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On Geodesic Subspaces of Group Spaces

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Introduction

The purpose of this paper is to investigate the properties of the geodesic subspaces of group spaces, mainly, by means of the exponential mapping of the Lie algebra into the Lie group. Let G be a connected real or complex Lie group with the Lie algebra $\mathfrak{G}$, and let S be a geodesic subspace (cf. §2 for definition) of the group space G with the associated linear subspace $\mathfrak{S}_x(x \in S)$ of $\mathfrak{G}$, then $\mathfrak{S}_x(x \in S)$ are all Lie triple systems (Lemma 2). In §1 we shall state the fundamental concepts which are used in the following sections. In §2 we shall define the geodesic subspaces of a group space and consider the linear subspaces $\mathfrak{S}_x$ and the relations among them. The results obtained in §2 are as follows:

1. If there exists a geodesic arc on $S: y(t)=x \exp tA$ having the end points x and y, then it holds

$$\mathfrak{S}_y=\exp\left(-\tfrac{1}{2}\operatorname{ad} A\right)\cdot\mathfrak{S}_x.$$

2. Under the same assumption as in 1, if A is a regular element of $\mathfrak{S}_x$ (cf. §1 for definition), then the intersection $\mathfrak{S}_x \cap \mathfrak{S}_y$ of $\mathfrak{S}_x$ and $\mathfrak{S}_y$ is given as

$$\mathfrak{S}_x \cap \mathfrak{S}_y=\{Y;\ Y \in \mathfrak{S}_x,\ \operatorname{ad} A \cdot Y \in \mathfrak{S}_x\}.$$

In §3, we shall consider the analogous property (Lemma 5) for the geodesic subspaces of Schreier's Theorem in the theory of topological groups, and the results obtained by means of this property are follows:

3. For any elements x and y of S, $\mathfrak{S}_y$ is transferred from $\mathfrak{S}_x$ by an inner automorphism of $\mathfrak{G}$.

4. For any elements x and y of S, there exists an analytic curve in G through x and y, which lies on $x \exp \mathfrak{S}_x$ in a sufficiently small neighborhood of x in G.

From the result 4, by means of the method of Schröder ([8])[1] the following result is obtained.

5. Let $\mathfrak{S}$ be a complex geodesic subspace of a complex linear Lie group, then we have $S \subset x\,\overline{\exp \mathfrak{S}_x}$, where $\overline{\exp \mathfrak{S}_x}$ denotes the closure of $\exp \mathfrak{S}_x$ in G.

1) Numbers in brackets refer to the references at the end of the paper.

§1. Preliminaries

Let G be a connected real or complex Lie group and let $\mathfrak{G}$ be its Lie algebra, which is identified with the tangent space to G at the identity element e. For any $X \in \mathfrak{G}$, $\{\exp tX;\ t$ real or complex$\}$ denotes the unique one parameter subgroup of G whose direction at the identity element e is X, and $\exp X$ denotes its element for $t=1$. Thus we can consider the so-called exponential mapping: $X \to x = \exp X$ of $\mathfrak{G}$ into G. The exponential mapping is an analytic mapping of the analytic manifold $\mathfrak{G}$ into the analytic manifold G. Let $\mathfrak{V}$ be a finite dimensional linear space over the field of real numbers or the field of complex numbers and let $\mathfrak{E}$ denote the linear space of endomorphisms of $\mathfrak{V}$. If G is a group of endomorphisms in $\mathfrak{E}$, then the exponential mapping is expressed by $\exp X = \Sigma_0^\infty X^n/n!$. By a geodesic of the group space (i.e., the underlying space of a Lie group) we shall mean a coset of one parameter subgroup of G: $\{x \exp tX;\ t$ real or complex$\}$, x being an element of G. Moreover, $\mathrm{Ad}\, x$ denotes the differential of the inner automorphism $g \to xgx^{-1} (g \in G)$ at the identity element e, and is an automorphism of $\mathfrak{G}$. $\mathrm{ad}\, X$ denotes the inner derivation of $\mathfrak{G}$, i.e., $\mathrm{ad}\, X: W \to [X, W] (W \in \mathfrak{G})$. It is well known that $\mathrm{Ad} \exp X = \exp \mathrm{ad}\, X$.

As a characteristic property of Lie groups, there exist the symmetric neighborhoods U and $U_1 (U \cdot U \subset U_1)$ of the identity element e of G, and the symmetric neighborhoods $\mathfrak{U}$ and $\mathfrak{U}_1 (\mathfrak{U} \subset \mathfrak{U}_1)$ of the zero element 0 of $\mathfrak{G}$, satisfying the following properties:

(1) The mapping $\exp: \mathfrak{U}_1 \to U_1 = \exp \mathfrak{U}_1$ is an analytic isomorphism (i.e., exp is a homeomorphism of $\mathfrak{U}_1$ with U_1, both exp and its reciprocal mapping $\exp^{-1}$ are everywhere analytic).

(2) For any two elements $x = \exp X$ and $y = \exp Y$ $(X, Y \in \mathfrak{U})$ of U, there exists the unique element Z of $\mathfrak{U}_1$ such that

$$xy = \exp X \exp Y = \exp Z,$$

where Z can be expressed by the so-called SCH series (Schur-Campbell-Hausdorff series) ([2], [5], [9]).

For an element $a = \exp A$ $(A \in \mathfrak{G})$ of G we can define $Y(t) \in \mathfrak{G}$ by the relation:

$$\exp A \exp Y(t) = \exp (A + tX), \quad (X \in \mathfrak{G}),$$

and $Y(t)$ is expressed by

$$Y(t) = t\,(\exp(-\mathrm{ad}\, A) - I)/{-\mathrm{ad}\, A} \cdot X + O(t^2),$$

where $(\exp(-\mathrm{ad}\, A) - I)/{-\mathrm{ad}\, A}$ means $\sum_1^\infty (-\mathrm{ad}\, A)^{m-1}/m!$. Here if we set

$$Z = (dY/dt)_{t=0} \text{ and } \chi(A) = (\exp(-\mathrm{ad}\, A) - I)/{-\mathrm{ad}\, A},$$

then we have $Z = \chi(A) X$ ([4], p. 157).

By Ado-Cartan's theorem, it is well known that a Lie algebra has a faithful representation ([1], [3]) and so we may consider $\mathfrak{G}$ as a linear Lie algebra. Therefore, as for the local properties of a Lie group G, we may take G as a linear Lie group without loss of generality. Then the exponential mapping is the exponential function of an endomorphism (i.e. the exponential matrix function). For this reason, the above formulas are also verified, because they are clear for a linear Lie group.

A linear subspace $\mathfrak{S}$ of $\mathfrak{G}$ is said to be a Lie triple system, if $[X, [Y, Z]] \in \mathfrak{S}$ for any X, Y and $Z \in \mathfrak{S}$. This condition is equivalent to that $(\mathrm{ad}\, X)^2 Y \in \mathfrak{S}$ for any X and $Y \in \mathfrak{S}$ ([7]). An element A of $\mathfrak{G}$ is called a regular element of $\mathfrak{G}$, if $\chi(A)$ has an inverse, (i.e., for which $\mathrm{ad}\, A$ has no eigen values such as $2l\pi\sqrt{-1}$ (l: non-zero integers). The set of regular elements is denoted by $\mathfrak{G}_0$, and an element of $\mathfrak{G}_s = \mathfrak{G} - \mathfrak{G}_0$ is called a singular element of $\mathfrak{G}$. And similarly, an element A of a Lie triple system $\mathfrak{S}$ in $\mathfrak{G}$ is called a regular element of $\mathfrak{S}$, if the restriction $(\mathrm{ad}\, A)^2/\mathfrak{S}$ of $(\mathrm{ad}\, A)^2$ on $\mathfrak{S}$ has no eigen values such as $-4l^2\pi^2$ (l: non-zero integers). The set of regular elements of $\mathfrak{S}$ is denoted by $\mathfrak{S}_0$, and an element of $\mathfrak{S}_s = \mathfrak{S} - \mathfrak{S}_0$ is called a singular element of $\mathfrak{S}$.

§2. Geodesic subspaces

In this paper we shall define the geodesic subspaces of group spaces as follows:

DEFINITION. *Let S be a submanifold (cf. [4] for definition) of a group manifold of a Lie group G satisfying the following condition*:

To every element x of S there corresponds a linear subspace $\mathfrak{S}_x$ of $\mathfrak{G}$ such that if $\mathfrak{U}_x$ is a small neighborhood of 0 *in $\mathfrak{G}$, the elements $x \exp X$, for $X \in \mathfrak{S}_x \cap \mathfrak{U}_x$, form a neighborhood of x in S.*

Then S is called a geodesic subspace of a group space G.

The dimension of S at x is equal to the dimension of $\mathfrak{S}_x$. Since a manifold has the common dimension at its all elements ([4]), for all the elements x of S the linear spaces $\mathfrak{S}_x$ have the same dimension. The linear subspaces $\mathfrak{S}_x$ of $\mathfrak{G}$ may be called the tangent space to S at x. An open submanifold of a subgroup of G forms, clearly, a geodesic subspace of G.

In this section we shall consider the linear subspaces $\mathfrak{S}_x$ $(x \in S)$ and the relation among them.

LEMMA 1. *Let $\mathfrak{U}_x$ be a neighborhood of* 0 *in $\mathfrak{G}$ such that the elements $x \exp X$, for $X \in \mathfrak{S}_x \cap \mathfrak{U}_x$, form a neighborhood of x in S. If $y = x \exp A$, $x, y \in S$ and $A \in \mathfrak{S}_x \cap \mathfrak{U}_x$, then it holds $\chi(A)\mathfrak{S}_x \subset \mathfrak{S}_y$. If A is a regular element of $\mathfrak{G}$, then it holds $\chi(A)\mathfrak{S}_x = \mathfrak{S}_y$.*

PROOF. For any element $X \in \mathfrak{S}_x$, there exist the elements $Y(t) \in \mathfrak{G}$ which are determined by the relation:

$$\exp A \exp Y(t) = \exp (A + tX),$$

where $|t|$ is small enough to $A + tX \in \mathfrak{S}_x \cap \mathfrak{U}_x$. As mentioned in §1, $Y(t)$ is expressed by

$$Y(t) = t\chi(A)X + O(t^2).$$

And we have

$$y \exp Y(t) = x \exp (A + tX)$$

and so there exists a positive number ε such that $Y(t) \in \mathfrak{S}_y$ for $t : |t| \leqq \varepsilon$. Hence we have

$$\chi(A)X = \lim_{t \to 0} Y(t)/t \in \mathfrak{S}_y,$$

therefore we have $\chi(A)\mathfrak{S}_x \subset \mathfrak{S}_y$. If A is regular in $\mathfrak{G}$, then $\dim \chi(A)\mathfrak{S}_x = \dim \mathfrak{S}_x = \dim \mathfrak{S}_y$, and hence it holds $\chi(A)\mathfrak{S}_x = \mathfrak{S}_y$. Thus the lemma is proved.

LEMMA 2. *Let S be a geodesic subspace of a group space G, then the linear spaces $\mathfrak{S}_x$ for any elements of S are all Lie triple systems.*

PROOF. Let $\mathfrak{U}_x$ be a neighborhood of 0 in $\mathfrak{G}$ such that the elements $x \exp X$, for $X \in \mathfrak{S}_x \cap \mathfrak{U}_x$, form a neighborhood of x in S. By taking $\mathfrak{U}_x$ as a sufficiently small neighborhood of 0 in $\mathfrak{G}$, we may assume that any elements of $\mathfrak{S}_x \cap \mathfrak{U}_x$ are regular in $\mathfrak{G}$. Then, for any element y such that $y = x \exp A$, $A \in \mathfrak{S}_x \cap \mathfrak{U}_x$, it holds $\chi(A)\mathfrak{S}_x = \mathfrak{S}_y$. And then we have $x = y \exp(-A)$, and $-A \in \mathfrak{S}_y$. For, the element $y \exp (t-1)A$, $(= x \exp tA)$, lies in S for $t : |t| \leqq 1$, and if $|t-1|$ is sufficiently small, this element lies in a small neighborhood of y in S, and hence $A \in \mathfrak{S}_y$. Here it does not necessarily happen that A lies in $\mathfrak{U}_y$, but there exists a positive number η_1 such that tA lies in $\mathfrak{U}_y$ for $t : |t| \leqq \eta_1$. If we set $y(t) = y \exp(-tA)$, then $y, y(t) \in S$ and $-tA \in \mathfrak{S}_y \cap \mathfrak{U}_y$ for $t : |t| \leqq \eta_1$; and therefore by Lemma 1 we have

$$\chi(-tA)\mathfrak{S}_y = \mathfrak{S}_{y(t)} \quad \text{for} \quad t : |t| \leqq \eta_1.$$

(If A is in a sufficiently small neighborhood of 0 in $\mathfrak{G}$, then $-tA$ are always regular for $t : |t| \leqq \eta_1$). Furthermore, $y(t) = x \exp (1-t)A$, $(y(t) \in S)$, and $(1-t)A \in \mathfrak{S}_x \cap \mathfrak{U}_x$ for $t : |t| \leqq \eta_2$; therefore we have

$$\chi((1-t)A)\mathfrak{S}_x = \mathfrak{S}_{y(t)} \quad \text{for} \quad t : |t| \leqq \eta_2.$$

From these facts it follows that

$$\chi((1-t)A)\mathfrak{S}_x = \chi(-tA)\chi(A)\mathfrak{S}_x \quad \text{for} \quad t : |t| \leqq \eta_1, \eta_2.$$

This is rewritten as

$$\chi((1-t)A)^{-1}\chi(-tA)\chi(A)\mathfrak{S}_x = \mathfrak{S}_x.$$

Let $\pi(A, t) = \chi((1-t)A)^{-1}\chi(-tA)\chi(A)$, then from the definition of $\chi(A)$ it follows that

$$\pi(A, t) = \varphi(\operatorname{ad} A, t),$$

where

$$\varphi(z, t) = (t-1)z/(e^{(t-1)z} - 1) \cdot (e^{tz} - 1)/tz \cdot (e^{-z} - 1)/-z.$$

After some computations we know that

$$\begin{aligned}
\varphi(z, t) &= \tfrac{1}{2}(t-1)z/\sinh \tfrac{1}{2}(t-1)z \cdot \sinh \tfrac{1}{2} tz/\tfrac{1}{2} tz \cdot \sinh \tfrac{1}{2} z/\tfrac{1}{2} z \\
&= 1 + t(\tfrac{1}{2} z \coth \tfrac{1}{2} z - 1) + O(t^2) \\
&= 1 + t(z/(e^z - 1) + \tfrac{1}{2} z - 1) \\
&= 1 + t(\textstyle\sum_1^\infty (-1)^m B_m z^{2m}/(2m)!) + O(t^2),
\end{aligned}$$

where B_m are Bernoulli's numbers, $B_1=\frac{1}{6}$, $B_2=-\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=-\frac{1}{30}, \cdots$. (It is easily seen that $\varphi(z, t)$ is an even function of z). Consequently, we have

$$\pi(A, t)=I+t(\sum_1^\infty(-1)^m B_m(\text{ad } A)^{2m}/(2m)!)+O(t^2),$$

where I denotes the identity transformation. From the condition:

$$\pi(A, t)\mathfrak{S}_x=\mathfrak{S}_x \quad \text{for} \quad t:|t|\leqq\eta_1, \eta_2$$

it follows that

$$(\sum_1^\infty(-1)^m B_m(\text{ad } A)^{2m}/(2m)!)\mathfrak{S}_x\subset\mathfrak{S}_x.$$

This must be valid, even if A is replaced by $sA(|s|<1)$, therefore we have

$$(\text{ad } A)^2\mathfrak{S}_x\subset\mathfrak{S}_x \quad \text{for} \quad A\in\mathfrak{S}_x\cap\mathfrak{U}_x,$$

since this condition is linear with respect to A, we have

$$(\text{ad } A)^2\mathfrak{S}_x\subset\mathfrak{S}_x \quad \text{for} \quad A\in\mathfrak{S}_x.$$

This asserts that $\mathfrak{S}_x$ is a Lie triple system ([7]), thus the lemma is proved.

LEMMA 3. *Let $\mathfrak{U}_x$ be a neighborhood of 0 in $\mathfrak{G}$ such that the elements $x \exp X$, for $X\in\mathfrak{S}_x\cap\mathfrak{U}_x$, form a neighborhood of x in S, and any elements of $\mathfrak{S}_x\cap\mathfrak{U}_x$ are regular in $\mathfrak{G}$. If $y=x\exp A$ and $A\in\mathfrak{S}_x\cap\mathfrak{U}_x$, then it holds*

$$\mathfrak{S}_y=\exp(-\tfrac{1}{2}\text{ ad } A)\mathfrak{S}_x.$$

PROOF. Under the assumptions, by Lemma 1 it holds that $\mathfrak{S}_y=\chi(A)\mathfrak{S}_x$. It is easily seen that

$$\begin{aligned}\chi(A)&=(\exp(-\text{ad } A)-I)/-\text{ad } A\\ &=\exp(-\tfrac{1}{2}\text{ ad } A)\cdot(\sinh\tfrac{1}{2}\text{ ad } A/\tfrac{1}{2}\text{ ad } A)\end{aligned}$$

where

$$\sinh\tfrac{1}{2}\text{ ad } A/\tfrac{1}{2}\text{ ad } A=\sum_0^\infty(\tfrac{1}{2}\text{ ad } A)^{2m}/(2m+1)!.$$

Since, by Lemma 2, $(\text{ad } A)^2\mathfrak{S}_x\subset\mathfrak{S}_x$, it is clear that

$$(\sinh\tfrac{1}{2}\text{ ad } A/\tfrac{1}{2}\text{ ad } A)\mathfrak{S}_x\subset\mathfrak{S}_x;$$

and since, by the assumption, A is a regular element of $\mathfrak{G}$,

$$\dim(\sinh\tfrac{1}{2}\text{ ad } A/\tfrac{1}{2}\text{ ad } A)\mathfrak{S}_x=\dim\mathfrak{S}_x.$$

Therefore we have

$$(\sinh\tfrac{1}{2}\text{ ad } A/\tfrac{1}{2}\text{ ad } A)\mathfrak{S}_x=\mathfrak{S}_x.$$

Thus we have

$$\begin{aligned}\mathfrak{S}_y&=\chi(A)\mathfrak{S}_x\\ &=\exp(-\tfrac{1}{2}\text{ ad } A)\cdot(\sinh\tfrac{1}{2}\text{ ad } A/\tfrac{1}{2}\text{ ad } A)\mathfrak{S}_x\\ &=\exp(-\tfrac{1}{2}\text{ ad } A)\cdot\mathfrak{S}_x,\end{aligned}$$

which completes the proof.

THEOREM 1. *Let S be a geodesic subspace of a group space G. If there exists a geodesic arc on $S: y(t)=x\exp tA$ having the end points x and y, (i.e., $y(0)=x$, $y(1)=y$ and $y(t)\in S$ for $t:0\leqq t\leqq 1$), then it holds*

$$\mathfrak{S}_y=\exp(-\tfrac{1}{2}\text{ ad } A)\cdot\mathfrak{S}_x.$$

PROOF. Let U_z be a neighborhood of z in S such that $U_z = z\exp(\mathfrak{S}_z \cap \mathfrak{U}_z)$ and $\mathfrak{U}_z$ satisfies the assumptions in Lemma 3. Since the geodesic arc is compact in S, from the open covering $U_{y(t)}$ $(0 \leqq t \leqq 1)$ of this goedesic arc, we can choose a finite open covering $U_{y(t_i)}$ $(i=0, 1, 2, \cdots, m,\ t_0=0,\ t_m=1)$ such that $U_{y(t_i)} \cap U_{y(t_{i+1})}$ contains a point $y(t_i')$ $(t_i \leqq t_i' \leqq t_{i+1})$ of this geodesic arc. Then we have

$$\begin{aligned} y(t_i') &= y(t_i)\exp s(t_{i+1}-t_i)A, \\ &= y(t_{i+1})\exp(s-1)\ (t_{i+1}-t_i)A, \qquad (0 \leqq s \leqq 1) \\ &\quad (i=0, 1, 2, \cdots, m). \end{aligned}$$

It is clear that $A \in \mathfrak{S}_{y(t_i)}$ $(i=0, 1, 2, \cdots, m)$, so by Lemma 3 we have

$$\begin{aligned} \mathfrak{S}_{y(t_i')} &= \exp(-\tfrac{1}{2}s(t_{i+1}-t_i)\operatorname{ad}A)\cdot\mathfrak{S}_{y(t_i)} \\ &= \exp(-\tfrac{1}{2}(s-1)(t_{i+1}-t_i)\operatorname{ad}A)\cdot\mathfrak{S}_{y(t_{i+1})}, \end{aligned}$$

and hence

$$\mathfrak{S}_{y(t_{i+1})} = \exp(-\tfrac{1}{2}(t_{i+1}-t_i)\operatorname{ad}A)\cdot\mathfrak{S}_{y(t_i)}$$

Therefore we have

$$\mathfrak{S}_y = \exp(-\tfrac{1}{2}\operatorname{ad}A)\cdot\mathfrak{S}_x$$

which completes the proof.

It is easily seen that $(\sinh\tfrac{1}{2}\operatorname{ad}A/\tfrac{1}{2}\operatorname{ad}A)\mathfrak{S}_x = \mathfrak{S}_x$, if and only if A is regular in $\mathfrak{S}_x$. Therefore we have a corollary of Theorem 1.

COROLLARY. *Under the assumptions of Theorem 1, if A is regular in $\mathfrak{S}_x$, then $\mathfrak{S}_y = \chi(A)\mathfrak{S}_x$; and if A is singular in $\mathfrak{S}_x$, then $\chi(A)\mathfrak{S}_x \subsetneqq \mathfrak{S}_y$.*

THEOREM 2. *Let S be a geodesic subspace of a group space G. If there exists a geodesic arc on $S: y(t) = x\exp tA$ having the end points x and y, and if A is a regular element of $\mathfrak{S}_x$, then the intersection $\mathfrak{S}_x \cap \mathfrak{S}_y$ of $\mathfrak{S}_x$ and $\mathfrak{S}_y$ is given as:*

$$\mathfrak{S}_x \cap \mathfrak{S}_y = \{Y;\ Y \in \mathfrak{S}_x,\ \operatorname{ad}A\cdot Y \in \mathfrak{S}_x\}.$$

And moreover $\mathfrak{S}_x = \mathfrak{S}_y$, if and only if $[A, Y] \in \mathfrak{S}_x$ for any elements Y of $\mathfrak{S}_x$.

PROOF. Since A is a regular element of $\mathfrak{S}_x$, by the corollary of Theorem 1, we have $\mathfrak{S}_y = \chi(A)\mathfrak{S}_x$. Let now $Y \in \mathfrak{S}_x \cap \mathfrak{S}_y$, then from $Y \in \mathfrak{S}_y$ it follows that $Y = \chi(A)X$, $X \in \mathfrak{S}_x$, and moreover Y must belong to $\mathfrak{S}_x$. Since

$$\begin{aligned} X = \chi(A)^{-1}Y &= \operatorname{ad}A/(\exp\operatorname{ad}A - I)Y \\ &= (I - \tfrac{1}{2}\operatorname{ad}A + \textstyle\sum_1^\infty(-1)^{m-1}B_m(\operatorname{ad}A)^{2m}/(2m)!)Y, \end{aligned}$$

it is easily seen that $X \in \mathfrak{S}_x \cap \mathfrak{S}_y$, if and only if $Y \in \mathfrak{S}_x$ and $\chi(A)^{-1}Y \in \mathfrak{S}_x$; since $\mathfrak{S}_x$ is a Lie triple system, (i.e., $(\operatorname{ad}A)^2 Y \in \mathfrak{S}_x$ for $Y \in \mathfrak{S}_x$), the condition: $\chi(A)^{-1}Y \in \mathfrak{S}_x$ and $Y \in \mathfrak{S}_x$ is equivalent to the condition: Y and $\operatorname{ad}A\cdot Y \in \mathfrak{S}_x$. Therefore we have

$$\mathfrak{S}_x \cap \mathfrak{S}_y = \{Y;\ Y \text{ and } \operatorname{ad}A\cdot Y \in \mathfrak{S}_x\}.$$

Moreover, clearly, $\mathfrak{S}_x = \mathfrak{S}_y$, if and only if $\mathfrak{S}_x \cap \mathfrak{S}_y = \mathfrak{S}_x$; and hence $\mathfrak{S}_x = \mathfrak{S}_y$, if and only if $[A, Y] \in \mathfrak{S}_x$ for any elements $Y \in \mathfrak{S}_x$. Thus the theorem is proved.

Furthermore, if we require that $\mathfrak{S}_x=\mathfrak{S}_y$ for any element y of $x \exp(\mathfrak{S}_x \cap \mathfrak{U}_x)$, then this requirement is equivalent to that $[A, Y] \in \mathfrak{S}_x$ for any elements A and Y of $\mathfrak{S}_x$, i.e., $\mathfrak{S}_x$ is a Lie subalgebra of $\mathfrak{G}$.

REMARK. From our standpoint, the linearity of $\mathfrak{S}_x$ is assumed from the beginning, but in the recent paper of K. Morinaga and F. Mitsudo ([6]), the linearity of $\mathfrak{S}_x$ is deduced from the local convexity and the extensivity of S.

§3. Some fundamental properties of geodesic subspaces

In this section we shall consider some fundamental properties of geodesic subspaces of group spaces.

LEMMA 4. *Let $\mathfrak{S}$ be a homogeneous subspace of Lie algebra $\mathfrak{G}$, then, for any elements Y and Z in a sufficiently small neighborhood of 0 in $\mathfrak{G}$, the element* $\exp Y \exp Z \exp Y$ *belongs to* $\exp \mathfrak{S}$, *if and only if $\mathfrak{S}$ is a Lie triple system of $\mathfrak{G}$.*

PROOF. Assume that

$$\exp tY \exp tZ \exp tY \in \exp \mathfrak{S},$$

for a sufficiently small t. By the SCH series we have

$$\exp tY \exp tZ \exp tY$$
$$=\exp [t(2Y+Z)+\tfrac{1}{6}t^3([[Y, Z], Z]-[Y, [Y, Z]])+O(t^4)].$$

Therefore it must be valid that

$$t(2Y+Z)+\tfrac{1}{6}t^3([[Y, Z], Z]-[Y, [Y, Z]])+O(t^4) \in \mathfrak{S}$$

for a sufficiently small t, and hence we have

$$2Y+Z \in \mathfrak{S} \quad \text{and} \quad [[Y, Z], Z]-[Y, [Y, Z]] \in \mathfrak{S}.$$

Since $\mathfrak{S}$ is a homogeneous subspace of $\mathfrak{G}$, if $Y, Z \in \mathfrak{S}$, then $\frac{1}{2}aY, bZ \in \mathfrak{S}$ (a and b are arbitrary constants), and therefore we may take $\frac{1}{2}aY$ and bZ in place of Y and Z respectively in the above consideration, then we see that

$$Y, Z \in \mathfrak{S} \text{ implies } aY+bZ \in \mathfrak{S},$$

that is, $\mathfrak{S}$ is a linear subspace of $\mathfrak{G}$. Next, in the condition:

$$[[Y, Z], Z]-[Y, [Y, Z]] \in \mathfrak{S},$$

we may take $-Z$ in place of Z, then we have

$$[[Y, Z], Z]+[Y, [Y, Z]] \in \mathfrak{S},$$

Since now $\mathfrak{S}$ is a linear subspace of $\mathfrak{G}$, from these two conditions it follows that $[Y, [Y, Z]] \in \mathfrak{S}$. This asserts that $\mathfrak{S}$ is Lie triple system in $\mathfrak{G}$ ([7]).

Conversely, assume that $\mathfrak{S}$ is a Lie triple system in $\mathfrak{G}$. If Y and Z lie in a sufficiently small neighborhood of 0 in $\mathfrak{S}$, in order that

$$\exp Y \exp Z \exp Y \in U_1,$$

where U_1 is the neighborhood mentioned in §1 of the identity element of G, then we may define the elements $Z(t)$ in $\mathfrak{G}$ by the relation:

$$\exp tY \exp Z \exp tY=\exp Z(t)$$

for $t: |t| \leqq 1$. Then we have

$$dZ/dt = \mathrm{ad}\, Z \cdot \coth \tfrac{1}{2} \mathrm{ad}\, Z \cdot Y \quad \text{and} \quad Z(0) = Z,$$

where

$$\mathrm{ad}\, Z \cdot \coth \tfrac{1}{2} \mathrm{ad}\, Z = 2(1 + \textstyle\sum_1^\infty (-1)^m B_m (\mathrm{ad}\, Z)^{2m}/(2m)!)$$

and this is an even function of ad Z. By the same reasoning as in G. D. Mostow's paper ([7]), we have

$$\exp Y \exp Z \exp Y \in \exp \mathfrak{S}.$$

Thus the lemma is proved.

LEMMA 5. *Let S be a geodesic subspace of G and let $\mathfrak{S}_x$ $(x \in \mathfrak{S})$ be the associated Lie triple systems. Then, for any two elements x and y of S, and for a previously given neighborhood $\mathfrak{U}$ of 0 in $\mathfrak{G}$, there exists a series of finite elements $y_0 = x$, $y_1, y_2, \cdots, y_m = y$ of S such that $y_{i+1} = y_i \exp X_{i+1}$, $X_{i+1} \in \mathfrak{S}_{y_i} \cap \mathfrak{U}$ and $y_i \exp tX_{i+1} \subset S$ for $t: |t| \leqq 1$, $(i = 0, 1, 2, \cdots, m-1)$.*

PROOF. Let R be the set of the elements y of S which are constructed in the following manner:

$$y_{i+1} = y_i \exp X_{i+1}, \quad X_{i+1} \in \mathfrak{S}_{y_i} \cap \mathfrak{U},$$

and $\quad y_i \exp tX_{i+1} \subset S \quad$ for $\quad t: |t| \leqq 1,\ y_0 = x,\ y_m = y,\ (i = 0, 1, 2, \cdots, m-1).$

Then we can prove that R is open in S. For any element y of R, there exists a neighborhood of y in S such as $y \exp(\mathfrak{S}_y \cap \mathfrak{U}_y \cap \mathfrak{U})$. If z is any element of $y \exp(\mathfrak{S}_y \cap \mathfrak{U}_y \cap \mathfrak{U})$, then we have

$$z = y \exp X_{m+1}, \quad X_{m+1} \in \mathfrak{S}_y \cap \mathfrak{U},$$

and $$y \exp tX_{m+1} \subset S \quad \text{for} \quad t: |t| \leqq 1,$$

that is, $z \in R$. This means that R is open in S.

Next let us assume that for any element U of $S - R$ there exists a neighborhood of u in S such as $u \exp(\mathfrak{S}_u \cap \mathfrak{U}_u \cap \mathfrak{U})$, which contains an element v of R, i.e., $v \in u \exp(\mathfrak{S}_u \cap \mathfrak{U}_u \cap \mathfrak{U})$. From this it follows that $u \in v \exp(\mathfrak{S}_v \cap \mathfrak{U})$; therefore we have $u \in R$. This contradicts the assumption that u is an element of $S - R$. Hence R is closed in S. Since S is connected, we can conclude that $R = S$; this proves the lemma.

THEOREM 3. *Let S be a geodesic subspace of a group space G. For any elements x and y of S, $\mathfrak{S}_y$ is transferred from $\mathfrak{S}_x$ by an inner automorphism of $\mathfrak{G}$.*

PROOF. By Lemma 5, for any elements x and y of S, we have

$$y = x \exp X_1 \exp X_2 \cdots \exp X_m,$$

where $X_1, X_2, \cdots, X_m$ are contained in a previously given neighborhood $\mathfrak{U}$ of 0 in $\mathfrak{G}$ such that $X_{i+1} \in \mathfrak{S}_{y_i}$, $y_i = x \exp X_1 \exp X_2 \cdots X_i$, $y_0 = x$ and $y_m = y$ $(i = 0, 1, 2, \cdots, m-1)$. Therefore, by Theorem 1 we have

$$\begin{aligned}\mathfrak{S}_{y_{i+1}} &= \exp(-\tfrac{1}{2} \mathrm{ad}\, X_{i+1}) \mathfrak{S}_{y_i} \\ &= \exp(-\tfrac{1}{2} \mathrm{ad}\, X_{i+1}) \exp(-\tfrac{1}{2} \mathrm{ad}\, X_i) \cdots \exp(-\tfrac{1}{2} \mathrm{ad}\, X_1) \cdot \mathfrak{S}_x,\end{aligned}$$

consequently,

$$\mathfrak{S}_y = \exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_m)\exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_{m-1})\cdots\exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_1)\cdot\mathfrak{S}_x.$$

That is, $\mathfrak{S}_y$ is transferred from $\mathfrak{S}_x$ by an inner automorphism. Thus, the theorem is proved.

The associated linear subspaces $\mathfrak{S}_x$ to S at any elements x of S are all Lie triple systems, and they are mutually isomorphic as the Lie triple systems.

THEOREM 4. *Let S be a geodesic subspace of G, then for any elements x and y of S, there exists an analytic curve in G through x and y, which lies on $x\exp\mathfrak{S}_x$ in a sufficiently small neighborhood of x in G.*

PROOF. Since

$$\mathfrak{S}_{y_{i+1}} = \exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_{i+1})\exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_i)\cdots\exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_1)\cdot\mathfrak{S}_x,$$

the expression obtained in Lemma 5 is rewritten as:

$$y = x\exp\tfrac{1}{2}\mathring{X}_1\exp\tfrac{1}{2}\mathring{X}_2\cdots\exp\tfrac{1}{2}\mathring{X}_{m-1}\exp\mathring{X}_m\exp\tfrac{1}{2}\mathring{X}_{m-1}\cdots\exp\tfrac{1}{2}\mathring{X}_2\exp\tfrac{1}{2}\mathring{X}_1,$$

where $\mathring{X}_1, \mathring{X}_2, \cdots, \mathring{X}_m$ are the elements of $\mathfrak{S}_x$ which are obtained from $X_1, X_2, \cdots, X_m$ by the relations:

$$\begin{cases} \mathring{X}_1 = X_1 \\ \mathring{X}_{i+1} = \exp\tfrac{1}{2}\,\mathrm{ad}\,X_1\exp\tfrac{1}{2}\,\mathrm{ad}X_2\cdots\exp\tfrac{1}{2}\,\mathrm{ad}\,X_{i+1}, \quad (i=1,2,\cdots,m-1). \end{cases}$$

If we consider the curve in G:

$$y(t) = x\exp\tfrac{1}{2}t\mathring{X}_1\exp\tfrac{1}{2}t\mathring{X}_2\cdots\exp\tfrac{1}{2}t\mathring{X}_{m-1}\exp t\mathring{X}_m \cdot\exp\tfrac{1}{2}t\mathring{X}_{m-1}\cdots\exp\tfrac{1}{2}t\mathring{X}_2\exp\tfrac{1}{2}t\mathring{X}_1,$$

then it is easily seen that $y(0)=x$, $y(1)=y$, and $y(t)$ is an analytic curve in G. By means of Lemma 4, we see that $y(t)$ lies on $x\exp\mathfrak{S}_x$ for a sufficiently small neighborhood of x in G. Thus the theorem is proved.

REMARK. If $X_1, X_2, \cdots, X_m$ are the regular elements of $\mathfrak{G}$, then also $\mathring{X}_1, \mathring{X}_2, \cdots, \mathring{X}_m$ are the regular elements of $\mathfrak{G}$. In fact, from the relations between $X_1, X_2, \cdots, X_m$ and $\mathring{X}, \mathring{X}_2, \cdots, \mathring{X}_m$, it is easily verified that

$$\mathrm{ad}\,\mathring{X}_{i+1} = \exp\tfrac{1}{2}\,\mathrm{ad}\,X_1\cdots\exp\tfrac{1}{2}\,\mathrm{ad}\,X_i\,\mathrm{ad}\,X_{i+1}\exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_i)\cdots\exp(-\tfrac{1}{2}\,\mathrm{ad}\,X_1),$$
$$(i=1,2,\cdots,m-1) \quad\text{and}\quad \mathrm{ad}\,\mathring{X}_1 = \mathrm{ad}\,X_1.$$

Therefore, $\mathrm{ad}\,X_i$ and $\mathrm{ad}\,\mathring{X}_i$ have the same eigen values, which proves the assertion.

THEOREM 5. *Let S be a complex geodesic subspace of a complex linear group space* (i.e., *the underlying space of a complex linear Lie group*) *and let $\mathfrak{S}_x$ be the associated Lie triple system at $x\in S$. Then we have $S\subset x\,\overline{\exp\mathfrak{S}_x}$, where $\overline{\exp\mathfrak{S}_x}$ means the closure of* $\exp\mathfrak{S}_x$ *in G.*

PROOF. By Theorem 4, for any element y of S, there exists an analytic curves in G through x and y, which lies on $x\exp\mathfrak{S}_x$ in a sufficiently small neighborhood of x in G. By the method of K. Schröder ([8]) which is based on the analytic continuation of the logarithmic matrix function along an analytic curve, we can prove $S\subset x\,\overline{\exp\mathfrak{S}_x}$.

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