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On the Lie Triple System and its Generalization

By

Kiyosi YAMAGUTI

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In a Lie triple system $\mathfrak{T}$[1] (L.t.s.) over a base field Φ,[2] let D be a vector space generated from the set of linear mappings $x \rightarrow \Sigma[a_i b_i x]$ and D' be a vector subspace generated from such set of linear mappings as $[abx]=0$ for all x in $\mathfrak{T}$, then the factor space $\mathfrak{D}(\mathfrak{T}) \equiv D/D'$ has the structure of Lie algebra of linear mappings of $\mathfrak{T}$ (inner derivations algebra of $\mathfrak{T}$).

The following theorem was first established by N. Jacobson [3] and improved under weaker assumptions than his in [6].

THEOREM. *L.t.s. $\mathfrak{T}$ can be 1-to-1 imbedded into a Lie algebra $\mathfrak{L}$ in such a way that the given composition $[abc]$ in $\mathfrak{T}$ coincides with the product $[[ab]c]$ defined in $\mathfrak{L}$ and $\mathfrak{L}=\mathfrak{T}\oplus\mathfrak{D}(\mathfrak{T})$.*

$\mathfrak{L}$ is called a standard enveloping Lie algebra of $\mathfrak{T}$.

E. Cartan proved that Lie algebra is semi-simple if and only if the determinant $|g_{ij}|$ is not zero. In §1 we shall generalize this result to L.t.s., and prove some other properties. In §2, we shall define the general Lie triple system which has the geometrical meaning, and prove that the general Lie triple system can be imbedded into a Lie algebra.

§1. Some properties of Lie triple systems.

At the standard imbedding of L.t.s. $\mathfrak{T}$, $\mathfrak{T}$ and inner derivation algebra $\mathfrak{D}(\mathfrak{T})$ satisfy the following relation

$$(1.1) \qquad [\mathfrak{T}, \mathfrak{T}]=\mathfrak{D}(\mathfrak{T}),\ [\mathfrak{T}, \mathfrak{D}(\mathfrak{T})] \subseteqq \mathfrak{T},\ [\mathfrak{D}(\mathfrak{T}), \mathfrak{D}(\mathfrak{T})] \subseteqq \mathfrak{D}(\mathfrak{T}).$$

Conversely

PROPOSITION 1.1. *Let $\mathfrak{L}$ be a Lie algebra and $\mathfrak{A}$, $\mathfrak{B}$ complementary subspaces of $\mathfrak{L}$ such that $[\mathfrak{A}, \mathfrak{A}]=\mathfrak{B}$, $[\mathfrak{A}, \mathfrak{B}] \subseteqq \mathfrak{A}$, $[\mathfrak{B}, \mathfrak{B}] \subseteqq \mathfrak{B}$. Assume that there is not a non-zero element b such that $[b, \mathfrak{A}]=(0)$. Then there is a L.t.s. $\mathfrak{T}$ such that $\mathfrak{T}$ is L.t.s. isomorphic to $\mathfrak{A}$ and $\mathfrak{D}(\mathfrak{T})$ Lie isomorphic to $\mathfrak{B}$, where $\mathfrak{D}(\mathfrak{T})$ is an inner derivation algebra of $\mathfrak{T}$.*

PROOF. Put $\mathfrak{A} \equiv \mathfrak{T}$ then $\mathfrak{T}$ is a L.t.s. since $[[\mathfrak{T}, \mathfrak{T}]\mathfrak{T}] \subseteqq \mathfrak{T}$. Any element of $\mathfrak{B}$ can be written in the form $\Sigma[a, b]$ $a, b \in \mathfrak{A}$. Then it is easy to see that the mapping $\Sigma[a, b] \rightarrow \Sigma D_{(a, b)}$ $(a, b \in \mathfrak{A})$ is the Lie isomorphism of $\mathfrak{B}$ onto $\mathfrak{D}(\mathfrak{T})$.

1) The notations and terminologies used in this paper are to be found in [4] and [6].
2) Throughout the paper we shall assume that the characteristic of the base field is 0.

The one of assumptions which has used in Prop. 1.1 can be replaced as follows by others if $dim\, \mathfrak{B} < \infty$.

THEOREM 1.1.[3] *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be complementary subspaces of a Lie algebra* $\mathfrak{L}$ *such that* $[\mathfrak{A}, \mathfrak{A}] \subseteqq \mathfrak{B}$, $[\mathfrak{A}, \mathfrak{B}] \subseteqq \mathfrak{A}$, $[\mathfrak{B}, \mathfrak{B}] \subseteqq \mathfrak{B}$, *if* $dim\, \mathfrak{B} < \infty$ *then next two conditions are equivalent.*

(i) *In* $\mathfrak{B}$ *there is no ideal* $(\neq 0)$ *of* $\mathfrak{L}$.

(ii) *If* $[b, \mathfrak{A}] = (0)$, $b \in \mathfrak{B}$ *then* $b = 0$.

PROOF. (i)$\Rightarrow$(ii). Suppose that there exists an element $b \neq 0$ in $\mathfrak{B}$ which satisfies $[b, \mathfrak{A}] = (0)$. If we put $[b, \mathfrak{B}^k] = [\cdots[[b, \overbrace{\mathfrak{B}]\mathfrak{B}]\cdots\mathfrak{B}}^{k \text{ times}}]$, then we have $[[b, \mathfrak{B}]\mathfrak{A}] \subseteqq [[b, \mathfrak{A}]\mathfrak{B}] + [b[\mathfrak{B}, \mathfrak{A}]] = (0)$ and using the induction and Jacobi identity we have $[[b, \mathfrak{B}^k]\mathfrak{A}] = (0)$. Now denote by $\mathfrak{B}_k$ the vector space generated by $b + [b, \mathfrak{B}] + \cdots + [b, \mathfrak{B}^k]$, $k = 0,1,2,\cdots$, where $\mathfrak{B}_0$ is the space generated by a single element b. If $[b, \mathfrak{B}^{k+1}]$ includes the elements which are independent from $\mathfrak{B}_k$, we add $[b, \mathfrak{B}^{k+1}]$ to $\mathfrak{B}_k$ and obtain $\mathfrak{B}_{k+1}$. Since $\mathfrak{B}_k \subseteqq \mathfrak{B}$ for all k and $dim\, \mathfrak{B} < \infty$, $[b, \mathfrak{B}^{n+1}]$ are linearly dependent on $\mathfrak{B}_n$ for some n. Then $[\mathfrak{B}_n, \mathfrak{L}] \subseteqq [\mathfrak{B}_n, \mathfrak{A}] + [\mathfrak{B}_n, \mathfrak{B}] \subseteqq \mathfrak{B}_n$. Hence $\mathfrak{B}$ includes an ideal $\mathfrak{B}_n$ of $\mathfrak{L}$, therefore $\mathfrak{B}_n = (0)$ which contradicts our assumption since a non-zero element b belongs to $\mathfrak{B}_n$.

(ii)$\Rightarrow$(i). Suppose that there is an ideal $\mathfrak{C} \neq (0)$ of $\mathfrak{L}$ in $\mathfrak{B}$. Let b be any non-zero element of $\mathfrak{C}$. Then $[b, \mathfrak{A}] \subseteqq \mathfrak{B} \cap \mathfrak{A} = (0)$, which is a contradiction.

Now we shall assume that the $dim\, \mathfrak{T} < \infty$ and take a base $(X_1, \cdots, X_r, X_{r+1}, \cdots, X_n)$ of the standard enveloping Lie algebra $\mathfrak{L}$ of $\mathfrak{T}$ such that the set $(X_1, \cdots, X_r)$ and $(X_{r+1}, \cdots, X_n)$ form the bases of $\mathfrak{T}$ and $\mathfrak{D}(\mathfrak{T})$ respectively. Throughout the whole discussion in the next theorem let the indices run as follows,

$$\begin{aligned} i, j, k, \cdots &= 1,2, \cdots, r, \\ \alpha, \beta, \gamma, \cdots &= r+1,\ r+2, \cdots, n, \\ A, B, C, \cdots &= 1,2, \cdots, n, \end{aligned}$$

and we put $G_{AB} = \sum_{S=1}^{n} C_{SAB}{}^{S} = \sum_{S,T=1}^{n} C_{SA}{}^{T} C_{TB}{}^{S}$ where $C_{AB}{}^{C}$ are the structure constants of $\mathfrak{L}$.

THEOREM 1.2. *Let* $\mathfrak{L}$ *be a standard enveloping Lie algebra of L.t.s.* $\mathfrak{T}$ *then the following three conditions are equivalent.*

(i) $\mathfrak{T}$ *is a semi-simple L.t.s.*

(ii) $\mathfrak{L}$ *is a semi-simple Lie algebra.*

(iii) $det\, | G_{ij} | \neq 0$.

PROOF. (i)$\Rightarrow$(ii) was proved in [4]. (ii)$\Rightarrow$(iii). From (1) $G_{i\alpha} = \sum_{S,T} C_{Si}{}^{T} C_{T\alpha}{}^{S} = \sum_{j,\beta} C_{ji}{}^{\beta} C_{\beta\alpha}{}^{j} + \sum_{j,\beta} C_{\beta i}{}^{j} C_{j\alpha}{}^{\beta} = 0$. Hence $| G_{AB} | = | G_{ij} | \cdot | G_{\alpha\beta} |$ and we obtain $| G_{ij} | \neq 0$.

3) A. Fujimoto has proved this result in the case where the dimensions of $\mathfrak{A}$ and $\mathfrak{B}$ are both finite [1].

(iii)$\Rightarrow$(i). Now assume that $\mathfrak{T}$ is not the semi-simple L.t.s. then we have a non-zero solvable ideal $\mathfrak{a}$ in $\mathfrak{T}$:

$$\mathfrak{a}\equiv\mathfrak{a}^{(0)}\supsetneqq\mathfrak{a}^{(1)}\supsetneqq\cdots\supsetneqq\mathfrak{a}^{(s-1)}\supsetneqq\mathfrak{a}^{(s)}=(0) \text{ where } \mathfrak{a}^{(l+1)}\equiv[\mathfrak{T},\mathfrak{a}^{(l)},\mathfrak{a}^{(l)}].$$

Since $\mathfrak{a}$ is an ideal in $\mathfrak{T}$, $\mathfrak{a}^{(l)}$ are so in $\mathfrak{T}$ [4, Lemma 2.1]. Our proof is now divided into the following two cases:

Case 1. $\mathfrak{a}^{(s-1)}=\mathfrak{T}$: $\mathfrak{a}^{(s)}=[\mathfrak{T},\mathfrak{T},\mathfrak{T}]=(0)$ and $\mathfrak{D}(\mathfrak{T})=(0)$ because $\mathfrak{L}$ is standard enveloping algebra. Therefore we obtain $|G_{ij}|=0$ which contradicts (iii).

Case 2. $\mathfrak{a}^{(s-1)}\neq\mathfrak{T}$: Without loss of generality we may assume that X_{m+1}, $X_{m+2},\cdots,X_r$ $(m<r)$ the bases of $\mathfrak{a}^{(s-1)}$ and indices $a,b,c,\cdots;\ p,q,t,\cdots$ run from 1 to m; 1 to $r-m$ respectively, then $\sum_i C_{i,\,m+p,\,m+q}{}^i=0$ since $[\mathfrak{T},\mathfrak{a}^{(s-1)},\mathfrak{a}^{(s-1)}]=(0)$. Moreover by using the fact that any element X_α in $\mathfrak{D}(\mathfrak{T})$ can be written $\sum\lambda_{ij}[X_i,X_j]$ and $[X_{m+t},X_{m+q}]\equiv D_{(m+t,\,m+q)}=0$ because $[[\mathfrak{a}^{(s-1)},\mathfrak{a}^{(s-1)}]\mathfrak{T}]\subseteqq[[\mathfrak{T},\mathfrak{a}^{(s-1)}]\mathfrak{a}^{(s-1)}]=(0)$ we obtain $[[X_\alpha,X_{m+p}]X_{m+q}]=\sum\lambda_{ij}[[[X_i,X_j]X_{m+p}]X_{m+q}]=\sum\lambda_{m+t}[X_{m+t},X_{m+q}]=0$, hence $\sum_\alpha C_{\alpha,\,m+p,\,m+q}{}^\alpha=0$. Therefore $G_{m+p,\,m+q}=0$. Next we have $\sum_j C_{j,\,a,\,m+p}{}^j=\sum_{b=1}^{m}C_{b,\,a,\,m+p}{}^b+\sum_{q=1}^{r-m}C_{m+q,\,a,\,m+p}{}^{m+q}=0$. If X_α belongs to $\mathfrak{D}(\mathfrak{T})$, then $X_\alpha=\sum\lambda_{bc}[X_b,X_c]+\sum\mu_{d,\,m+q}[X_d,X_{m+q}]$, $[[X_\alpha,X_a]X_{m+p}]=\sum\lambda_{bc}[[[X_b,X_c]X_a]X_{m+p}]=\sum\lambda_e[X_e,X_{m+p}]$. Hence $\sum_\alpha C_{\alpha,\,a,\,m+p}{}^\alpha=0$, and $G_{a,\,m+p}=0$. Therefore we have $|G_{ij}|=0$, which contradicts (iii).

It can be proved that Lie semi-simple and L.t.s. semi-simple are equivalent for Lie algebra and Theorem 1.2 reduces to the theorem of E. Cartan for Lie algebra.

THEOREM 1.3. *If the inner derivations algebra of L.t.s. $\mathfrak{T}$ is semi-simple, then $\mathfrak{T}$ is a direct sum of a semi-simple subsystem and an abelian ideal in $\mathfrak{T}$.*

PROOF. Let $\mathfrak{T}=\mathfrak{S}\oplus\mathfrak{R}$ be the Levi decomposition of $\mathfrak{T}$, where $\mathfrak{S}$ is semi-simple subsystem and $\mathfrak{R}$ radical of $\mathfrak{T}$. $\mathfrak{D}(\mathfrak{R})\equiv\{\sum D_{(r,\,s)};\ r,s\in\mathfrak{R}\}$ is an ideal of inner derivation algebra $\mathfrak{D}(\mathfrak{T})$. Because for all $a,b,x\in\mathfrak{T}$, $r,s\in\mathfrak{R}$

$$\begin{aligned}[D_{(a,\,b)},D_{(r,\,s)}]x&=[ab[rsx]]-[rs[abx]]\\&=[[abr]sx]-[[abs]rx]=(D_{(r',\,s)}+D_{(r,\,s')})x, \quad \text{where } r',s'\in\mathfrak{R}.\end{aligned}$$

Moreover $\mathfrak{D}(\mathfrak{R})$ is a homomorphic image of a solvable Lie algebra $[\mathfrak{R},\mathfrak{R}]$ by the mapping $\sum[r,s]\to\sum D_{(r,\,s)}$, $(r,s\in\mathfrak{R})$. Hence $\mathfrak{D}(\mathfrak{R})=(0)$ and $[\mathfrak{R},\mathfrak{R},\mathfrak{T}]=(0)$, that is, $\mathfrak{R}$ is abelian in $\mathfrak{T}$.

In the analogical way to Hochschild [2, Theorem 4.1] we obtain the following:

PROPOSITION 1.2. *The radical $\mathfrak{R}$ of L.t.s. $\mathfrak{T}$ is characteristic, that is, $D\mathfrak{R}\subseteqq\mathfrak{R}$ for every derivation D of $\mathfrak{T}$.*

§2. General Lie triple systems.

DEFINITION 2.1. *A general Lie triple system* (general L.t.s.) is a vector space $\mathfrak{B}$ over a field Φ which is closed with respect to a trilinear product $[a,b,c]$ and a bilinear product $a\circ b$ such that

(1) $[a, a, b]=0$,
(2) $a\circ a=0$,
(3) $[a, b, c]+[b, c, a]+[c, a, b]-(a\circ b)\circ c-(b\circ c)\circ a-(c\circ a)\circ b=0$,
(4) $[a\circ b, c, d]+[b\circ c, a, d]+[c\circ a, b, d]=0$,
(5) $[[a, b, c], d, e]+[[b, a, d], c, e]+[b, a, [c, d, e]]+[c, d, [a, b, e]]=0$,
(6) $[a, b, c\circ d]+d\circ[a, b, c]+c\circ[b, a, d]=0$.

Lie algebra is a general L.t.s. by putting $a\circ b=[a, b]$, $[a, b, c]=[[a, b]c]$ and L.t.s. becomes a general L.t.s. if we define $a\circ b=0$ for every element of it.

DEFINITION 2.2. A *derivation* of a general L.t.s. $\mathfrak{B}$ is a linear mapping D of $\mathfrak{B}$ into $\mathfrak{B}$ such that

$$D[xyz]=[Dx, y, z]+[x, Dy, z]+[x, y, Dz],$$
$$D(x\circ y)=(Dx)\circ y+x\circ(Dy).$$

For any two elements a, b in general L.t.s. $\mathfrak{B}$, the linear mapping $D_{(a, b)}$: $x\rightarrow[abx]$, $x\in\mathfrak{B}$, is a derivation of $\mathfrak{B}$. If we define a sum $D_{(a, b)}+D_{(c, d)}$ and a scalar product $\alpha D_{(a, b)}$, $\alpha\in\Phi$, as usual, then the set of these derivations becomes a vector space D over Φ. Also $D'=\{D_{(a, b)}; D_{(a, b)}x=0 \text{ for all } x\in\mathfrak{B}\}$ is a vector subspace of D. Then in the factor space $\mathfrak{D}(\mathfrak{B})\equiv D/D'$, using (5) we have

(2.1) $[D_{(a, b)}, D_{(c, d)}]\overset{\text{def}}{=}D_{(a, b)}D_{(c, d)}-D_{(c, d)}D_{(a, b)}=D_{([abc], d)}-D_{([abd], c)}\in\mathfrak{D}(\mathfrak{B})$.

Hence $\mathfrak{D}(\mathfrak{B})$ makes a Lie algebra, we shall call it the inner derivation algebra of $\mathfrak{B}$.

K. Nomizu has proved the following proposition [5, pp. 61–62]. But we shall here treat it by an algebraic method. His (19.2) and (19.3) correspond with our (3) and (4) respectively.

PROPOSITION 2.1. *The general L.t.s. $\mathfrak{B}$ can be 1-to-1 imbedded into a Lie algebra $\mathfrak{L}$ in such a way that $\mathfrak{L}$ is a direct sum of $\mathfrak{B}$ and its inner derivation algebra.*

PROOF. We shall denote by $\mathfrak{L}$ the direct vector sum of general L.t.s. $\mathfrak{B}$ and its inner derivation algebra $\mathfrak{D}(\mathfrak{B})$, and define the product for any element a, b, c, d, e, f in $\mathfrak{B}$ as follows:

$$[a, b]=a\circ b-D_{(a, b)},$$
$$[D_{(a, b)}, c]=-[c, D_{(a, b)}]=[a, b, c],$$
$$[D_{(a, b)}, D_{(c, d)}]=D_{([abc], d)}-D_{([abd], c)},$$

and for $u=a+\sum D_{(b, c)}$ and $v=d+\sum D_{(e, f)}$

$$[a+\sum D_{(b, c)}, d+\sum D_{(e, f)}]=[a, d]+\sum[a, D_{(e, f)}]+\sum[D_{(b, c)}, d]+\sum[D_{(b, c)}, D_{(e, f)}].$$

Then the above defined product is bilinear and skew-symmetric. That this product satisfies the Jacobi identity can be proved by making use of the identities (1), (2),$\cdots$,(6) and (2.1) in the same way in the previous paper [6, Theorem 2.1].

Next we shall consider the geometrical meaning of this proposition. In a space L_r with a linear connection in which the covariant derivatives of the torsion and the curvature tensor are vanished, that is $\nabla_m S_{jk}^{\cdot\cdot l}=0$, $\nabla_m R_{ijk}^{\cdot\cdot\cdot l}=0$ where $S_{jk}^{\cdot\cdot l}$ and $R_{ijk}^{\cdot\cdot\cdot l}$ are torsion and curvature tensor respectively, we have the following identities:

(i) $\quad R_{ijk}^{\cdot\cdot\cdot l}+R_{jik}^{\cdot\cdot\cdot l}=0,$

(ii) $\quad S_{jk}^{\cdot\cdot l}+S_{kj}^{\cdot\cdot l}=0,$

(iii) $\quad R_{ijk}^{\cdot\cdot\cdot l}+R_{jki}^{\cdot\cdot\cdot l}+R_{kij}^{\cdot\cdot\cdot l}-4(S_{ij}^{\cdot\cdot m}S_{mk}^{\cdot\cdot l}+S_{jk}^{\cdot\cdot m}S_{mi}^{\cdot\cdot l}+S_{ki}^{\cdot\cdot m}S_{mj}^{\cdot\cdot l})=0,$

(iv) $\quad S_{ij}^{\cdot\cdot m}R_{mke}^{\cdot\cdot\cdot l}+S_{jk}^{\cdot\cdot m}R_{mie}^{\cdot\cdot\cdot l}+S_{ki}^{\cdot\cdot m}R_{mje}^{\cdot\cdot\cdot l}=0,$

(v) $\quad R_{efm}^{\cdot\cdot\cdot l}R_{ijk}^{\cdot\cdot\cdot m}-R_{efi}^{\cdot\cdot\cdot m}R_{mjk}^{\cdot\cdot\cdot l}-R_{efj}^{\cdot\cdot\cdot m}R_{imk}^{\cdot\cdot\cdot l}-R_{efk}^{\cdot\cdot\cdot m}R_{ijm}^{\cdot\cdot\cdot l}=0,$

(vi) $\quad R_{efm}^{\cdot\cdot\cdot l}S_{jk}^{\cdot\cdot m}-R_{efj}^{\cdot\cdot\cdot m}S_{mk}^{\cdot\cdot l}-R_{efk}^{\cdot\cdot\cdot m}S_{jm}^{\cdot\cdot l}=0,$

where (iii) and (iv) are Bianchi's identities and (v) and (vi) are Ricci's identities.

In L_r we shall denote by $X_1, X_2, \cdots, X_r$ the base vectors of a tangent space T_p at a point p. And if we define the trilinear product $[X_i, X_j, X_k]$ by $[X_i, X_j, X_k]=\frac{1}{4}R_{ijk}^{\cdot\cdot\cdot l}X_l$ and a bilinear product $X_j\circ X_k$ by $X_j\circ X_k=S_{jk}^{\cdot\cdot l}X_l$ in T_p then T_p has the structure of general L.t.s.. Using above proposition, L_r can be realized as a subspace in a group space.

PROPOSITION 2.2. *The general L.t.s. has the structure of abstract L.t.s. with respect to the composition* $\{a, b, c\}\equiv[a, b, c]-(a\circ b)\circ c$. *Moreover it has the structure of L.t.s. if the following condition be satisfied*:

$$[a\circ b, e, c\circ d]-[c\circ d, e, a\circ b]-[a\circ b, c, d]\circ e+[a\circ b, d, c]\circ e=0.$$

PROOF. In general L.t.s. $\mathfrak{B}$ put $\{a, b, c\}\equiv[a, b, c]-(a\circ b)\circ c$ then it is clear that this product is trilinear and $\{a, a, b\}=0$ and $\{a, b, c\}+\{b, c, a\}+\{c, a, b\}=0$. Hence $\mathfrak{B}$ is an abstract L.t.s.. Next, using identities (1), (2), $\cdots$, (6) after some calculating we obtain the following

$$\begin{aligned}&\{\{abc\}de\}+\{\{bad\}ce\}+\{ba\{cde\}\}+\{cd\{abe\}\}\\ &\quad=-[a\circ b, e, c\circ d]-[e, c\circ d, a\circ b]+[a\circ b, c, d]\circ e+[d, a\circ b, c]\circ e,\end{aligned}$$

this proves our assertion.

COROLLARY. *A space provided with a linear connection such that the covariant derivatives of the torsion and the curvature tensor equal to zero become a space with affine connection which is symmetric in the sense of Cartan, if the following condition be satisfied:*

$$S_{ij}^{\cdot\cdot p}S_{kl}^{\cdot\cdot q}(R_{mpq}^{\cdot\cdot\cdot h}-R_{mqp}^{\cdot\cdot\cdot h})+S_{ij}^{\cdot\cdot p}S_{qm}^{\cdot\cdot h}(R_{pkl}^{\cdot\cdot\cdot q}-R_{plk}^{\cdot\cdot\cdot q})=0.$$

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Mathematical Institute,
Hiroshima University