologize E by adopting submodules $\{\mathfrak{m}^n E\}$ $(n = 1, 2, \cdots)$ as a fundamental system of neighborhoods of 0 in E. Thus topologized module is referred to as an $\mathfrak{m}$-*adic module* provided that $\mathfrak{m}$ has a finite ideal basis. In the case where E is finite over an $\mathfrak{m}$-adic Zariski ring A, $E \otimes \hat{A}$ may be considered as an $\mathfrak{m}$-adic completion of E and any submodule F of E is a closed subspace of E, therefore $E \cap (F \otimes \hat{A}) = F$[2)] [2, exposé 18].

By virtue of the above lemma, we can prove the following proposition which is a formal generalization of Lemmas 3 and 4 in [9].

PROPOSITION 1. *Let E be finite module over an $\mathfrak{m}$-adic Zariski ring A. Let P and Q be a prime and a primary submodules of E belonging to the same prime ideal $\mathfrak{p}$ of A. And let $Q \otimes_A \hat{A} = \hat{Q}_1 \cap \cdots \cap \hat{Q}_n$ be an irredundant primary representation of $Q \otimes_A \hat{A}$ in $E \otimes_A \hat{A}$. Then*

i) *$Q \otimes_A \hat{A}$ and $P \otimes_A \hat{A}$ have the same prime divisors.*

ii) *$\hat{\mathfrak{p}}_i \cap A = \mathfrak{p}$ for $i = 1, \cdots, n$ where $\hat{\mathfrak{p}}_i$ denotes the prime divisor of $\hat{Q}_i$.*

iii) *$\hat{Q}_i \cap E = Q$ for $i = 1, \cdots, n$.*

PROOF. Since $(P \otimes \hat{A}) : (E \otimes \hat{A}) = (P : E) \otimes \hat{A} = \mathfrak{p}\hat{A}$, $(Q \otimes \hat{A}) : (E \otimes \hat{A}) = (Q : E) \otimes \hat{A} = \mathfrak{q}A$ where $\mathfrak{p} = P : E$ and $\mathfrak{q} = Q : E$ and since $\hat{\mathfrak{p}}_i (i = 1, \cdots, n)$ are the prime divisors of $\mathfrak{q}\hat{A}$, i) and ii) are immediate consequences of Lemma 3 in [9].

As for iii), clearly $\hat{Q}_i \cap E \supseteq (Q \otimes \hat{A}) \cap E = Q$ by the above remark. If $\hat{Q}_i \cap E \supsetneqq Q$ for some $i (1 \leq i \leq n)$, there exists an element x such that $x \in \hat{Q}_i \cap E$ and $x \notin Q$. Set $\mathfrak{q}' = Q : Ax$. Then we see easily that $\mathfrak{q}'$ is $\mathfrak{p}$-primary as in the ideal theory. Therefore

$$\mathfrak{q}'\hat{A} = (Q : Ax) \otimes \hat{A} = (Q \otimes \hat{A}) : (Ax \otimes \hat{A}) = (\hat{Q}_1 \cap \cdots \cap \hat{Q}_n) : (Ax \otimes \hat{A})$$
$$= (\hat{Q}_1 : (Ax \otimes \hat{A})) \cap \cdots \cap (\hat{Q}_i : (Ax \otimes \hat{A})) \cap \cdots \cap (\hat{Q}_n : (Ax \otimes \hat{A})).$$

By shortening this representation we have an irredundant representation of $\mathfrak{q}'\hat{A}$. Since $\hat{Q}_i : (Ax \otimes \hat{A}) = \hat{A}$, i) contradicts the uniqueness of associated prime divisors of $\mathfrak{q}'\hat{A}$.

Chevalley's lemma may be generalized as follows:

PROPOSITION 2. *Let E be a finite module over a complete semi-local ring A with Jacobson radical $\mathfrak{m}$. Let $E_1 \supset E_2 \supset \cdots$ be a descending sequence of submodules of E such that $\cap E_n = 0$. Then*

$E_n \subseteq \mathfrak{m}^{\sigma(n)} \cdot E$ where $\sigma(n) \to \infty$ as $n \to \infty$. [8, Proposition 2, p. 9]

PROOF. Since $E/\mathfrak{m}^n E$ is finite over an Artin ring $A/\mathfrak{m}^n$, $E/\mathfrak{m}^n E$ satisfies the descending condition for its submodules. On the other hand, since A is

2) We may consider E, F and $F \otimes \hat{A}$ the submodules of $E \otimes \hat{A}$.

complete in the $\mathfrak{m}$-adic topology. Therefore our Proposition can be proved in the same way as was given in [8].

§ 3. Hilbert functions and multiplicities.

Let E be a finite module over a semi-local ring A. And let $\mathfrak{q}$ be a defining ideal of A, i.e. $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some integer n, where $\mathfrak{m}$ denotes the Jacobson radical of A. We consider a finite $A/\mathfrak{q}^n$-module $E/\mathfrak{q}^n E$. Since $A/\mathfrak{q}^n$ is an Artin ring, $E/\mathfrak{q}^n E$ has a finite length $l(E/\mathfrak{q}^n E)$. Godement has shown in [2, Exposé 16] that, for n sufficiently large, $l(E/\mathfrak{q}^n E)$ is a polynomial in n, whose degree is independent of the choice of $\mathfrak{q}$ and is equal to rank E[3] (= Sup corank $\mathfrak{p}$, where $\mathfrak{p}$ runs over the associated prime divisors of 0 in E). This polynomial will be called *the Hilbert function of* $\mathfrak{q}$ *in* E and denoted by $\chi_E(\mathfrak{q}, n)$. Then we may write it as

$$\chi_E(\mathfrak{q}, n) = l(E/\mathfrak{q}^n E) = \frac{e_E(\mathfrak{q})}{d!} n^d + \cdots,$$

where $d = \text{rank } A$ and $e_E(\mathfrak{q})$ is an integer $\geqq 0$. We call $e_E(\mathfrak{q})$ *the multiplicity of* $\mathfrak{q}$ *in* E. Remark that if rank $A >$ rank $A/(0:E) =$ rank E, we have, by our definition, $e_E(\mathfrak{q}) = 0$. In the case when $E = A$, we have of course $e_E(\mathfrak{q}) = e(\mathfrak{q})$ where $e(\mathfrak{q})$ stands for the multiplicity of $\mathfrak{q}$ in the sense of Samuel [8, p. 28]. And we see easily that $e_{A/\mathfrak{a}}(\mathfrak{q}) = e(\mathfrak{q} + \mathfrak{a}/\mathfrak{a})$ for any ideal $\mathfrak{a}$ in A such that rank $A =$ rank $A/\mathfrak{a}$.

On the other hand we have a following device, due to Nagata, which reduces the study of finite modules over semi-local rings to that of semi-local rings [6, Addenda p. 225].

Let again E be a finite module over a semi-local ring A and let $\mathfrak{q}$ be a defining ideal of A. We may consider the A-module $B = A + E$ (direct) as a ring by defining $E^2 = 0$. Then we see easily that B becomes a semi-local ring of the same rank as that of A. If $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ be maximal ideals of A then $\mathfrak{m}_i + E$ ($i = 1, \cdots, r$) are the maximal ideals of B and $\mathfrak{q}B$ is also a defining ideal of B and

$$B/\mathfrak{q}B = (A+E)/(\mathfrak{q} + \mathfrak{q}E) \approx (A/\mathfrak{q}) + (E/\mathfrak{q}E) \text{ (direct as } A\text{-modules).}$$

Therefore $l(B/\mathfrak{q}B) = l(A/\mathfrak{q}) + l(E/\mathfrak{q}E)$, hence

$$\chi_E(\mathfrak{q}, n) = P_{\mathfrak{q}B}(n) - P_{\mathfrak{q}}(n) \quad \text{and} \quad e_E(\mathfrak{q}) = e(\mathfrak{q}B) - e(\mathfrak{q})$$

where $P_{\mathfrak{q}B}(n)$ and $P_{\mathfrak{q}}(n)$ stand for the Hilbert functions of $\mathfrak{q}$ in the sense of Samuel.

With the aid of this remark we may generalize the theorem of Lech [5, Theorem 2, p. 303] in the following form:

THEOREM 2. (LECH). *Let E be a finite module over a semi-local ring A of* rank d *and let $\mathfrak{q}$ be a defining ideal of A which is generated by a system of pa-*

3) Godement calls it "hauteur de E" and denotes it by $h(E)$ [2, Exposé 16].

rameters in A. Put $\mathfrak{q}=(x_1,\cdots,x_d)$. *Then we have*

$$e_E(\mathfrak{q})=\lim_{n_i\to\infty} l(E/(x_1^{n_1},\cdots,x_d^{n_d})E)/n_1\cdots n_d.$$

PROOF. We shall first remark that, for any $a\in\mathfrak{q}$, we have

$$\begin{aligned}\chi_{E/aE}(\mathfrak{q}/aA,n)&=l(E/(\mathfrak{q}^nE+aE))=l(E/\mathfrak{q}^nE)-l((\mathfrak{q}^nE+aE)/\mathfrak{q}^nE)\\&=\chi_E(\mathfrak{q},n)-l(aE/\mathfrak{q}^nE\cap aE)=\chi_E(\mathfrak{q},n)-l(E/(\mathfrak{q}^nE:aA)),\end{aligned}$$

since $f^{-1}(\mathfrak{q}^nE\cap aE)=\mathfrak{q}^nE:aA$ where f is the A-homomorphism $E\to aE$ defined by $f(x)=ax\ (x\in E)$. Hence

$$\begin{aligned}\chi_{E/x_1^{n_1}E}(\mathfrak{q}/x^{n_1}A,n)&=\chi_E(\mathfrak{q},n)-l(E/\mathfrak{q}^nE:x_1^{n_1}A)\geqq\chi_E(\mathfrak{q},n)-l(E/\mathfrak{q}^{n-n_1}E)\\&=\chi_E(\mathfrak{q},n)-\chi_E(\mathfrak{q},n-n_1)=\frac{n_1e_0}{(d-1)!}n^{d-1}+\cdots\cdots,\end{aligned}$$

where $e_0=e_E(\mathfrak{q})$. Put $e_1=e_{E/x_1^{n_1}E}(\mathfrak{q}/x_1^{n_1}A)$. Then $e_1\geqq n_1e_0$. Therefore, by induction, we see easily

$$l(E/(x_1^{n_1},\cdots,x_d^{n_d})E)=e_{E/(x_1^{n_1},\cdots,x_d^{n_d})E}(\mathfrak{q}/(x_1^{n_1},\cdots,x_d^{n_d}))\geqq n_d\cdots n_1\,e_0.$$

Hence
$$l(E/(x_1^{n_1},\cdots,x_d^{n_d})E)/n_1\cdots n_d\geqq e_E(\mathfrak{q}).$$

To prove the converse we shall first consider the case when $A=E$. Let $\mathfrak{m}_1,\cdots,\mathfrak{m}_r$ be the maximal ideals of A and let $\mathfrak{q}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_r$ be a normal decomposition of $\mathfrak{q}$. Consider the chain of $\mathfrak{m}_i$-primary ideals $\mathfrak{m}_i=\mathfrak{q}'_{i1}\supset\mathfrak{q}'_{i2}\supset\cdots\cdots\supset\mathfrak{q}'_{it_i}=\mathfrak{q}_i$ such that each inclusion is strict and no $\mathfrak{m}_i$-primary ideals can be inserted between $\mathfrak{q}'_{i\nu}$ and $\mathfrak{q}'_{i\nu+1}$ ($i=1,\cdots,r$ and $\nu=1,\cdots,t_i-1$). From this we obtain a composition series from A to $\mathfrak{q}$:

$$A\supset\mathfrak{m}_1=\mathfrak{q}'_{11}\supset\cdots\supset\mathfrak{q}'_{1t_1}=\mathfrak{q}_1\supset\mathfrak{q}_1\mathfrak{q}'_{21}\supset\cdots\supset\mathfrak{q}_1\mathfrak{q}'_{2t_2}=\mathfrak{q}_1\mathfrak{q}_2\supset\cdots\supset\mathfrak{q}_1\mathfrak{q}_2\cdots\mathfrak{q}_r=\mathfrak{q}.$$

Put $K_i=A/\mathfrak{m}_i$ and $\mathfrak{q}_{i\nu}=\mathfrak{q}_1\cdots\mathfrak{q}_{i-1}\mathfrak{q}'_{i\nu}$ ($i=1,\cdots,r$ and $0\leqq\nu_i<t_i$). Then $\mathfrak{q}_{i\nu}/\mathfrak{q}_{i\nu+1}$ is isomorphic to K_i, because $\mathfrak{m}_i\mathfrak{q}_{i\nu}\subset\mathfrak{q}_{i\nu+1}$ and cleary we have

$$l(A/\mathfrak{a}+\mathfrak{q}^n)=\sum_{i=1}^{r}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{t_i-1}l(\mathfrak{a}+\mathfrak{q}_{i\nu}\mathfrak{q}^\mu/\mathfrak{a}+\mathfrak{q}_{i\nu+1}\mathfrak{q}^\mu).$$

Therefore we can continue the proof in the same way as was given in [5] replacing $K[X_1,\cdots,X_d]$ by a finite number of polynomial rings $K_i[X_1,\cdots,X_d]$ ($i=1,\cdots,r$) where K means the residue field of a local ring. Then we get $\lim_{n_i\to\infty} l(A/(x_1^{n_1},\cdots,x_d^{n_d}))/n_1\cdots n_d\leqq e(\mathfrak{q})$.

The general case now follows from this with the aid of the above remark. In fact, we have

$$l(B/(x_1^{n_1},\cdots,x_d^{n_d})B)=l(A/(x_1^{n_1},\cdots,x_d^{n_d})A)+l(E/(x_1^{n_1},\cdots,x_d^{n_d})E)$$

(with the same notations as in the above remark). Hence

$$l(E/(x_1^{n_1},\cdots,x_d^{n_d})E)/n_1\cdots n_d\leqq l(B/(x_1^{n_1},\cdots,x_d^{n_d})B)/n_1\cdots n_d-e(\mathfrak{q})$$

by the first part of the proof. Therefore

$$\begin{aligned}\lim_{n_i\to\infty} l(E/(x_1^{n_1},\cdots,x_d^{n_d})E)/n_1\cdots n_d&\leqq\lim_{n_i\to\infty} l(B/(x_1^{n_1},\cdots,x_d^{n_d})B)/n_1\cdots n_d-e(\mathfrak{q})\\&\leqq e(\mathfrak{q}B)-e(\mathfrak{q})=e_E(\mathfrak{q}).\end{aligned}$$

§ 4. Fundamental theorems on multiplicities.

We first show that $e_E(\mathfrak{q})$ has the property of additivity.

THEOREM 3. *Let* $\mathfrak{q}$ *be a defining ideal of a semi-local ring* A. *And let* E, E' *and* E'' *be finite* A-*modules such that*

(1) $$0 \to E' \to E \to E'' \to 0 \quad \text{(exact)}.$$

Then, $$e_E(\mathfrak{q}) = e_{E'}(\mathfrak{q}) + e_{E''}(\mathfrak{q}).$$

PROOF. From the exact sequence (1), we have

$$E' \otimes_A (A/\mathfrak{q}^n) \to E \otimes_A (A/\mathfrak{q}^n) \to E'' \otimes_A (A/\mathfrak{q}^n) \to 0 \quad \text{(exact)},$$

i.e.

(2) $$E'/\mathfrak{q}^n E' \to E/\mathfrak{q}^n E \xrightarrow{i} E''/\mathfrak{q}^n E'' \to 0 \quad \text{(exact)}.$$

Hence $$l(E'/\mathfrak{q}^n E') + l(E''/\mathfrak{q}^n E'') \geqq l(E/\mathfrak{q}^n E).$$

We express these lengths as Hilbert functions and compare the leading coefficients. Then we get $e_{E'}(\mathfrak{q}) + e_{E''}(\mathfrak{q}) \geqq e_E(\mathfrak{q})$.

Next we consider the converse ineqality. From the exact sequence (2), $E' + \mathfrak{q}^n E/\mathfrak{q}^n E$ is contained in the Ker i. Hence $l(E/\mathfrak{q}^n E) \geqq l(E''/\mathfrak{q}^n E'') + l(E' + \mathfrak{q}^n E/\mathfrak{q}^n E)$. Since $E' + \mathfrak{q}^n E/\mathfrak{q}^n E \approx E'/\mathfrak{q}^n E \cap E'$ and $\mathfrak{q}^n E \cap E' = \mathfrak{q}^{n-r}(\mathfrak{q}^r E \cap E') \subseteq \mathfrak{q}^{n-r} E'$ for $n \geqq r$ by Lemma 1, we get $l(E' + \mathfrak{q}^n E/\mathfrak{q}^n E) \geqq l(E'/\mathfrak{q}^{n-r} E')$. Therefore

$$l(E/\mathfrak{q}^n E) \geqq l(E''/\mathfrak{q}^n E'') + l(E'/\mathfrak{q}^{n-r} E').$$

From this, the same consideration as above shows that $e_E(\mathfrak{q}) \geqq e_{E'}(\mathfrak{q}) + e_{E''}(\mathfrak{q})$.

From our Theorem 3, we obtain immediately the following

COROLLARY. *Let* E_i $(i = 1, \cdots, n)$ *be finite* A-*modules such that*

$$0 \to E_1 \to E_2 \to \cdots \to E_{n-1} \to E_n \to 0 \quad \text{(exact)}.$$

Then we have

$$\sum_{i=1}^{n} (-1)^i e_{E_i}(\mathfrak{q}) = 0.$$

The special case of the next lemma was treated in [6, Theorem 2, p. 206] and [8, Proposition 2, p. 31] and is called the "extension formula" by Nagata.

LEMMA 3. *Let* $\mathfrak{q}$ *be a defining ideal of a semi-local ring* A *and let* E *be a finite* A-*module. Assume that there exist an element* c *in* A $(c \neq 0)$ *and a system of elements* $x_1, \cdots, x_r$ *in* E *which satisfy the following conditions*:

i) c *is not a zero divisor in* A.

ii) $x_1, \cdots, x_r$ *are linearly independent over* A.

iii) cE *is contained in* $Ax_1 + \cdots + Ax_r$.

Then, for any finite A-*module* F, *we have*

$$r.e_F(\mathfrak{q}) = e_{E \otimes F}(\mathfrak{q}).$$

In particular $$r.e(\mathfrak{q}) = e_E(\mathfrak{q}).$$

PROOF. Put $G = Ax_1 + \cdots + Ax_r$. From the exact sequence of A-modules

$$0 \to G \to E \to E/G \to 0,$$

we deduce the exact sequence

$$G \otimes_A F \to E \otimes_A F \to (E/G) \otimes_A F \to 0.$$

Since we may consider $(E/G) \otimes F$ as a finite A/cA-module and since rank $A/cA <$ rank A (by our assumption i)), we have $e_{(E/G)\otimes F}(\mathfrak{q}) = 0$. Therefore, by the similar consideration to the first part of the proof of Theorem 3, we get

$$e_{E\otimes F}(\mathfrak{q}) \leqq e_{G\otimes F}(\mathfrak{q}) = r.e_F(\mathfrak{q}) \quad \text{(by our assumption ii)).}$$

Next we consider the exact sequences

$$0 \to cE \to G \to G/cE \to 0 \quad \text{and} \quad E \to cE \to 0.$$

Then, by the same argument as above, we see

$$r.e_F(\mathfrak{q}) = e_{G\otimes F}(\mathfrak{q}) = e_{cE\otimes F}(\mathfrak{q}) \quad \text{and} \quad e_{E\otimes F}(\mathfrak{q}) \geqq e_{cE\otimes F}(\mathfrak{q}).$$

Combining these inequalities, we get

$$e_{E\otimes F}(\mathfrak{q}) \leqq e_{G\otimes F}(\mathfrak{q}) = r.e_F(\mathfrak{q}) = e_{cE\otimes F}(\mathfrak{q}) \leqq e_{E\otimes F}(\mathfrak{q}).$$

THEOREM 4. (REDUCTION THEOREM).

Let A be a semi-local ring whose zero ideal is a primary ideal belonging (say) to the prime ideal $\mathfrak{p}$. And let E be a finite A-module. Then, for any defining ideal $\mathfrak{q}$ of A, we have

$$e_E(\mathfrak{q}) = e_{A/\mathfrak{p}}(\mathfrak{q})\, l(E \otimes_A A_{\mathfrak{p}}).^{4)}$$

In particular

$$e(\mathfrak{q}) = e(\mathfrak{q} + \mathfrak{p}/\mathfrak{p})\, l(A_{\mathfrak{p}}). \quad [6, \text{Theorem 9, p. 219}]$$

PROOF. We proceed to prove the theorem by induction on the length of $(0)A$. In the case when $l((0)A) = 1$, i.e. $(0)A = \mathfrak{p}$, we have to prove

$$e_E(\mathfrak{q}) = e(\mathfrak{q})\, l(E \otimes_A K)$$

where K is a quotient field of A. Our proof will be completed in this case, by Lemma 3, if we can find $x_1 \cdots, x_s \in E$ and $c \in A$ ($c \neq 0$) such that $x_1, \cdots, x_s$ are linearly independent over A and $cE \subseteq Ax_1 + \cdots Ax_s$, where $s = l(E \otimes_A K) = \dim_K E \otimes_A K$ (as a vector space over K).

Let $x_1, \cdots, x_s$ be elements in E such that the set $x_i \otimes 1$ ($i = 1, \cdots, s$) forms a basis of a vector space $E \otimes K$ over K. Put $E = Ay_1 + \cdots + Ay_n$. Then, we can find $d_i (\neq 0)$, $a_1, \cdots, a_s$ in A such that

$$d_i(y_i \otimes 1) = a_1(x_1 \otimes 1) + \cdots + a_s(x_s \otimes 1).$$

Since Ker $(E \to E \otimes K)$ is a submodule N of E generated by all elements z in E such that $bz = 0$ for some $b \neq 0$ in A and since N is finitely generated, we have

$$b_i(d_i y_i - (a_1 x_1 + \cdots + a_s x_s)) = 0$$

for some $b_i \neq 0$ in A, i.e. $b_i d_i y_i = b_i a_1 x_1 + \cdots + b_i a_s x_s$. Clearly $x_1, \cdots, x_s$ are linearly

4) Since $A_{\mathfrak{p}}$ is an Artin ring, $E \otimes_A A_{\mathfrak{p}}$ has the finite length.

independent over A, therefore $c=\prod_{i=1}^{n} b_i d_i$, $x_1, \cdots, x_s$ satisfy the conditions.

Now we assume the theorem is true in the case when $l((0)A)=n-1$, and consider the case when $l((0)A)=n$. Let $\mathfrak{p}=\mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_{n-1} \supset \mathfrak{q}_n=(0)$ be a chain of $\mathfrak{p}$-primary ideals and assume that each inclusion is strict and no $\mathfrak{p}$-primary ideals can be inserted between $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ $(i=1,\cdots,n-1)$. Put $\mathfrak{q}'=\mathfrak{q}_{n-1}$. Then $E/\mathfrak{q}'E$ is a finite $A/\mathfrak{q}'$-module and $l((0)A/\mathfrak{q}')=n-1$. Therefore, by our induction hypothesis, we see that

(1) $$e_{E/\mathfrak{q}'E}(\mathfrak{q}) = e_{(A/\mathfrak{q}')/(\mathfrak{p}/\mathfrak{q}')}(\mathfrak{q})\, l((E/\mathfrak{q}'E)\otimes_{A/\mathfrak{q}'}(A/\mathfrak{q}')_{\mathfrak{p}/\mathfrak{q}'})$$
$$= e_{A/\mathfrak{p}}(\mathfrak{q})l((E/\mathfrak{q}'E)\otimes(A_{\mathfrak{p}}/\mathfrak{q}'A_{\mathfrak{p}}))^{5)} = e_{A/\mathfrak{p}}(\mathfrak{q})l((E\otimes_A A_{\mathfrak{p}})/\mathfrak{q}'A_{\mathfrak{p}}(E\otimes_A A_{\mathfrak{p}})).$$

On the other hand, from the exact sequence

(2) $$0\to\mathfrak{q}'E\to E\to E/\mathfrak{q}'E\to 0,$$

we have, by Theorem 3,

(3) $$e_E(\mathfrak{q})=e_{\mathfrak{q}'E}(\mathfrak{q})+e_{E/\mathfrak{q}'E}(\mathfrak{q}).$$

Since $\otimes_A A_{\mathfrak{p}}$ is an exact functor, from (2) we have

$$0\to\mathfrak{q}'E\otimes A_{\mathfrak{p}}\to E\otimes A_{\mathfrak{p}}\to(E/\mathfrak{q}'E)\otimes A_{\mathfrak{p}}\to 0 \quad \text{(exact)}.$$

Therefore,

(4) $$l(E\otimes A_{\mathfrak{p}}) = l(\mathfrak{q}'E\otimes A_{\mathfrak{p}})+l((E/\mathfrak{q}'E)\otimes A_{\mathfrak{p}})$$
$$= l(\mathfrak{q}'A_{\mathfrak{p}}(E\otimes A_{\mathfrak{p}}))+l((E\otimes A_{\mathfrak{p}})/\mathfrak{q}'A_{\mathfrak{p}}(E\otimes A_{\mathfrak{p}})).$$

Hence, by (1), (3) and (4), in order to prove our theorem, it remains to show the following equality:

(5) $$e_{\mathfrak{q}'E}(\mathfrak{q}) = e_{A/\mathfrak{p}}(\mathfrak{q})l(\mathfrak{q}'A_{\mathfrak{p}}(E\otimes A_{\mathfrak{p}})).$$

Since $\mathfrak{p}\mathfrak{q}'=(0)$, we may consider $\mathfrak{q}'E$ as an $A/\mathfrak{p}$-module. We consider the vector space $M=\mathfrak{q}'E\otimes_{A/\mathfrak{p}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ over the field $K=A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Then, clearly

$$l(\mathfrak{q}'A_{\mathfrak{p}}(E\otimes A_{\mathfrak{p}})) = \dim_K M = s \quad \text{(say)}.$$

Take elements $x_1,\cdots,x_s$ in $\mathfrak{q}'E$ such that $x_1\otimes 1,\cdots,x_s\otimes 1$ form a basis of M over K. Then, by the same argument as above, we can find an element $c\neq 0$ in $A/\mathfrak{p}$ such that $cm\in(A/\mathfrak{p})x_1+\cdots+(A/\mathfrak{p})x_s$ holds for any element m in $\mathfrak{q}'E$. Hence, by Lemma 3, we have

$$e_{\mathfrak{q}'E}(\mathfrak{q}) = s\cdot e_{A/\mathfrak{p}}(\mathfrak{q})$$

which proves (5) and concludes the proof of the theorem.

LEMMA 4. *Let E be a finite module over a semi-local ring A. And let $\mathfrak{b}$ and $\mathfrak{c}$ be ideals of A such that* rank $A/\mathfrak{b}>$ rank $A/\mathfrak{c}$. *Then, for any defining ideal $\mathfrak{q}$ of A, we have*

i) corank $\mathfrak{b}=$ corank $\mathfrak{b}\cap\mathfrak{c}=$ corank $\mathfrak{b}\mathfrak{c}$.

ii) $e_{E/\mathfrak{b}E}(\mathfrak{q}+\mathfrak{b}/\mathfrak{b})=e_{E/(\mathfrak{b}\cap\mathfrak{c})E}(\mathfrak{q}+(\mathfrak{b}\cap\mathfrak{c})/\mathfrak{b}\cap\mathfrak{c})=e_{E/\mathfrak{b}\mathfrak{c}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c})$.

5) $(E/\mathfrak{q}'E)\otimes_{A/\mathfrak{q}'}(A_{\mathfrak{p}}/\mathfrak{q}'A_{\mathfrak{p}})$ may be identified with $(E/\mathfrak{q}'E)\otimes_A(A_{\mathfrak{p}}/\mathfrak{q}'A_{\mathfrak{p}})$.

[8, Proposition 3, p. 32]

PROOF. Consider the exact sequence of $A/\mathfrak{b}\mathfrak{c}$ -modules
$$0 \to \mathfrak{b}E/\mathfrak{b}\mathfrak{c}E \to E/\mathfrak{b}\mathfrak{c}E \to E/\mathfrak{b}E \to 0.$$
Then, by Theorem 3,
$$e_{E/\mathfrak{b}\mathfrak{c}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}) = e_{\mathfrak{b}E/\mathfrak{b}\mathfrak{c}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}) + e_{E/\mathfrak{b}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}).$$
Since $\mathfrak{b}E/\mathfrak{b}\mathfrak{c}E$ may be considered as a finite $A/\mathfrak{c}$-module and since rank $A/\mathfrak{c} <$ rank $A/\mathfrak{b} =$ rank $A/\mathfrak{b}\mathfrak{c}$, we have $e_{\mathfrak{b}E/\mathfrak{b}\mathfrak{c}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}) = 0$. Therefore
$$e_{E/\mathfrak{b}\mathfrak{c}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}) = e_{E/\mathfrak{b}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}) = e_{E/\mathfrak{b}E}(\mathfrak{q}+\mathfrak{b}/\mathfrak{b}).$$
From $\mathfrak{b}E \supset (\mathfrak{b}\cap\mathfrak{c})\, E \supset \mathfrak{b}\mathfrak{c}E$, we have obviously
$$e_{E/\mathfrak{b}E}(\mathfrak{q}+\mathfrak{b}/\mathfrak{b}) \leq e_{E(\mathfrak{b}\cap\mathfrak{c})E}(\mathfrak{q}+(\mathfrak{b}\cap\mathfrak{c})/\mathfrak{b}\cap\mathfrak{c}) \leq e_{E/\mathfrak{b}\mathfrak{c}E}(\mathfrak{q}+\mathfrak{b}\mathfrak{c}/\mathfrak{b}\mathfrak{c}).$$

THEOREM 5. (THEOREM OF ADDITIVITY).

Let E be a finite module over a semi-local ring A and let $\mathfrak{q}$ be a defining ideal of A. Let $(0)A = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_n$ be a normal decomposition of zero ideal of A with associated prime divisors $\mathfrak{p}_i$ $(i=1,\cdots,n)$. Assume that corank $\mathfrak{p}_i =$ rank A $(i=1,\cdots,r)$ *and* corank $\mathfrak{p}_{r+j} <$ rank A $(j=1,2,\cdots,n-r)$. *Then we have*
$$e_E(\mathfrak{q}) = \sum_{i=1}^{r} e_{E/\mathfrak{q}_iE}(\mathfrak{q}).$$
In particular
$$e(\mathfrak{q}) = \sum_{i=1}^{r} e(\mathfrak{q}+\mathfrak{q}_i/\mathfrak{q}_i). \quad \text{[6, Theorem 3, p. 207]}$$

PROOF. First we shall show that we may assume $r=n$. Put $\mathfrak{b} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ and $\mathfrak{c} = \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_n$. Then $(0)A = \mathfrak{b} \cap \mathfrak{c}$ and corank $A/\mathfrak{b} >$ corank $A/\mathfrak{c}$. Hence, by the preceding lemma 4,
$$e_E(\mathfrak{q}) = e_{E/(\mathfrak{b}\cap\mathfrak{c})E}(\mathfrak{q}) = e_{E/\mathfrak{b}E}(\mathfrak{q}).$$
Since $E/\mathfrak{b}E$ is finite over a semi-local ring $A/\mathfrak{b}$ and any prime divisor of $(0)A/\mathfrak{b}$ has the same corank, we see easily
$$e_{E/\mathfrak{b}E}(\mathfrak{q}) = \sum_{i=1}^{r} e_{E/\mathfrak{q}_iE}(\mathfrak{q})$$
by our assumptions.

Now we denote by K the total quotient ring of A, then there exist primitive idempotents $x_1,\cdots,x_r \in K$ such that Ax_i is isomorphic to $A/\mathfrak{q}_i$ for any i. Then, by Lemma 3, we have $e_E(\mathfrak{q}) = e_{E\otimes F}(\mathfrak{q})$ where $F = Ax_1 + \cdots + Ax_r$. Since $E\otimes_A F \approx E\otimes_A(A/\mathfrak{q}_1 + \cdots + A/\mathfrak{q}_r) \approx E/\mathfrak{q}_1E + \cdots + E/\mathfrak{q}_rE$ (direct), then
$$e_{E\otimes F}(\mathfrak{q}) = \sum_{i=1}^{r} e_{E/\mathfrak{q}_iE}(\mathfrak{q}).$$
Therefore, combining these relations, we have
$$e_E(\mathfrak{q}) = \sum_{i=1}^{r} e_{E/\mathfrak{q}_iE}(\mathfrak{q}).$$

COROLLARY. *Let A be a semi-local ring and let $\mathfrak{q}$ be a defining ideal of A. Then, for any finite A-module E, we have*

$$e_E(\mathfrak{q}) = \sum_{\mathfrak{p}} e_{A/\mathfrak{p}}(\mathfrak{q}) l(E \otimes_A A_{\mathfrak{p}})$$

where $\mathfrak{p}$ runs over all prime ideals of A such that corank $\mathfrak{p} =$ rank A. *In particular*

$$e(\mathfrak{q}) = \sum_{\mathfrak{p}} e(\mathfrak{q} + \mathfrak{p}/\mathfrak{p}) l(A_{\mathfrak{p}}).$$

[6, Corollary 1 of Theorem 9, p. 220]

PROOF. Being the same notations in the theorem, we have

$$e_E(\mathfrak{q}) = \sum_{i=1}^{r} e_{E/\mathfrak{q}_i E}(\mathfrak{q})$$

and, by Theorem 4,

$$\begin{aligned} e_{E/\mathfrak{q}_i E}(\mathfrak{q}) &= e_{A/\mathfrak{p}_i}(\mathfrak{q}) l((E/\mathfrak{q}_i E) \otimes_{A/\mathfrak{q}_i} (A/\mathfrak{q}_i)_{\mathfrak{p}_i/\mathfrak{q}_i}) = e_{A/\mathfrak{p}_i}(\mathfrak{q}) l((E/\mathfrak{q}_i E) \otimes_{A/\mathfrak{q}_i} (A_{\mathfrak{p}_i}/\mathfrak{q}_i A_{\mathfrak{p}_i})) \\ &= e_{A/\mathfrak{p}_i}(\mathfrak{q}) l((E \otimes_A A_{\mathfrak{p}_i})/\mathfrak{q}_i (E \otimes_A A_{\mathfrak{p}_i}))^{5)} = e_{A/\mathfrak{p}_i}(\mathfrak{q}) l(E \otimes_A A_{\mathfrak{p}_i}) \end{aligned}$$

for any i.

The next lemma is the special case of the transition theorem.

LEMMA 5. *Let $\mathfrak{p}$ be a prime ideal of an $\mathfrak{m}$-adic Zariski ring A and let $\mathfrak{P}$ be an isolated prime divisor of $\mathfrak{p}\hat{A}$ ($\hat{A}$ means the $\mathfrak{m}$-adic completion of A). Then, for any finite A-module E, we have*

$$l(\hat{E} \otimes_A (\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})) = l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}}) l(E \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})),$$

where $\hat{E}$ is the $\mathfrak{m}$-adic completion of E.

PROOF. To prove the lemma it is sufficient to show the following equalities:

i) $l(E \otimes_A (A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})) = l(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) l(E \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))$, where $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A.

ii) $l(E \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})) = l(\hat{E} \otimes_A (\hat{A}_{\mathfrak{P}}/\mathfrak{P}\hat{A}_{\mathfrak{P}}))$.

In fact,

$$l(\hat{E} \otimes (\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})) = l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}}) l(\hat{E} \otimes (\hat{A}_{\mathfrak{P}}/\mathfrak{P}\hat{A}_{\mathfrak{P}})) = l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}}) l(E \otimes (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})).$$

Now we shall prove i) by induction on the length of $\mathfrak{q}$. Since in the case when $l(\mathfrak{q}) = 1$, i.e. $\mathfrak{q} = \mathfrak{p}$, the equality is obvious, we consider the case when $l(\mathfrak{q}) > 1$. By passing to the residue module we may assume $\mathfrak{q} = (0)$. Let $\mathfrak{p} = \mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_{n-1} \supset \mathfrak{q}_n = \mathfrak{q} = (0)$ be a chain of $\mathfrak{p}$-primary ideals and assume that each inclusion is strict and no $\mathfrak{p}$-primary ideals can be inserted between $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1} (i = 1, \cdots, n-1)$. Put $\mathfrak{q}' = \mathfrak{q}_{n-1}$. Then, by our induction hypothesis, $l(E \otimes (A_{\mathfrak{p}}/\mathfrak{q}'A_{\mathfrak{p}})) = l(A_{\mathfrak{p}}/\mathfrak{q}'A_{\mathfrak{p}}) l(E \otimes (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))$. Hence, to prove i), it is sufficient to show that $l(E \otimes \mathfrak{q}'A_{\mathfrak{p}}) = l(\mathfrak{q}'A_{\mathfrak{p}}) l(E \otimes (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))$. Since $\mathfrak{p}\mathfrak{q}' = (0)$ and $\mathfrak{q}'A_{\mathfrak{p}}$ is principal, we see that $\mathfrak{q}'A_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-module. Therefore $l(\mathfrak{q}'A_{\mathfrak{p}}) = 1$ and $l(E \otimes \mathfrak{q}'A_{\mathfrak{p}}) = l(E \otimes (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))$.

As for ii), clearly $l(E\otimes(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))=l((E/\mathfrak{p}E)\otimes_{A/\mathfrak{p}}K)$ and $l(\hat{E}\otimes_{\hat{A}}(\hat{A}_{\mathfrak{P}}/\mathfrak{P}\hat{A}_{\mathfrak{P}}))=l((\hat{E}/\mathfrak{P}\hat{E})\otimes_{\hat{A}/\mathfrak{P}}L)$ where K and L are the quotient fields of $A/\mathfrak{p}$ and $\hat{A}/\mathfrak{P}$ respectively. Since $\mathfrak{P}\cap A=\mathfrak{p}$, we may consider $A/\mathfrak{p}$ as a subring of $\hat{A}/\mathfrak{P}$ and K as a subfield of L and we have

$$(\hat{E}/\mathfrak{P}\hat{E})\otimes_{\hat{A}/\mathfrak{P}}L\approx((E/\mathfrak{p}E)\otimes_{A/\mathfrak{p}}(\hat{A}/\mathfrak{P}))\otimes_{\hat{A}/\mathfrak{P}}L\approx((E/\mathfrak{p}E)\otimes_{A/\mathfrak{p}}K)\otimes_K L.$$

Therefore

$$l(\hat{E}\otimes_{\hat{A}}L)=l(((E/\mathfrak{p}E)\otimes_{A/\mathfrak{p}}K)\otimes_K L)=l((E/\mathfrak{p}E)\otimes_{A/\mathfrak{p}}K)=l(E\otimes_A K).$$

THEOREM 6. (TRANSITION THEOREM).

Let A be an $\mathfrak{m}$-adic Zariski ring with a completion $\hat{A}$ and let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of A and $\mathfrak{P}$ be an isolated prime divisor of $\mathfrak{p}\hat{A}$. Then, for any finite A-module E, we have

i) $l(\hat{E}\otimes_{\hat{A}}(\hat{A}_{\mathfrak{P}}/\mathfrak{q}\hat{A}_{\mathfrak{P}}))=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})l(E\otimes_A(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}))$,

ii) $\chi_{\hat{E}\otimes\hat{A}_{\mathfrak{P}}}(\mathfrak{q}\hat{A}_{\mathfrak{P}},n)=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})\chi_{E\otimes A_{\mathfrak{p}}}(\mathfrak{q}A_{\mathfrak{p}},n)$,

iii) $e_{\hat{E}\otimes\hat{A}_{\mathfrak{P}}}(\mathfrak{q}\hat{A}_{\mathfrak{P}})=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})e_{E\otimes A_{\mathfrak{p}}}(\mathfrak{q}A_{\mathfrak{p}})$,

where $\hat{E}$ means an $\mathfrak{m}$-adic completion of E, i.e. $\hat{E}=E\otimes\hat{A}$. [6, Theorem 7, p. 217] or [9, Theorem 2, p. 96]

PROOF. We proceed to prove i) by applying induction to the length of $\mathfrak{q}$. Since our theorem is true in the case when $l(\mathfrak{q})=1$, i.e. $\mathfrak{q}=\mathfrak{p}$, by Lemma 5, we may assume $l(\mathfrak{q})>1$. By passing to the residue module, we may further assume $\mathfrak{q}=(0)$. Now, let $\mathfrak{q}'$ be a $\mathfrak{p}$-primary ideal such that no $\mathfrak{p}$-primary ideal exists between (0) and $\mathfrak{q}'$. Then our induction hypothesis shows that

$$l(\hat{E}\otimes(\hat{A}_{\mathfrak{P}}/\mathfrak{q}'\hat{A}_{\mathfrak{P}}))=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})l(E\otimes(A_{\mathfrak{p}}/\mathfrak{q}'A_{\mathfrak{p}})).$$

Therefore, to prove i) it is enough to show

$$l(\hat{E}\otimes\mathfrak{q}'\hat{A}_{\mathfrak{P}})=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})l(E\otimes\mathfrak{q}'A_{\mathfrak{p}}).$$

Since $\mathfrak{p}\mathfrak{q}'=(0)$, we have $\mathfrak{q}'\hat{A}_{\mathfrak{P}}\approx\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}}$ and

$$\begin{aligned}l(\hat{E}\otimes\mathfrak{q}'\hat{A}_{\mathfrak{P}})&=l(\hat{E}\otimes(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}}))=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})l(\hat{E}\otimes(\hat{A}_{\mathfrak{P}}/\mathfrak{P}\hat{A}_{\mathfrak{P}}))\\&=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})l(E\otimes(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))=l(\hat{A}_{\mathfrak{P}}/\mathfrak{p}\hat{A}_{\mathfrak{P}})l(E\otimes\mathfrak{q}'A_{\mathfrak{p}}).\end{aligned}$$

Now that i) is proved, ii) and iii) follow from i) immediately.

Finally we shall prove the so-called "associative formula."

THEOREM 7. (ASSOCIATIVE FORMULA).

Let E be a finite module over a semi-local ring A of rank d *and let $\{x_1,\cdots,x_d\}$ be a system of parameters of A. Put $\mathfrak{q}=(x_1,\cdots,x_d)$ and $\mathfrak{a}=(x_1,\cdots,x_s)$ $(s\leq d)$. Then we have*

$$e_E(\mathfrak{q})=\sum_{\mathfrak{p}}e_{A/\mathfrak{p}}(\mathfrak{q}+\mathfrak{p}/\mathfrak{p})e_{E\otimes A_{\mathfrak{p}}}(\mathfrak{a}A_{\mathfrak{p}})$$

where $\mathfrak{p}$ runs over all (minimal) prime divisors of $\mathfrak{a}$ such that rank $\mathfrak{p}+$ corank

$\mathfrak{p} = d$. *In particular*

$$e(\mathfrak{q}) = \sum_{\mathfrak{p}} e(\mathfrak{q}+\mathfrak{p}/\mathfrak{p}) e(\mathfrak{a}A_{\mathfrak{p}}).$$

[5, Theorem 1, p. 301] or [6, Theorem 8, p. 217].

PROOF. Let $\mathfrak{p}_1,\cdots,\mathfrak{p}_m$ be all prime divisors of $\mathfrak{a}$ such that rank $\mathfrak{p}_i+$ corank $\mathfrak{p}_i = d$. For any integer $t>0$, $\mathfrak{a}=(x_1,\cdots,x_s)$ and $(x_1^t,\cdots,x_s^t)$ have the same prime divisors. Put $\mathfrak{q}^* = (x_1^t,\cdots,x_s^t,x_{s+1},\cdots,x_d)$, $A'=A/(x_1^t,\cdots,x_s^t)$, $\mathfrak{q}'=\mathfrak{q}^*/(x_1^t,\cdots,x_s^t)$, $\mathfrak{p}'_i = \mathfrak{p}_i/(x_1^t,\cdots,x_s^t)$ and $E'=E/(x_1^t,\cdots,x_s^t)E$. Then, by Corollary of Theorem 5, we have

$$e_{E'}(\mathfrak{q}') = \sum_{i=1}^{m} e(\mathfrak{q}'+\mathfrak{p}'_i/\mathfrak{p}'_i) l(E'\otimes A'_{\mathfrak{p}_i}) = \sum_{i=1}^{m} e(\mathfrak{q}+\mathfrak{p}_i/\mathfrak{p}_i) l(E'\otimes A'_{\mathfrak{p}'_i}).$$

We denote by $\bar{x}_i$ $(i=s+1,\cdots,d)$ the residue of x_i modulo $(x_1^t,\cdots,x_s^t)$. Then $\mathfrak{q}' = \bar{x}_{s+1},\cdots,\bar{x}_d)$. Therefore, by Theorem 2, we have

$$e_{E'}(\mathfrak{q}') = \lim_{n\to\infty} l(E'/\bar{x}_{s+1}^n,\cdots,\bar{x}_d^n)E')/n^{d-s} = \lim_{n\to\infty} l(E/(x_1^t,\cdots,x_s^t,x_{s+1}^n,\cdots,x_d^n)E)/n^{d-s}.$$

Hence

$$\lim_{n\to\infty} l(E/(x_1^t,\cdots,x_s^t,x_{s+1}^n\cdots,x_d^n)E)/n^{d-s} = \sum_{i=1}^{m} e(\mathfrak{q}+\mathfrak{p}_i/\mathfrak{p}_i) l(E'\otimes A'_{\mathfrak{p}'_i})$$
$$= \sum_{i=1}^{m} e(\mathfrak{q}+\mathfrak{p}_i/\mathfrak{p}_i) l((E\otimes A_{\mathfrak{p}_i})/(x_1^t,\cdots,x_s^t)(E\otimes A_{\mathfrak{p}_i})).$$

Dividing by t^s, we get

$$e_E(\mathfrak{q}) = \lim_{n,t\to\infty} l(E/(x_1^t,\cdots,x_s^t,x_{s+1}^n,\cdots,x_d^n)E)/t^s n^{d-s}$$
$$= \sum_{i=1}^{m} e(\mathfrak{q}+\mathfrak{p}_i/\mathfrak{p}_i) \lim_{t\to\infty} l(E\otimes A_{\ i}/(x_1^t,\cdots,x_s^t))/t^s$$
$$= \sum_{i=1}^{m} e(\mathfrak{q}+\mathfrak{p}_i/\mathfrak{p}_i) e_{E\otimes A_{\mathfrak{p}_i}}(\mathfrak{a}A_{\cdot i}).$$

§ 5. Complete tensor product of modules.

Let A and A' be an $\mathfrak{m}$-adic and an $\mathfrak{m}'$-adic Zariski rings and let E and E' be finite modules over A and A' respectively. Assume A and A' contain a common subfield K. Put $G=(E/\mathfrak{m}^n E)\otimes_K(E'/\mathfrak{m}'^n E')$. Then the canonical homomorphism $\phi_n: G_n \to G_{n-1}$ is defined and the system $\{G_n, \phi_n\}$ $(n=1,2,\cdots)$ constitute an inverse system of $A\otimes_K A'$-modules. We denote by $E\hat{\otimes}_K E'$ its projective limit and call it *the complete tensor product of* E *and* E' *over* K.[6] We remark that $E\hat{\otimes} E'$ is nothing but a completion of $E\otimes_K E'$ in the $(\mathfrak{m},\mathfrak{m}')$-adic topology.[7] This follows immediately from the following lemma.

6) In this section, unless otherwise specified, we consider the tensor products and complete tensor products of modules over K.

7) The canonical mappings $A\to A\otimes A'$ and $A'\to A\otimes A'$ are injective. Therefore we can identify A and A' with their images.

LEMMA 6. (*With the same notations and assumptions as above.*)

i) $E\otimes E'$ *is a finite* $A\otimes A'$*-module.*

ii) $E\otimes E'$ *is an* $(\mathfrak{m}, \mathfrak{m}')$*-adic module.*

PROOF. i) Since the function $f: (A\otimes A')\times(E\otimes E')\to E\otimes E'$ defined by $f(a\otimes a', x\otimes x') = ax\otimes a'x'$ is well-defined, $E\otimes E'$ is considered as an $A\otimes A'$-module. Put $E = Ax_1+\cdots+Ax_n$ and $E' = A'x'_1+\cdots+A'x'_m$, then, we see easily that $E\otimes E' = \sum_{i=1}^{n}\sum_{j=1}^{m}(A\otimes A')(x_i\otimes x'_j)$.

ii) Put $G = E\otimes E'$. We first show that $\cap\mathfrak{m}^nG = 0$. Let $\xi\in\cap\mathfrak{m}^nG$ and let $\xi = y_1\otimes y'_1+\cdots+y_t\otimes y'_t$ where $y_i\in E$ and $y'_i\in E'(i=1,\cdots,t)$ and y'_i are linearly independent over K. Since $\xi\in\mathfrak{m}^nG$ and since $\mathfrak{m}^nG\approx(\mathfrak{m}^nE)\otimes E'$, we have $y_i\in\mathfrak{m}^nE$ for any n. Therefore $y_i\in\cap\mathfrak{m}^nE=0$. From this we see that $\cap\mathfrak{m}^nG+\mathfrak{m}'^iG=\mathfrak{m}'^iG$, since $E'/\mathfrak{m}'^iE$ is a finite module over an $\mathfrak{m}'/\mathfrak{m}'^i$-adic Zariski ring $A'/\mathfrak{m}'^i$ and since $E\otimes(E'/\mathfrak{m}'^iE')$ is isomorphic to $E\otimes E'/\mathfrak{m}'^i(E\otimes E')=G/\mathfrak{m}'^iG$. Therefore

$$\bigcap_n(\mathfrak{m}\otimes A'+A\otimes\mathfrak{m}')^nG=\bigcap_n\mathfrak{m}^nG+\mathfrak{m}'^nG\subseteq\bigcap_i\bigcap_n(\mathfrak{m}^nG+\mathfrak{m}'^iG)=\bigcap_i\mathfrak{m}'^iG=0.$$

Let again E and E' be finite A- and A'-modules and let $E\hat{\otimes}E'$ and $A\hat{\otimes}A'$ be the completions of $E\otimes E'$ and $A\otimes A'$ in the $(\mathfrak{m}, \mathfrak{m}')(A\otimes A')$-adic topology respectively. The canonical $A\otimes A'$-bilinear mapping $(A\otimes A')\times(E\otimes E')\to E\otimes E'$ may be extended by continuity to a mapping $(A\hat{\otimes}A')\times(E\hat{\otimes}E')\to E\hat{\otimes}E'$ which makes $E\hat{\otimes}E'$ an $A\hat{\otimes}A'$-module. The canonical injection mapping $E\otimes E'\to E\hat{\otimes}E'$ can be extended, therefore, by linearlity to an $A\hat{\otimes}A'$-homomorphism:

$$\varepsilon:\quad (E\otimes E')\otimes_{A\otimes A'}(A\hat{\otimes}A')\to E\hat{\otimes}E'.$$

We shall prove that $\hat{\otimes}_K$ is an exact functor. Namely,

PROPOSITION 3. *Let A and A' be an $\mathfrak{m}$- and $\mathfrak{m}'$-adic Zariski rings. Assume A and A' have the common subfield K. Let E, F and G be finite A-modules such that*

$$(1)\qquad 0\to E\to F\to G\to 0\quad\text{(exact).}$$

Then, for any finite A'-module F', we have the exact sequence of $A\hat{\otimes}A'$-modules.

$$(2)\qquad 0\to E\hat{\otimes}F'\to F\hat{\otimes}F'\to G\hat{\otimes}F'\to 0.$$

PROOF. From (1), we have the exact sequence of A-modules:

$$E/\mathfrak{m}^nE\to F/\mathfrak{m}^nF\to G/\mathfrak{m}^nG\to 0.$$

Since $\otimes_K$ is right exact,

$$(E/\mathfrak{m}^nE)\otimes(F'/\mathfrak{m}'^nF')\to(F/\mathfrak{m}^nF)\otimes(F'/\mathfrak{m}'^nF')\to(G/\mathfrak{m}^nG)\otimes(F'\mathfrak{m}'^nF')\to 0$$

is eact as $A\otimes A'$-modules, i.e.,

$$E\otimes F'/(\mathfrak{m}^n, \mathfrak{m}'^n)\to F\otimes F'/(\mathfrak{m}^n, \mathfrak{m}'^n)\to G\otimes F'/(\mathfrak{m}^n, \mathfrak{m}'^n)^{8)}\to 0.$$

On the other hand we have the exact sequence of $A\otimes A'$-modules:

$$0\to E\otimes F'\to F\otimes F'\to G\otimes F'\to 0.$$

8) This means $G\otimes F'/(\mathfrak{m}^n, \mathfrak{m}'^n)(G\otimes F')$.

Since

$$(\mathfrak{m}^n, \mathfrak{m}'^n)(F\otimes F')\cap(E\otimes F') = ((\mathfrak{m}^n F)\otimes F' + F\otimes(\mathfrak{m}'^n F'))\cap(E\otimes F')$$
$$= (\mathfrak{m}^n F\cap E)\otimes F' + E\otimes \mathfrak{m}'^n F' \quad \text{(by the following Lemma 7)}$$
$$= (\mathfrak{m}^{n-r}(\mathfrak{m}^r F\cap E))\otimes F' + E\otimes \mathfrak{m}'^n F' \quad \text{(by Lemma 1)}$$
$$\subset \mathfrak{m}^{n-r} E\otimes F' + E\otimes \mathfrak{m}'^{n-r} F', \quad \text{for } n\geqq r,$$

we see easily that the above sequence (2) is exact as $A\hat{\otimes} A'$-modules. In order to prove Proposition 3, therefore, it remains to prove the following

LEMMA 7. *Let K be a field and let E and F be vector spaces over K, Then, for subspaces E' and E'' of E* and *F' of F, we have*

$$(E'\otimes F + E\otimes F')\cap(E''\otimes F) = (E'\cap E'')\otimes F + E''\cap F'.$$

PROOF. Since $(E'\cap E'')\otimes F'$ is contained in all subspaces of $E\otimes F$ in question, by passing to the factor space, we may assume $E'\cap E'' = 0$ and $F' = 0$. Therefore it is enough to show that $(E'\otimes F)\cap(E''\otimes F) = 0$. But this is obvious, since $E'\cap E'' = 0$.

PROPOSITION 4. *(With the same notations and assumptions as in Proposition 3.)*

$E\hat{\otimes}E'$ is a finite $A\hat{\otimes}A'$-module and $E\hat{\otimes}E' \approx (E\otimes E')\otimes_{A\otimes A'}(A\hat{\otimes}A')$.

PROOF. Let $L_1\to L\to E\to 0$ and $L'_1\to L'\to E'\to 0$ be the exact sequences of finite A-and A'-modules where L, L_1, L' and L'_1 are free. Then

$$L_1\otimes L' + L\otimes L'_1 \text{ (direct)} \to L\otimes L' \to E\otimes E' \to 0$$

is exact as $A\otimes A'$-modules [3, Proposition 4. 3, p. 24]. Put $B = A\hat{\otimes}A'$. Since $\hat{\otimes}$ is exact by Proposition 3, we have the following commutative diagram where both rows are exact:

$$\begin{array}{ccccc} (L_1\otimes L')\otimes B + (L\otimes L'_1)\otimes B & \to & (L\otimes L')\otimes B & \to & (E\otimes E')\otimes B^{9)} \to 0 \\ \downarrow \varepsilon_1 & & \downarrow \varepsilon_2 & & \downarrow \varepsilon_3 \\ L_1\hat{\otimes}L' + L\hat{\otimes}L'_1 & \to & L\hat{\otimes}L' & \to & E\hat{\otimes}E' \to 0 \end{array}$$

Since ε_1 and ε_2 are obviously bijective as B-modules, we see easily that ε_3 is bijective.

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9) This means $(E\otimes E')\otimes_{A\otimes A'} B$.

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