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A Condition Under Which Simple Closed Curves Bound Discs

By

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1. Introduction

A 3-manifold is a separable metric space M such that each point of M lies in an open set whose closure is a 3-cell (a set homeomorphic to the unit sphere plus its interior in euclidean 3-space E^3). We may assume without loss of generality that M has a certain triangulation by E. E. Moise's work. Moreover, every thing will be considered from the semi-linear point of view. For example, a curve means a polygonal one.

Let N be a connected subset in M, a, b points on N and C a curve from a to b such that $C-(a+b)\subset M-N$. Let us suppose that there exists a curve C' on N joining a to b and homotopic to C in M. If the image of the interior of Q under the homotopic mapping is in $M-N$, we shall say C is *ρ-homotopic* to C' in M with respect to N, where Q is the fundamental square of the homotopic mapping. If for any a, b and C there exists such a C', we denote this fact by $\pi_1(M-N, N)=1$.

In this paper the following theorem will be proved (§ 2).

THEOREM. *Let M be a 3-manifold, compact or not, with boundary which may be empty and L a simple closed curve in M. A necessary and sufficient condition that L bounds a disc in M is that there exists a neighborhood U of L such that L is homotopic to zero in U and* $\pi_1(U-L, L)=1$.

The proof of this theorem is based on Dehn's lemma, ([2], [5], [7]) i.e. if M is a 3-manifold as in the theorem and D is a Dehn disc (§ 2) in M, then bd D[1] bounds a disc.

In the last section we shall refer to knots in 3-sphere S^3 as an application of the theorem.

2. The proof of the theorem

We shall first note several definitions used in this section. Let $\widetilde{D}$ be a triangulated disc, and $f:\widetilde{D}\rightarrow M$ an unhomeomorphic mapping of $\widetilde{D}$ into M such

(1) bd = boundary.

that $f(\sigma)$ is a rectilinear 2-simplex in a 3-simplex in M for each 2-simplex σ of $\tilde{D}$. Then $f(\tilde{D})$ is called a *singular disc* and the pair $(\tilde{D}, f)$ the *diagram* of $f(\tilde{D})$. If L is the closed curve $f(\mathrm{bd}\,\tilde{D})$, then we shall say that L bounds $f(\tilde{D})$ in M and L is the boundary of $f(\tilde{D})$, denoted by $L=\mathrm{bd}\,f(\tilde{D})$. A *Dehn disc* [5] is a singular disc such that no point of the boundary of it is singular, i.e. there is a small neighborhood of L in $f(\tilde{D})$, which is an annulus with L as one of two components of its boundary. (Hence L is simple.)

Necessity. Let D be a disc bounded by L. There exists a thin closed neighborhood U of D in M which is a strong deformation retract ([3], p. 30) of a 3-cell V whose interior contains D (V may be an ideal figure not contained in M). Since $\pi_1(V-L, L)=1$, we have also $\pi_1(U-L, L)=1$.

Sufficiency. It suffices to prove that L bounds a Dehn disc in M by Dehn's lemma.

(i) Singular disc D_1. Since L is homotopic to zero in M, there exists a continuous mapping of a disc $\tilde{D}_1$ into M which takes $\mathrm{bd}\,\tilde{D}_1$ onto L. We may suppose that the mapping is semi-linear, hence L bounds a singular disc D_1 which may have singularities on $\mathrm{bd}\,D_1=L$. In addition, we may assume that the singularities of D_1 consist of the following (cf. [2], pp. 147–148, [4], p. 249, [5], pp. 3–5): double curves along which two sheets

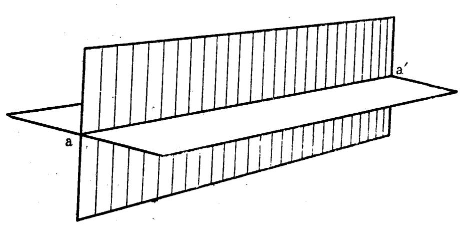

Fig. 1

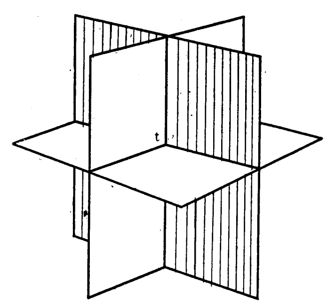

Fig. 2

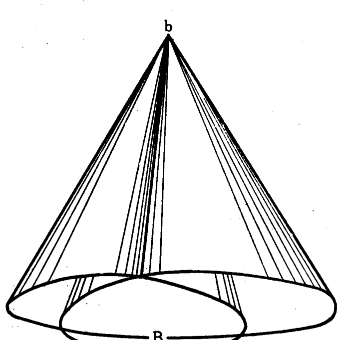

Fig. 3_1

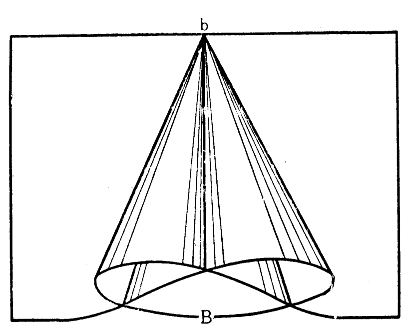

Fig. 3_2

cross, (aa' in Fig. 1), triple points at which three sheets cut (t in Fig. 2), (so that it is contained in the interior of D_1), branch points (b in Figs. 3_1 and 3_2) and piercing points (p, p' in Figs. 7 and 8). A point b is called a branch point of D_1 if a sufficiently small 2-sphere in M with b as its center cuts D_1 in a single non-simple curve B where in the small domain in M bounded by the sphere there is no singular point (i.e. triple point or branch point or piercing point) other than b. If B is (not) closed, it is called to be of the 1st (2nd) kind as in Fig. 3_1 (3_2). Any branch point of the 1st (2nd) kind is in the interior (on the boundary) of D_1. The number of double points of B is called the *multiplicity* of b, denoted by $m(b)$ ([5], p. 5).

A piercing point p lies on L and its local aspect is as follows: in a vicinity of the origin $(0, 0, 0)$ corresponding to p, L corresponds to x-axis, and D_1 to (y, z)-plane plus the half-plane $\{(x, y, z), y \geqq 0, z=0\}$.

A singular disc is said to be *normal* if its singularities are at most double curves, triple points and piercing points.

(ii) Removing branch points of the 2nd kind. This will be done by making small cuts of D_1, which will be defined in the following, along double curves starting at branch points of the 2nd kind and ending at points of B. Let b be a branch point of the 2nd kind and the pair $(\tilde{D}_1, f_1)$ the diagram of D_1 (§ 1). Let us denote $f_1^{-1}(B)$, $f_1^{-1}(b)$, $f_1^{-1}(L)$ by $\tilde{B}$, $\tilde{b}$, $\tilde{L}$, respectively. Let be_i $(i \geqq 1)$ be double curves which issue from b and end at points e_i on B. We may assume that $\tilde{D}_1$ is a square plus its interior and B is a semi-circle in $\tilde{D}_1$ whose center $\tilde{b}$ is on $\tilde{L}$ (Fig. 4_1). $f_1^{-1}(e_i)$ consists of two points $\tilde{e}_i$ and $\tilde{e}'_i$ belonging to $\tilde{B}$. And it may be assumed that $\tilde{b}\tilde{e}_i + \tilde{b}\tilde{e}'_i = f_1^{-1}(be_i)$. We construct a new diagram $(\tilde{D}'_1, f'_1)$ from $(\tilde{D}_1, f_1)$ as follows (Fig. 4_2): let $\tilde{d}_1$ and $\tilde{d}_2$ be the end points of $\tilde{B}$ (where $\tilde{d}_1$, $\tilde{e}_1$, $\tilde{e}'_1$, $\tilde{d}_2$

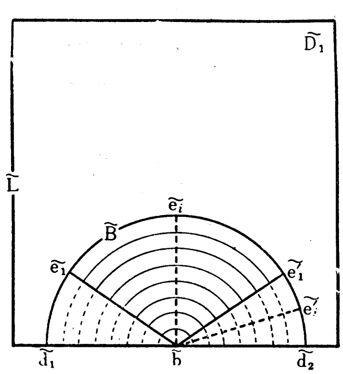

Fig. 4_1

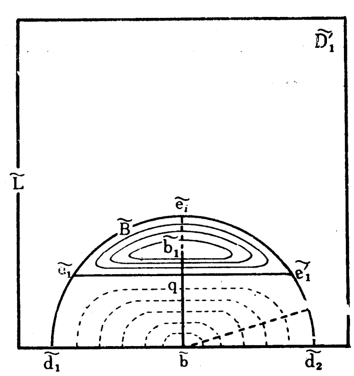

Fig. 4_2

are arranged in clock-wise order) and Δ the disc in $\tilde{D}_1$ bounded by $\tilde{B}$ and the subarc of $\tilde{L}$ containing $\tilde{b}$. The segment $\tilde{e}_1\tilde{e}'_1$ divides Δ into two discs Δ_1, Δ_2 where Δ_1 is bounded by $\tilde{e}_1\tilde{e}'_1$ and the subarc λ of $\tilde{B}$ whose end points are $\tilde{e}_1$ and $\tilde{e}_1'$. Now let $\tilde{b}_1$ be a point in the interior of Δ_1. If $\tilde{e}_i$ (or $\tilde{e}_i'$) is on λ or $\tilde{B}-\lambda$ ($i\neq 1$), we join it to $\tilde{b}_1$ or $\tilde{b}$ by a segment. Thus we have a new diagram $\tilde{D}_1'$.

Now we shall define a mapping f_1' of $\tilde{D}_1'$ into M. Let $f_1'(x)=f_1(x)$ for each $x\in(\tilde{D}_1-\text{int}\,\Delta)^{(2)}+\tilde{L}$; $f_1'(\tilde{e}_1\tilde{e}_1')=e_1$; $f_1'(\tilde{b}_1\tilde{e}_i)=be_i$ if $\tilde{e}_i\in\lambda$ and $f_1'(\tilde{b}\tilde{e}_i)=be_i$ if $\tilde{e}_i\notin\lambda$, for $i\neq 1$ (the same is true for $\tilde{e}_i'$). And we have $f_1'(\tilde{b}_1 q)=f_1'(\tilde{b}q)=be_1$ where $q=\tilde{b}\tilde{b}_1\cdot\tilde{e}_1\tilde{e}_1'$. Furthermore $\tilde{b}q$ divides Δ_2 into discs Δ_{21} and Δ_{22} where $\text{bd}\,\Delta_{21}=\tilde{b}\tilde{d}_1+$(the subarc of $\tilde{B}$ with end points $\tilde{d}_1$ and $\tilde{e}_1$)$+\tilde{e}_1 q+q\tilde{b}$ or $\Delta_{22}=\overline{\Delta_2-\Delta_{21}}^{(3)}$. Let S_1, S_2 and S_3 be the sectors of $\tilde{D}_1$ bounded by $\tilde{b}\tilde{e}_1+\lambda+\tilde{e}_1'\tilde{b}$, $\tilde{b}\tilde{d}_1+$(the subarc of $\tilde{B}$ with end points $\tilde{d}_1$ and $\tilde{e}_1$)$+\tilde{e}_1\tilde{b}$ and $\tilde{b}\tilde{d}_2+$(the subarc of $\tilde{B}$ with end points $\tilde{d}_2$ and $\tilde{e}_1'$)$+\tilde{e}_1'\tilde{b}$, respectively. Then we shall define f_1' in such a way as $f_1'(S_i)=f_1(S_i)$ ($i=1, 2, 3$). The two families of curves in Figs. 4_1 and 4_2 show this correspondence.

Hence we have a singular disc $D_1'=f_1'(\tilde{D}_1')$, with $(\tilde{D}_1', f_1')$ as its diagram. By a small modification of D_1', which leaves fixed every thing outside $f_1'(\Delta)$, we obtain a new normal singular disc D_1'' (§ 2, (i)) with a diagram $(\tilde{D}_1', f_1'')$, such that (i) to the point b of D_1 correspond two points $f_1''(\tilde{b})$, $f_1''(\tilde{b}_1)$ of D_1'', (2) e_1 is a branch point of the 1st kind and the multiplicity of e_1 is equal to 1, $m(f_1''(\tilde{b}))\geqq 0$ and $m(f_1''(\tilde{b}_1))\geqq 0$, and $m(f_1''(\tilde{b}))+m(f_1''(\tilde{b}_1))<m(b)$, (3) D_1'' is obtained from D_1 by deformation. The above operation is called a cut of D_1 along the double curve be_1 and it reduces the multiplicity of b.

By repeating this operation for branch points of the 2nd kind, we have a singular disc D_2 with boundary L which has not branch point of the 2nd kind and has the diagram $(\tilde{D}_2, f_2)$.

(iii) Reduction of multiplicity of branch points of the 1st kind ([5], p. 6, Lemma (3.1)). As in (ii), we illustrate this operation by diagrams

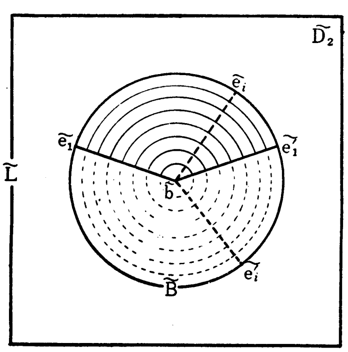

Fig. 5_1

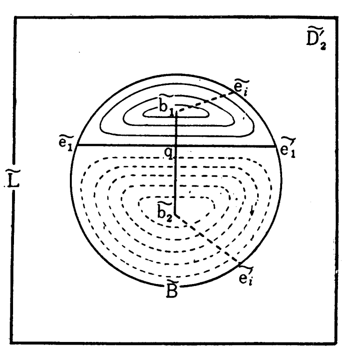

Fig. 5_2

(2) int=interior.

(3) $\bar{A}$ means the closure of A.

(Figs. 5_1 and 5_2). Let b be a branch point of the 1st kind whose multiplicity is larger than 1, i.e. $m(b)>1$, and let $\tilde{B}, \tilde{b}, \tilde{e}_i, \tilde{e}'_i$ and $\tilde{L}$ have the similar meanings as in (ii). We note that $\tilde{B}$ is closed, i.e. it is a circle in the interior of $\tilde{D}_2$ and the number of double curves issued from b is $\geqq 2$. Let $\varDelta$ be the disc in $\tilde{D}_2$ bounded by $\tilde{B}$ and $\tilde{b}_1(\tilde{b}_2)$ a point in the interior of the disc $\varDelta_1(\varDelta_2)$ into which $\varDelta$ is divided by the segment $\tilde{e}_1\tilde{e}'_1$. If $\tilde{e}_i \in \varDelta_1(\tilde{e}_i \in \varDelta_2)$, we join $\tilde{e}_i$ to $\tilde{b}_1(\tilde{b}_2)$ by the segment $\tilde{e}_i\tilde{b}_1(\tilde{e}_i\tilde{b}_2)$. The same is true for $\tilde{e}'_i$.

We shall define a continuous mapping f'_2 of the new diagram into M as follows: let $f'_2(x)=f_2(x)$ for each $x \in \tilde{D}'_2-\operatorname{int}\varDelta$, $f'_2(\tilde{b}_1)=f'_2(\tilde{b}_2)=b$, $f'_2(\tilde{e}_1\tilde{e}'_1)=e_1$, $f'_2(\tilde{b}_1\tilde{e}_i)=be_i$ if $\tilde{e}_i \in \operatorname{bd}\varDelta_1$ and $f'_2(\tilde{b}_2\tilde{e}_i)=be_i$ if $\tilde{e}_i \in \operatorname{bd}\varDelta_2$, for $i \neq 1$ (the same is true for $\tilde{e}'_i$). We put $f'_2(\tilde{b}_1 q)=f'_2(\tilde{b}_2 q)=be_1$ where $q=\tilde{b}_1\tilde{b}_2\cdot\tilde{e}_1\tilde{e}'_1$. Furthermore f'_2 is defined on the domain $\operatorname{int}(\varDelta_1-\tilde{b}_1 q)$ ($\operatorname{int}(\varDelta_2-\tilde{b}_2 q)$), in such a way as $f'_2(\operatorname{int}(\varDelta_1-\tilde{b}_1 q))$ ($f'_2(\operatorname{int}(\varDelta_2-\tilde{b}_2 q))$) is the image under f_2 of the sector of $\varDelta$ bounded by $\tilde{b}\tilde{e}_1+\tilde{b}\tilde{e}'_1+$(a subarc of $\tilde{B}$ with end points $\tilde{e}_1$ and $\tilde{e}'_1$).

Thus we have a singular disc D'_2 with its diagram $(\tilde{D}'_2, f'_2)$ and with L as its boundary. By a small modification such that $f'_2(\tilde{D}'_2-\operatorname{int}\varDelta)$ is fixed, we have a new normal singular disc D''_2 with diagram $(\tilde{D}'_2, f''_2)$ such that (1) to the point b of D_2 correspond two points $f''_2(\tilde{b}_1)$ and $f''_2(\tilde{b}_2)$ of D''_2, (2) $f''_2(\tilde{e}_1)=e_1$ is a branch point of the 1st kind of D''_2 and $m(e_1)=1$ and (3) $m(f''_2(\tilde{b}'_1))\geqq 0$, $m(f''_2(\tilde{b}_2))\geqq 0$ and $m(f''_2(\tilde{b}_1))+m(f''_2(\tilde{b}_2))<m(b)$.

We shall call the above operation which constructs the diagram $(\tilde{D}'_2, f''_2)$ from $(\tilde{D}_2, f_2)$, a cut along be_1. Though the operation gives rise to new branch points with multiplicity 1, it reduces the multiplicity of b. By repeating a finite number of times the above process we finally obtain a normal singular disc D_3 with $(\tilde{D}_3, f_3)$ as its diagram and such that (1) each branch point of D_3 is of the 1st kind and its multiplicity is equal to 1, (2) $\operatorname{bd} D_3=L$, (3) D_3 is obtained from the initial disc D_1 by deformation.

(iv) Removing branch points from D_3. Let b be a branch point of D_3. Since $m(b)=1$, there issues only one double curve C from b. Let us start at b along C and go to the end. If we meet a triple point on our way, we pass by there. Then only two cases may happen.

a) We arrive at a branch point b' of the 1st kind. In this case the inverse image in $\tilde{D}_3$ of the double curve C through which we have passed is a simple closed curve contained in $\operatorname{int}\tilde{D}_3$ and which consists of two simple arcs $\tilde{C}$ and $\tilde{C}'$ joining $f_3^{-1}(b)$ to $f_3^{-1}(b')$. We remove from $\tilde{D}_3$ the interior of the disc in $\tilde{D}_3$ bounded by the simple closed curve $\tilde{C}+\tilde{C}'$ and gluing $\tilde{C}$ to $\tilde{C}'$ in such a way as each point $x \in \tilde{C}$ is identified with the point

$x' \in \tilde{C}'$ such that $f_3(x) = f_3(x')$. Hence we have a new diagram $\tilde{D}_3'$ from $\tilde{D}_3$. $f_3(\tilde{D}_3')$ is a new normal singular disc whose boundary is L and by this process, called a cut of D_3 along the double curve C, two branch points b, b' of D_3 and a double curve are removed from D_3.

b) Our way ends with a piercing point p. In this case the diagram is illustrated by Fig. 6_1. We shall construct a new diagram from the first one. We may suppose that the inverse image of $C(=\tilde{C}+\tilde{C}')$ is a segment perpendicular to a side of $\tilde{D}_3$ ($\tilde{p}\tilde{b}\tilde{p}'$ in Fig. 6_1). Let Δ be a disc in $\tilde{D}_3$

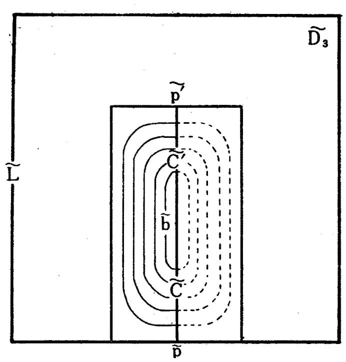

Fig. 6_1

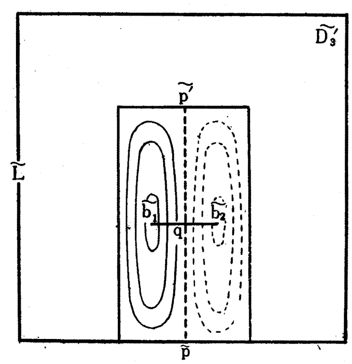

Fig. 6_2

whose boundary is a long rectangle in a small neighborhood in $\tilde{D}_3$ of $\tilde{C}+\tilde{C}'$ such that it does not contain any of inverse images of the other branch points, a short side of Δ is a subarc of $\tilde{L}$ whose interior contains $\tilde{p}$ and the opposite side is a segment in int $\tilde{D}_3$ whose interior contains $\tilde{p}'$ and such that both long sides are parallel to $C+C'$. The segment $\tilde{p}\tilde{p}'=\tilde{C}+\tilde{C}'$ divides Δ into two discs Δ_1 and Δ_2. Now let $\tilde{b}_1$ and $\tilde{b}_2$ be points in int Δ_1 and int Δ_2 respectively and let q be the point in which $\tilde{b}_1\tilde{b}_2$ meets $\tilde{p}\tilde{p}'$. We shall define a continuous mapping f_3' of $\tilde{D}_3'$ into M as follows: let $f_3'(x)=f_3(x)$ for each $x \in (\tilde{D}_3' - \text{int}\,\Delta) + \tilde{L}$, $f_3'(\tilde{b}_1)=f_3'(\tilde{b}_2)=b$, $f_3'(\tilde{p}\tilde{p}')=p$, $f_3'(\tilde{b}_1 q)=C=f_3'(\tilde{b}_2 q)$, $f_3'(\text{int}\,\Delta_1 - \tilde{b}_1 q)=f_3(\text{int}\,\Delta_1)$ and $f_3'(\text{int}\,\Delta_2 - \tilde{b}_2 q)=f_3(\text{int}\,\Delta_2)$. The two families of curves in Figs. 6_1 and 6_2 show the correspondence. Thus we have a new singular disc D_4' whose boundary is L. By a small deformation in a vicinity of C, which leaves fixed each point of L, we have from D_4' a normal singular disc D_4'' which is obtained by removing from D_3 a branch point of the 1st kind and a piercing point. The above operation is called a cut along C. Intuitively, it has the same meaning as cutting D_3 by scissors along C.

Repeating the above process a) or b), we can finally obtain a normal singular disc D_4 with $(\tilde{D}_4, f_4)$ as its diagram, having no branch point. Such normal singular disc is called to be canonical.

(v) A property of L. Let p, p' be two points of the oriented simple closed curve L. Each 1-simplex A_i of L has the orientation induced from the L's one. If C' is a curve on L joining p to p', it is denoted by

$$C' = A_{i_1}^{\varepsilon_1} A_{i_2}^{\varepsilon_2} \cdots A_{i_k}^{\varepsilon_k} \qquad (\varepsilon_i = \pm 1).$$

If all the powers ε_j of A_{i_j} are equal to $+1$ (or -1), C' is called to *have no turning point.*

Let C be a curve from p to p' such that $C-(a+b) \subset M-L$. We shall show that there exists a curve C' on L joining p to p' and having no turning point and such that C is ρ-homotopic to C' in M with respect to L (§1). Let T be a small tubular neighborhood of L in M. We may suppose that $C \cdot \overline{T}$ consists of two subarcs pp_1, $p'p_1'$ of C. For if not we have such a solid torus by retraction of T. Moreover it may be assumed that $\pi_1(U-T, \mathrm{bd}\, T)=1$, since $\pi_1(U-L, L)=1$ by the assumption of the theorem. Hence the curve $C_1 = C - (\overline{pp_1 + p'p_1'})$ in $U-T$ joining p_1 to p_1' is homotopic in $U-T$ to a curve C_1' on $\mathrm{bd}\, T$ from p_1 to p_1'. We can easily see that there exists a curve C' on L joining p to p' and having no turning point, such that $pp_1 + C_1' + p_1'p'$ is ρ-homotopic to C' in $\overline{T}$ with respect to L. The product of these homotopies is the desired homotopic mapping of C onto C'.

(vi) Removing piercing points from D_4. Let p be a piercing point of D_4. If we start from p along the double curve C issued from p and go straightly forward through triple points with which we may meet on our way, we must reach at a piercing point p' different from p since D_4 is canonical (see (iv)). Let K_1, K_2 be two long ribbon-shaped quadrilaterals which cross along C. We must distinguish two cases: a) a short side and its opposite one of K_1 are subarcs of L (Fig. 7) and b) a short side of K_1 and one of K_2 are subarcs of L (Fig. 8). In the case a) (b)), the pair p, p' of piercing points is called to be of the *1st* (*2nd*) *kind.*

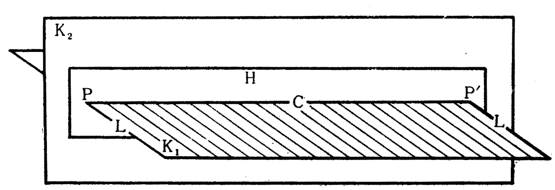

Fig. 7

a) Removing pairs of piercing points of the 1st kind.

We may assume that $L \cdot \mathrm{int}\, C = \phi$. By (v), there exists a singular disc D^* such that $\mathrm{int}\, D^* \subset M-L$ and $\mathrm{bd}\, D^* = C + C'$, where C' is a curve on L joining p to p' and having no turning point. We note that D^* has no piercing point.

Now let $\tilde{D}$, $\tilde{D}'$ be two squares on (x, y)-plane whose vertices are $(\frac{1}{2}, 0, 0)$, $(\frac{1}{2}, 1, 0)$, $(-\frac{1}{2}, 1, 0)$, $(-\frac{1}{2}, 0, 0)$; $(\frac{1}{2}+\varepsilon, 0, 0)$, $(\frac{1}{2}+\varepsilon, 1+\varepsilon, 0)$, $(-\frac{1}{2}-\varepsilon, 1+\varepsilon, 0)$,

$(-\frac{1}{2}-\varepsilon, 0, 0)$, respectively, where $\varepsilon>0$ may be chosen as small as we please. Then we denote the set $\{(x, y, z);\ (x, y, 0)\in\tilde{D}',\ |z|\leqq\varepsilon\}$ by N and the intersection of N and (x, z)-plane by $\tilde{H}$. Next let us consider a long ribbon-shaped quadrilateral H on K_2 such that $\operatorname{int} H\supset C$ and both long sides of H are "parallel" to C (Fig. 7). Let $(\tilde{D}^*, f)$ be the diagram of D^* such that $f((-\frac{1}{2}, 0, 0))=p'$, $f(\langle-\frac{1}{2}, \frac{1}{2}\rangle)=C$, $f(\operatorname{bd}\tilde{D}^*-\operatorname{int}\langle-\frac{1}{2}, \frac{1}{2}\rangle)=C'$ and $f(\tilde{H})=H$, where $\langle\xi, \eta\rangle$ means the interval on x-axis between $(\xi, 0, 0)$ and $(\eta, 0, 0)$. We extend the map $f:\tilde{D}^*\to D^*\subset M$ to a map $F:N\to M$ as follows: if $(x, y, 0)\in\operatorname{int}\tilde{D}$, the vertical line segment in N through $(x, y, 0)$ is mapped into the adjusted normal of D^* at $f((x, y, 0))$. Furthermore, let $(x, y, 0)\in\operatorname{bd}\tilde{D}^*-\operatorname{int}\langle-\frac{1}{2}, \frac{1}{2}\rangle$. Let $B(x, y, 0)$ be the disc which is a component of the intersection N' with the plane perpendicular to (x, y)-plane and containing $(0, 0, 0)$ and $(x, y, 0)$, where $N'=\{(x, y, z);\ (x, y, 0)\in\tilde{D}^*-\operatorname{int}\tilde{D},\ |z|\leqq\varepsilon\}$. It is sufficient to attach "continuously" $F(B(x, y, 0))$ to $f(x, y, 0)$. This operation is no other than thickening of D^*. Hence we have a singular disc $F(\operatorname{bd}N-\operatorname{int}\tilde{H})$ in M. In D_4 let us replace H by $F(\operatorname{bd}N-\operatorname{int}\tilde{H})=H_1$, then we obtain a new singular disc D_4' from D_4. Since C' has no turning point, ε is very small and $L\cdot\operatorname{int}D^*=\phi$, H_1 and L have no point in common. Hence a pair of piercing points p, p' is removed from D_4.

By a small modification which leaves fixed any point of a neighborhood of L, we have from D_4' a normal singular disc D_4'' such that $\operatorname{bd}D_4''=L$, the number of pairs of piercing points of D_4'' is less 1 than that of D_4 and D_4'' has no branch point of the 2nd kind. Furthermore, by the process (i), (ii), and (iii) we have from D_4'' a new normal singular disc having no branch point.

Repeating the above process, we have a normal singular disc D_5 which has neither pair of piercing points of the 1st kind nor branch point of the 2nd kind.

b) Removing pairs of piercing points of the 2nd kind from D_5.

Let p, p' be a pair of piercing points of the 2nd kind of D_5 and C the double curve along which we walk from p to p'. Let $D^*, \tilde{D}^*, f, C'$ and ε have the same meanings as in a). Let N' be the cone in E^3 whose base is the rectangle with the vertices $(\frac{1}{2}+\varepsilon, 0, \pm\varepsilon)$, $(-\frac{1}{2}-\varepsilon, 1+\varepsilon, \pm\varepsilon)$ and whose vertex is $(-\frac{1}{2}, 0, 0)$. Let Q be the triangle with its interior whose vertices are $(\frac{1}{2}+\varepsilon, 0, 0)$, $(\frac{1}{2}+\varepsilon, 1+\varepsilon, 0)$ and $(-\frac{1}{2}-\varepsilon, 1+\varepsilon, 0)$. Now we denote the set, $\{(x, y, z);\ (x, y, 0)\in Q,\ |z|\leqq\varepsilon\}+N'$ and the face of N which is the intersection of N with (x, z)-plane by N and $\tilde{H}$, respectively.

In a small neighborhood of C in K_2 we take a long wedge-shaped triangle H around C such that the vertex of its sharp angle is p, C is a subarc of the middle line issued from p and p' is in the interior of the disc H_2 on K_2 bounded by the triangle (Fig. 8). We extend $f:\tilde{D}^*\to D^*$, to a map $F:N\to M$ in such a way as $F(\tilde{H})=H$, $F(-\frac{1}{2},0,0)=p$, $F(\frac{1}{2},0,0)=p'$, $F(\langle -\frac{1}{2},\frac{1}{2}\rangle)=C$ and $F(\tilde{H})=H$.

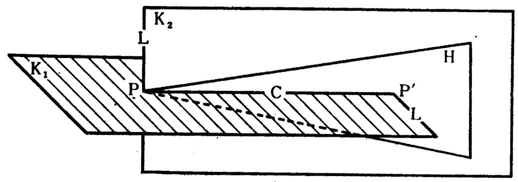

Fig. 8

This will be done in the same way as in a) and we replace H in D^* by $F(\mathrm{bd}\, N-\mathrm{int}\, \tilde{H})=H_1$. Thus we have a new singular disc D_5' from D_5. Since C' has no turning point, ε is very small and $L\cdot\mathrm{int}\, D=\phi$, H_1 is in contact with L at only one point p. Since p is neither piercing point nor branch point of D_5', by a small modification we have a normal singular disc D_5'' which has neither branch point of the 2nd kind nor pair of piercing points of the 1st kind and such that the number of pairs of piercing points of the 2nd kind is less 1 than that of D_5.

Repeating this process we have a Dehn disc in M having L as its boundary. Thus L bounds a disc in M by Dehn's lemma. Q.E.D.

3. Applications

1. By a torus manifold we mean a closed 3-manifold obtained by gluing $\mathrm{bd}\, T$ to $\mathrm{bd}\, T'$, where T and T' are solid tori. The study of the topological type of a given torus manifold M is conceivably reduced to that of a simple closed curve L in M, such that $\pi_1(M-L, L)=1$. For example, the following corollary which has been obtained in [1] is a consequence of the theorem.

COROLLARY 1. *Any simply connected torus manifold M is topologically* S^3.

PROOF. Let L be a core of T. Since M is simply connected, L is homotopic to zero in M. We see, in addition, that $\pi_1(M-L, L)=1$, for $\pi_1(T, \mathrm{bd}\, T')=1$. By the theorem, L bounds a disc D in M. We may, without loss of generality, assume that $T\cdot D$ is an annulus whose boundary is L plus a longitude l of T. Hence l is a meridian of T'. Therefore M is topologically S^3. Q.E.D.

In general, any orientable closed 3-manifold is homeomorphic to a 3-manifold obtained by gluing $\mathrm{bd}\, T_h$ to $\mathrm{bd}\, T_h'$, where h is an integer $\geqq 0$ and T_h, T_h' are solid tori of genus h (Henkelkörper von Geschlecht h [6], p. 219). Let us consider a retraction N of a solid torus of genus h which consists of h simple closed curves, $L_1, \cdots, L_h$, such that $L_i\cdot L_j=$ a point if $|i-j|=1$

and $=\phi$ if $|i-j|>1$, and such that $\pi_1(M-N, N)=1$. We may only consider N for a Heegaard diagram of M.

2. If a knot L in 3-sphere S^3 bounds a disc in S^3, it is called to be unknotted. By using this term, we have

COROLLARY 2. *L is unknotted if and only if* $\pi_1(S^3-L, L)=1$.

From this corollary we see that the following four conditions are equivalent: (i) L is unknotted, (ii) L is algebraically unknotted, i.e. $\pi_1(S^3-L)$ is free cyclic, (iii) the number of ends of $\pi_1(S^3-L)$ is 2 and (iv) $\pi_1(S^3-L, L)=1$. For the equivalence of (i)~(iii) refer to [5].

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Bibliography

[1] Bing, R. H.: Necessary and sufficient conditions that a 3-manifold be S^3, Ann. of Math. **68** (1958), 17-37.

[2] Dehn, M.: Ueber die Topologie des dreidimensionalen Raumes, Math. Ann., **69** (1910), 137-168.

[3] Eilenberg, S. and Steenrod, N.: Foundations of algebraic topology.

[4] Kneser, H.: Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten, Jber. Deutschen Math. Verein., **38** (1929), 248-260.

[5] Papakyriakopoulos, C. D.: On Dehn's lemma and asphericity of knots, Ann. of Math. **66** (1957), 1-26.

[6] Seifert, H. and Threlfall, W.: Lehrbuch der Topologie, Chelsea, 1947.

[7] Shapiro, A. and Whitehead, J. H. C.: A proof and extension of Dehn's lemma, Bull. Amer. Math. Soc. **64** (1958), 174-178.

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