JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, NO. 3, JANUARY, 1958

Note on the Paper "On the Singularity of General Linear Groups"

By

Takayuki NÔNO

(Received Sept. 26, 1957)

§1. **Introduction.** Let G be a connected Lie group and let $\mathfrak{G}$ be its Lie algebra, then there exists an analytic mapping, which is called the exponential mapping, of the analytic manifold $\mathfrak{G}$ into the analytic manifold G. The differential of the exponential mapping at an element A of $\mathfrak{G}$ is an endomorphism $\chi(A)$ of $\mathfrak{G}$:

$$A \to \chi(A) = (\exp(-\mathrm{ad}\, A) - I)/-\mathrm{ad}\, A,$$

where $(\exp(-\mathrm{ad}\, A) - I)/-\mathrm{ad}\, A = \sum_1^\infty (-\mathrm{ad}\, A)^{m-1}/m!$ and $\mathrm{ad}\, A \cdot W = [A, W]$ ([2][1], p. 157). The endomorphism $\chi(A)$ has an inverse, if and only if $\mathrm{ad}\, A$ has no eigen values such as $2l\pi\sqrt{-1}$ (l: non-zero integers). In the previous paper "On the Singularity of General Linear Groups" ([4]), the proof of Theorem 1, which asserts that the exponential mapping is locally homeomorphic at A, if and only if $\chi(A)$ has an inverse, is unsatisfactory for the following reason. Since $\chi(A)$ is the Jacobian at A of the exponential mapping, it is clear that if $\chi(A)$ has an inverse, then the exponential mapping is locally homeomorphic at A. For the case of the complex Lie groups, from the theorem concerning the Jacobian in several complex variables ([1], p. 179), it is deduced that if $\chi(A)$ has no inverse, then the exponential mapping is not locally homeomorphic at A. But, for the case of the real Lie groups, this argument is not available. In §2 of this note we shall complete the proof of this assertion (Theorem 1), and in §3 we shall make clear the correspondence, by the exponential mapping, between the neighborhoods A and $\exp A$ respectively.

§2. Let $\mathfrak{C}_{\mathfrak{G}}(A)$ denote the centralizer of A in $\mathfrak{G}$:

$$\mathfrak{C}_{\mathfrak{G}}(A) = \{X;\ X \in \mathfrak{G},\ \mathrm{ad}\, A \cdot X = 0\},$$

and let $\mathfrak{C}_{\mathfrak{G}}(\exp A) = \{X;\ X \in \mathfrak{G},\ \exp \mathrm{ad}\, A \cdot X = X\}$. In this section we shall prove the following theorem.

THEOREM 1. *The following conditions are mutually equivalent:*

(1) *The endomorphism $\chi(A)$ of $\mathfrak{G}$ has an inverse.*

(2) *The exponential mapping of $\mathfrak{G}$ into G is locally homeomorphic at A.*

(3) $\mathfrak{C}_{\mathfrak{G}}(A) = \mathfrak{C}_{\mathfrak{G}}(\exp A)$.

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1) Numbers in brackets refer to the references at the end of the paper.

PROOF. Since

$$(\exp(-\operatorname{ad} A)-I)=\chi(A)(-\operatorname{ad} A),$$

it is easily seen that (1) implies (3). And since

$$\operatorname{rank}(\exp(-\operatorname{ad} A)-I)= \operatorname{rank}(-\operatorname{ad} A)-p,$$

where p is the number of the blocks belonging to the eigen values such as $2l\pi\sqrt{-1}$ (l: non-zero integers) in Jordan's canonical form of the matrix $-\operatorname{ad} A$, if $\chi(A)$ has no inverse, i.e., $p\geqq 1$, then clearly $\mathfrak{C}_{\mathfrak{G}}(A)\subsetneqq\mathfrak{C}_{\mathfrak{G}}(\exp A)$, that is, (3) implies (1) ([4]). That (1) implies (2) is clear from the theorem of the implicit functions. So we have only to prove that (2) implies (1). Let us assume that $\chi(A)$ has no inverse, i.e., (by the equivalence of (1) and (3)) $\mathfrak{C}_{\mathfrak{G}}(A)\subsetneqq\mathfrak{C}_{\mathfrak{G}}(\exp A)$, then there exists an element $Y\neq 0$ belonging to $\mathfrak{C}_{\mathfrak{G}}(\exp A)-\mathfrak{C}_{\mathfrak{G}}(A)$. And if we consider the elements $A(t)=\exp t\operatorname{ad} Y\cdot A$ (t: real or complex), then clearly $\exp A(t)=\exp A$; since $Y\notin\mathfrak{C}_{\mathfrak{G}}(A)$, there exists a positive number ε such that $A(t)\neq A$ for $t:|t|<\varepsilon$ and $A(t_1)\neq A(t_2)$ for $t_1\neq t_2:|t_1|,|t_2|<\varepsilon$. That is, there exists an uncountable number of the elements $A(t)$ ($|t|<\varepsilon$) in a neighborhood of A in $\mathfrak{G}$ such that $\exp A(t)=\exp A$. The exponential mapping is not locally homeomorphic at A. Hence (2) implies (1). Thus the theorem is proved.

§3. In this section we shall consider the correspondence, under the exponential mapping, between the neighborhoods of A and $\exp A$ respectively. An element A of $\mathfrak{G}$ is said to be regular if $\chi(A)$ has an inverse, and to be singular if $\chi(A)$ has no inverse.

LEMMA 1. ([3]). *Assume that the blocks of the eigen value 1 in Jordan's canonical form of the matrix* $\exp\operatorname{ad} A$ *are diagonal. Then the elements* $\exp X(\exp A\exp Y)(\exp X)^{-1}$, *for X in a neighborhood of 0 in $\mathfrak{G}$ and for Y in a neighborhood of 0 in $\mathfrak{C}_{\mathfrak{G}}(\exp A)$, form a neighborhood of* $\exp A$ *in G.*

PROOF. Since the equation

$$\exp X(\exp A\exp Y)(\exp X)^{-1}=\exp A\exp Z$$

is written as

$$(\exp A)^{-1}\exp X\exp A\exp Y=\exp Z\exp X,$$

and moreover as

$$\exp(\exp(-\operatorname{ad} A)\cdot X)\exp Y=\exp Z\exp X,$$

the equation

$$\exp X(\exp A\exp Y)(\exp X)^{-1}=\exp A\exp Z$$

for any given element Z in a neighborhood of 0 in $\mathfrak{G}$, is reduced to

$$(\exp(-\operatorname{ad} A)-I)X+Y=Z,$$

neglecting the infinitesimals of higher orders of X, Y and Z. Let us decompose $\mathfrak{G}$ into its root spaces with respect to $\exp(-\operatorname{ad} A)$:

$$\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_\lambda+\mathfrak{G}_\mu+\mathfrak{G}_\nu+\cdots, \quad (\lambda,\mu,\nu,\cdots\neq 1)$$

then, by the assumption of the lemma, we have $\mathfrak{G}_1=\mathfrak{C}_{\mathfrak{G}}(\exp A)$ $(=\{W;(\exp(-\operatorname{ad} A)-I)W=0,\ W\in\mathfrak{G}\})$. If we put $X=X_1+X_2$ and $Z=Z_1+Z_2$ where

$X_1, Z_1 \in \mathfrak{C}_{\mathfrak{G}}(\exp A)$ and $X_2, Z_2 \in \mathfrak{G}_\lambda + \mathfrak{G}_\mu + \mathfrak{G}_\nu + \cdots$, then, since $(\exp(-\operatorname{ad} A) - I) X_1 = 0$, the above equation is reduced to $Y = Z_1$ and $(\exp(-\operatorname{ad} A) - I) X_2 = Z_2$; and, since $\exp(-\operatorname{ad} A) - I$ has an inverse on $\mathfrak{G}_\lambda + \mathfrak{G}_\mu + \mathfrak{G}_\nu + \cdots$, our equation is solved; thus the lemma is proved.

LEMMA 2. *Assume that the blocks of the eigen value* 0 *in Jordan's canonical form of the matrix* $\operatorname{ad} A$ *are diagonal. Then the elements* $(\exp \operatorname{ad} X)(A + Y)$, *for* X *in a neighborhood of* 0 *in* $\mathfrak{G}$ *and for* Y *in a neighborhood of* 0 *in* $\mathfrak{C}_{\mathfrak{G}}(A)$, *form a neighborhood of* A *in* $\mathfrak{G}$.

PROOF. The elements $(\exp \operatorname{ad} X)(A + Y)$, for X in a neighborhood of 0 in $\mathfrak{G}$ and for Y in a neighborhood in $\mathfrak{C}_{\mathfrak{G}}(A)$, are expressed as:

$$(\exp \operatorname{ad} X)(A + Y) = A + Y + \operatorname{ad} A \cdot X,$$

neglecting the infinitesimals of higher orders X and Y. Now let Z be an arbitrary element in a neighborhood of 0 in $\mathfrak{G}$, then we shall consider the equation: $(\exp \operatorname{ad} X)(A + Y) = A + Z$, that is, $-\operatorname{ad} A \cdot X + Y = Z$. Let us decompose $\mathfrak{G}$ into its root spaces with respect to $\operatorname{ad} A$:

$$\mathfrak{G} = \mathfrak{G}^0 + \mathfrak{G}^\alpha + \mathfrak{G}^\beta + \mathfrak{G}^\gamma + \cdots, \quad (\alpha, \beta, \gamma, \cdots \neq 0).$$

By the assumption we have $\mathfrak{C}_{\mathfrak{G}}(A) \equiv \{W;\ \operatorname{ad} A \cdot W = 0\} = \mathfrak{G}^0$. If we put $X = X_1 + X_2$ and $Z = Z_1 + Z_2$, where $X_1, Z_1 \in \mathfrak{G}^0$ and $X_2, Z_2 \in \mathfrak{G}^\alpha + \mathfrak{G}^\beta + \mathfrak{G}^\gamma + \cdots$, then the above equation is written as (since $\operatorname{ad} A \cdot X_1 = 0$),

$$-\operatorname{ad} A \cdot X_2 + Y = Z_1 + Z_2, \text{ i.e., } Y = Z_1 \text{ and } -\operatorname{ad} A \cdot X_2 = Z_2.$$

The restriction of $\operatorname{ad} A$ on $\mathfrak{G}^\alpha + \mathfrak{G}^\beta + \mathfrak{G}^\gamma + \cdots$ has an inverse, and hence, for any given Z we can obtain the desirable X and Y such that $(\exp \operatorname{ad} X)(A + Y) = A + Z$. Thus the lemma is proved.

By using these two lemmas, we have

THEOREM 2. *Assume that the blocks of the eigen value* 1 *in Jordan's canonical form of the matrix* $\exp \operatorname{ad} A$ *are diagonal, then there exists the following relation between a neighborhood* $\mathfrak{U}_A$ of A *in* $\mathfrak{G}$ *and a neighborhood* $\mathfrak{V}_{\exp A}$ *of* $\exp A$ *in* G:

If A *is regular, then* $\exp \mathfrak{U}_A = \mathfrak{V}_{\exp A}$.

If A *is singular, then* $\exp \mathfrak{U}_A \subsetneqq \mathfrak{V}_{\exp A}$.

PROOF. By the above two lemmas, we have

$\mathfrak{V}_{\exp A} = \{\exp X (\exp A \exp Y)(\exp X)^{-1};\ X$ in a neighborhood of 0 in $\mathfrak{G}$ and Y in a neighborhood of 0 in $\mathfrak{C}_{\mathfrak{G}}(\exp A)\}$

and

$\mathfrak{U}_A = \{(\exp \operatorname{ad} X)(A + W);\ X$ in a neighborhood of 0 in $\mathfrak{G}$ and W in a neighborhood of 0 in $\mathfrak{C}_{\mathfrak{G}}(A)\}$,

and therefore

$\exp \mathfrak{U}_A = \{\exp X (\exp A \exp W)(\exp X)^{-1};$ in a neighborhood of 0 in $\mathfrak{G}$ and W in neighborhood of 0 in $\mathfrak{C}_{\mathfrak{G}}(A)\}$.

If A is regular, then by Theorem 1, $\mathfrak{C}_{\mathfrak{G}}(A) = \mathfrak{C}_{\mathfrak{G}}(\exp A)$, therefore clearly we have $\mathfrak{V}_{\exp A} = \exp \mathfrak{U}_A$. If A is singular, then $\mathfrak{C}_{\mathfrak{G}}(A) \subsetneqq \mathfrak{C}_{\mathfrak{G}}(\exp A)$, and the elements: $\exp X(\exp A \exp W)(\exp X)^{-1}$, as is seen in the proof of Lemma 1, if and only if W runs over a neighborhood of 0 in $\mathfrak{C}_{\mathfrak{G}}(\exp A)$, for X in a neighborhood of 0 in $\mathfrak{G}$, form a neighborhood of $\exp A$ in G. That is, if A is singular, then $\exp \mathfrak{U}_A \subsetneqq \mathfrak{V}_{\exp A}$. Thus the theorem is proved.

I wish to express my hearty thanks to Prof. K. Morinaga for his kind guidance.

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Mathematical Institute,
Hiroshima University