JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, NO. 1, APRIL, 1957

Note on Fixed-Point Theorem

By

Akira TOMINAGA

(Received Jan. 25, 1957)

1. Introduction

The present note deals with relations between f.p.p. (the fixed point property) for some types of continua, for example, plane continua not separating the plane (§ 4), and continua whose arcwise components, which will be defined later, are finite in number (§ 6). By the way, we shall concern with multi-valued mappings on the n-simplex (§ 5).

2. Terminology and notations[5]

We shall assume that R^n is provided with a fixed Cartesian coordinate system. Then we adopt a metric, ρ, as follows: let $x=(\xi_1,\cdots,\xi_n)$, $y=(\eta_1,\cdots,\eta_n)$ be two points. Then we define $\rho(x, y)=\max\limits_{1\leq i\leq n}(|\xi_i-\eta_i|)$. $U(x, \varepsilon)$ is an ε-neighborhood of x under ρ. Let S be a subset of R^n, then $\mathfrak{J}(S)$, $\mathfrak{F}(S)$ are the interior, and the frontier of S respectively. The empty set is designated by ϕ.

The *ε-grating*, $\mathfrak{G}_\varepsilon$, is the collection of the principals, $\xi_i=\varepsilon m$, where $\varepsilon>0$ and $m=0, \pm 1, \pm 2,\cdots$. ε will be said the *mesh* of $\mathfrak{G}_\varepsilon$. The *n-cells on* $\mathfrak{G}_\varepsilon$ are the closures of the rectangular domains into which R^n is divided by $\mathfrak{G}_\varepsilon$. The *0-cells* (or *vertices*) and *1-cells on* $\mathfrak{G}_\varepsilon$ are the vertices and the edges on $\mathfrak{G}_\varepsilon$ respectively. By a *stepped arc on* $\mathfrak{G}_\varepsilon$ is meant a simple arc which is the union of 1-cells on $\mathfrak{G}_\varepsilon$. $\mathfrak{G}_{\varepsilon/k}$ (k: an integer ≥ 2) will be called a *refinement* of $\mathfrak{G}_\varepsilon$.

In this paper a *n-polytope*, P, means a point set which is the union of a finite set of n-cells on some fixed grating, $\mathfrak{G}_\varepsilon$. A cell in P means a cell on $\mathfrak{G}_\varepsilon$ which is contained in P. A vertex in $P\subset R^2$ is called to be *singular* if it is a common vertex which belongs to precisely two 2-cells in P. Thus a singular vertex is a boundary point of P.

Let p be a singular vertex of P and $\mathfrak{G}_{\varepsilon/k}$ a refinement of $\mathfrak{G}_\varepsilon$. Now let C_1, C_2 be two 2-cells on $\mathfrak{G}_{\varepsilon/k}$ contained in P and having p as their common vertex. Then we have a new polytope $\overline{P-(C_1\cup C_2)}$. This process will be called slicing of P at p by $\mathfrak{G}_{\varepsilon/k}$.

3. Approximations of plane continua by polytopes

DEFINITION 1. Let $\{S_i\}$ be a set sequence such that $S_i \supseteq S_{i+1}$ and $S=\bigcap_i S_i$. We say that S is approximated by $\{S_i\}$.

LEMMA 1. *Let M be a plane continuum and $\nu(M)$ the number of the bounded complementary domains of M. If $\nu(M)<\infty$, M can be approximated by a sequence, $\{S_i\}$, of connected polytopes such that $\nu(S_i)=\nu(M)$.*

PROOF. It is sufficient that for each $\varepsilon>0$ there exists a connected polytope, Q, such that $\nu(Q)=\nu(M)$ and $M\subseteq Q\subset U(M, \varepsilon)$. Since $\nu(M)=m<\infty$, there exists a finite sequence, D_i $(i=1, 2, \cdots, m)$, of the bounded complementary domains of M. Let D_0 be the unbounded complementary domain of M. If $p_i \in D_i$ $(i=0, 1, \cdots, m)$, then $\rho(p_i, M)>0$. Let Q' be the union of 2-cells on $\mathfrak{G}_\xi$ meeting M, where $\xi<\varepsilon/2$ and $<$ one half of $\min_{0\leq i\leq m} \rho(p_i, M)$. It is readily seen that Q' is contained in $U(M, \varepsilon)$ and $\nu(Q')=n\geq m$.

If $n=m$, put $Q=Q'$. Otherwise, there exist $n-m$ bounded components, D_j $(j=m+1, \cdots, n)$, such that for each j, $p_i \notin D_j$ $(i=1, \cdots, m)$. Let $p_j \in D_j$. Since each component of R^2-Q' is contained in some component of R^2-M, for each j there exists $D_{i(j)}$ $(0\leq i(j)\leq m)$ such that $D_j\subseteq D_{i(j)}$. We can find stepped arcs, α_j, on some grating, $\mathfrak{G}_\eta$, joining p_j to $p_{i(j)}$ and contained in $D_{i(j)}$. Let ζ be a positive number being less than both $\eta/3$ and one half of $\min_{m+1\leq j\leq n} \rho(\alpha_j, M)$. Now let P_j be the union of 2-cells on $\mathfrak{G}_\zeta$ meeting α_j. If R is the component of $Q'-\bigcup_{j=m+1}^{n} P_j$ containing M, put $Q=\bar{R}$. Q.E.D.

The condition in Lemma 1 is not sufficient, however we have

LEMMA 2. *Let $\{S_i\}$ be a sequence of connected polytopes such that $\nu(S_i)=m$ and S a continuum approximated by $\{S_i\}$. Then $\nu(S)\leq m$.*

PROOF. Suppose, on the contrary, that $\nu(S)>m$. There exists certain bounded complementary domain, D, of S containing no bounded complementary domain of S_i for every i. Let p be a point in D. Since $\bigcup_i S_i=S$, for some integer i_0, $S_{i_0}\not\ni p$. Furthermore, since $\mathfrak{F}(D)\subseteq S\subseteq S_{i_0}$, D contains a bounded complementary domain of S_{i_0}. This is impossible. Q.E.D.

LEMMA 3. *If Q is a 2-polytope not separating R^2 and having no singular vertex, Q is a 2-cell.*

LEMMA 4. *If S is a set in R^2 and K is the union of all the 2-cells on $\mathfrak{G}_\varepsilon$ meeting S, then $S\subset\mathfrak{J}(K)$.*

PROPOSITION 1. *In order that a continuum, M, does not separate R^2 it is necessary and sufficient that M can be approximated by a sequence of 2-polytopes which are homeomorphic with 2-cells. Here we can select such a sequence as M is contained in the interior of each 2-cell in the sequence.*

PROOF. The sufficiency follows immediately from Lemma 2. To prove the necessity it is sufficient that for each $\varepsilon>0$ there exists a 2-cell, Q, such that $M\subset\mathfrak{J}(Q)\subset Q\subset U(M, \varepsilon)$. Let Q' be the union of 2-cells on $\mathfrak{G}_{\varepsilon/2}$

meeting M. By Lemma 4, $M \subset \mathfrak{J}(Q')$.

a) The case where Q' does not separate R^2. $Q=Q'$ is a desired 2-cell since it has no singular vertex. For, suppose Q' has a singular vertex, p, and let C_1, C_2 be 2-cells on $\mathfrak{G}_{\varepsilon/2}$ contained in Q' and having p as their common vertex. If p would be a cut point of Q', $p \in M$ and hence $p \in \mathfrak{J}(Q')$, since both C_1 and C_2 meet M and M is a connected set contained in Q'. This contradicts to the fact that $p \in \mathfrak{F}(Q')$. Suppose p is not a cut point of Q'. There exists two points q_1, q_2 contained in C_1, C_2 respectively, such that p is an interior point of the segment $\overline{q_1q_2}$. q_1 and q_2 can be joined by a simple arc α in $Q'-p$ and we may suppose $\alpha+\overline{q_1q_2}$ is a simple closed curve. $\alpha+\overline{q_1q_2}$ separates some points (which situate in the vicinity of p) not belonging to Q', and therefore Q' separates R^2. This contradiction shows that Q' has no singular vertex. Thus Q' is a 2-cell by Lemma 1.

b) The case where Q' separates R^2. If Q' has singular points, we shall slice Q' at those points by a refinement of $\mathfrak{G}_{\varepsilon/2}$ and for simplicity we shall denote the resulted set by the same notation, Q'. The bounded complementary domains of Q' are finite in number and they are denoted by $D_1, D_2, \cdots, D_n$. We shall denote by D_0 the unbounded complementary domain of Q'. Let p_i be a point in D_i $(i=0, 1, \cdots, n)$. Since M does not separate R^2, for each i $(i=1, 2, \cdots, n)$ there exists a simple arc, α_i, in R^2-M joining p_i to p_0. Now let P_i be the union of 2-elements on $\mathfrak{G}_{\eta/2}$ meeting α_i where $\eta=\min_{1\leq i\leq n} \rho(\alpha_i, M)$. If R_0 is the component of $Q'=\bigcup_{i=1}^{n} P_i$ containing M, $Q=\overline{R}_0$ is a desired 2-cell. For let $R_0, R_1, \cdots, R_k$ be the components of $Q'-\bigcup_{i=1}^{n} P_i$, then R^2-R_0 is equal to $(R^2-Q')+\bigcup_{i=1}^{n} P_i+\bigcup_{j=1}^{k} R_j$. Since $(R^2-Q')+\bigcup_{i=1}^{n} P_i$ is connected and each R_j has a limit point in $\bigcup_{i=1}^{n} P_i$, R^2-R_0 is connected. Furthermore it is seen that $\mathfrak{J}(R^2-R_0)$ is connected, for $\mathfrak{J}(R^2-Q') \supset (R^2-Q')+\bigcup_{i=1}^{n} \mathfrak{J}(P_i)$ and $\overline{R}_j \cap (\bigcup_{i=1}^{n} P_i) \neq \phi$. Since $R^2-R_0 \supset R^2-Q \supset \mathfrak{J}(R^2-R_0)$, R^2-Q is connected, i.e. Q does not separate R^2. Furthermore Q has no singular vertex. For if there existed a singular vertex, p, in Q, it would be a cut point of Q, because Q does not separate R^2. On the other hand, since Q' has no singular vertex, $p \notin R_0$. Thus we should have a separation $Q-p=A \cup B$ and R_0 would be contained in A or B, say in A. Since $\overline{A} \subset Q$ and $\overline{A} \neq Q$, we have $\overline{R}_0 \neq Q$. This is impossible. By Lemma 3, Q is a desired 2-cell. Q.E.D.

The above is concerned with outer approximations of plane continua by a sequence of polytopes.

4. Plane continua not separating R^2

It is known that a plane continuum which does not contain a continuum which separates R^2 is tree-like[1] and that each tree-like continuum has f.p.p.[9] On the other hand a plane continuum, M, not separating R^2 has at least one fixed point for each orientation preserving topological mapping, f, of R^2 onto itself such that $f(M)=M$.[2],[6],[7],[4]

Let M be a plane continuum not separating R^2. By a 2-element in M is meant a component of $\mathfrak{J}(M)$. We shall show

LEMMA 5. *Let M be a plane continuum such that $M=\overline{C_1\cup C_2}$, where C_i are 2-elements of M having a.f.p.p., and such that M and $\overline{C}_2$ do not separate R^2. If f is a topological mapping of M onto itself, f has a fixed point.*

PROOF. If $f(C_i)=C_i$ ($i=1$ or 2), there exists a fixed point on $\overline{C}_i$. Let $f(C_1)=C_2$ and hence $f(C_2)=C_1$. Then $K=\overline{C}_1\cap\overline{C}_2$ is connected. For suppose K is not connected. Then there exists a separation, $K=K_1\cup K_2$, where both K_1 and K_2 are disjoint non-vacuous closed sets. Let $A_1=\overline{C}_1-U(K,\eta)$ and $A_2=\overline{C}_2-U(K,\eta)$, where $\eta=\rho(K_1,K_2)/3$. Then A_1 and A_2 are disjoint non-vacuous closed sets. Let $\mathfrak{G}$ be a grating whose mesh is not greater than $\rho(A_1,A_2)/3$. By the same way as in Prop. 1, we can construct a 2-cell, Q, such that $\mathfrak{J}(Q)\supset M$, from $\mathfrak{G}$. We may suppose that each 2-cell, E, (on a refinement of $\mathfrak{G}$) in the polytope Q meets M. Let B_i ($i=1,2$) be the connected sub-polytope of Q which is the union of E's meeting $\overline{C}_i$. Then $Q=B_1\cup B_2$, and $B_1\cap B_2$ is not connected. This contradicts the unicoherency of Q. Furtheremore K does not separate R^2. For, if K separates R^2, C_1 or C_2, say C_2, must be a bounded complementary domain of K, because M does not separate R^2. On the other hand, since $f(\overline{C}_1)=\overline{C}_2$ and $\mathfrak{F}(C_2)=K$, we have $\mathfrak{F}(C_1)=K$. This is impossible, because $\overline{C}_2$ does not separate R^2. Thus K is a tree-like continuum. Since $f(K)=K$, there exists a fixed point.

COROLLARY. *Let M be a plane continuum such that $M=\overline{C_1\cup C_2\cup\cdots\cup C_k}$, where C_i are 2-elements of M which have a.f.p.p., and such that M and each $\overline{C}_j$ do not separate R^2. If f is a topological mapping of M onto itself, f has a fixed point.*

THEOREM 1. *Let M be a plane continuum which does not separate R^2 and whose 2-elements are finite in number, and such that each 2-element has a.f.p.p. and its closure does not separate R^2. If f is a topological mapping of M onto itself, then f has a fixed point.*

PROOF. We construct a new tree-like continuum, M_1, from M as follows: let $C_1, C_2, \cdots$ be 2-elements of M. Define that $\overline{C}_i$ is equivalent to C_j if and only if for each $\varepsilon>0$, there exists a chain of an ε-covering of M which joins a point of C_i to one of C_j and such that each link meets an element of $\{C_i\}$. As a consequence, $\{C_i\}$ may be decomposed into classes. Let K_j be the closure of the union of points belonging to 2-elements which are equivalent. Then $K_j \cap K_l = \phi$ for $j \neq l$. Now let two points x and y of M be equivalent if and only if they are in the same K_j. Then M_1 is the set of the classes of the equivalence relation. The topology of M_1 is naturally induced by M, and M_1 becomes a tree-like continuum.

Since f is topological, $\{K_j\}$ is invariant under f which induces a continuous mapping, f_1, of M_1 into itself. It is suffice to inspect only the case $f_1(K_i) = K_j$ for some j. Q.E.D.

The theorem leads us to the question proposed by Fort.(3)

5. Multi-valued mappings

Let X be a metric space and $\mathfrak{R}(X)$ the collection of compact sets in X. $\mathfrak{R}(X)$ becomes a metric space by defining a metric, σ, as follows: for $F_1, F_2 \in \mathfrak{R}(X)$, $\sigma(F_1, F_2)=$inf. of α, such that $U(F_1, \alpha) \supseteq F_2$ and $U(F_2, \alpha) \supseteq F_1$. A multi-valued mapping, Φ, of X into itself is continuous if and only if, for each $x \in X$, $\Phi(x) \in \mathfrak{R}(X)$ and Φ is a continuous mapping of X into $\mathfrak{R}(X)$. The definition is equivalent to the Strother's one(8) for metric spaces.

Let $F \in \mathfrak{R}(R^n)$ and let $\mathfrak{G}_1, \mathfrak{G}_{1/2}, \cdots$ be the sequence of gratings. Let $C_1, C_2, \cdots, C_{m_1}$ be the n-cells on $\mathfrak{G}_1$ meeting F and x_i the center of C_i. Put $g_1 = \sum_{i=1}^{m_1} x_i / m_1$. Next let C_{ij} $(1 \leq j \leq k_i)$ be the n-cells on $\mathfrak{G}_{1/2}$ meeting F and contained in C_i. x_{ij} is the center of C_{ij}, and $g_2 = (\sum_{i=1}^{m_1}((\sum_{j=1}^{k_i} x_{ij})/k_i))/m_1$. By induction we have a fundamental sequence, $g_1, g_2, \cdots$, for F. Let g_F be the limit point of $\{g_i\}$, then for each $F \in \mathfrak{R}(R^n)$, g_F is uniquely determined. It is easily seen that the mapping, $F \rightarrow g_F$, of $\mathfrak{R}(R^n)$ onto X is continuous.

If $\mathfrak{S}(R^n)$ is the collection of $F \in \mathfrak{R}(R^n)$, such that $g_F \in F$ and if $\mathfrak{C}(R^n)$ is the collection of convex sets in $\mathfrak{R}(R^n)$, then $\mathfrak{S}(R^n) \supset \mathfrak{C}(R^n)$ and they are closed in $\mathfrak{R}(R^n)$.

THEOREM 2. *Let $Q \subset R^n$ be a set which is homeomorphic with the n-simplex, and let Φ be a multi-valued continuous mapping of Q into itself, such that for $x \in Q$, $\Phi(x) \in \mathfrak{S}(R^n)$. Then there exists a point, $x_0 \in Q$, such that $x_0 \in \Phi(x_0)$.*

PROOF. Since $\varphi : x \rightarrow g_{\Phi(x)}$ is a (one-valued) continuous mapping of Q into itself, there exists a point, $x_0 \in Q$, such that $x_0 = \varphi(x_0) = g_{\Phi(x_0)}$ by Brouwer's fixed-point theorem. Since $g_{\Phi(x_0)} \in \Phi(x_0)$, we have $x_0 \in \Phi(x_0)$. Q.E.D.

The above theorem is not a generalization of Neumann-Kakutani's theorem, since the latter theorem concerns with upper semi-continuous mappings.

6. Arcwise components

DEFINITION 2. An arcwise connected space, S, means a space, such that for each pair of points, $x, y \in S$, there exists an arc in S joining x to y.

DEFINITION 3. A space, S, is called to be almost arcwise connected if for each pair of points, $x, y \in S$ and for aribitrary neighborhoods, $U(x)$, $U(y)$, there exist $x' \in U(x)$, $y' \in U(y)$ and an arc in S joining x' to y'.

We shall remark here that locally connected continua (Peano continua) $\subset$ arcwise connected continua $\subset$ almost arcwise connected continua.

EXAMPLE 1. In the Cartesian plane, R^2, $M=\{(\xi_1, \xi_2) \mid (\xi_1 \in$ the Cantor ternary set on ξ_1-axis) & $(0 \leq \xi_2 \leq 1)\} \cup \{(\xi_1, \xi_2) \mid (0 \leq \xi_1 \leq 1)$ & $(\xi_2=0)\}$ is arcwise connected, but not locally connected.

EXAMPLE 2. In R^2, $M=\{(\xi_1, \xi_2) \mid (0<\xi_1 \leq 1/\pi)$ & $(\xi_2=\sin 1/\xi_1)\} \cup \{(\xi_1, \xi_2) \mid$ $(\xi_1=0)$ & $(-1 \leq \xi_2 \leq 1)\}$ is almost arcwise connected, but not arcwise connected.

DEFINITION 4. Two points x and y of a space, S, are called A-equivalent (in S) if there exists an arc in S joining x to y. The relation of A-equivalence in S is reflexive (if a point is considered as a degenerate arc), symmetric and transitive. The classes of A-equivalent points of S are called arcwise components of S.

There exists an almost arcwise connected continuum whose arcwise components are infinite in number, and a non-degenerate continuum whose arcwise components are all degenerate.

PROPOSITION 2. *A-eqvivalence are invariant under continuous mappings and the image under continuous mappings of an [almost] arcwise connected space is [almost] arcwise connected. The arcwise components of a space are invariant under topological mappings of the space.*

For the image under continuous mappings of an arc is arcwise connected.

DEFINITION 4. Let $M=\bigcup_\alpha N_\alpha$, where N_α are the arcwise components of M. We may think of the collection, $\mathfrak{N}=\{N_\alpha\}$, as constituting a complex, which is called the *A-complex* of M, if we let each N_α be called a "vertex", and a sub-collection, $\mathfrak{N}_1=\{N_{1_0}, N_{i_1}, \cdots, N_{i_m}\}$, of $\mathfrak{N}$ constitutes an m-simplex if $\bigcap_{j=0}^{m} \overline{N}_{i_j} \neq \phi$. And we shall speak henceforth of simplexes of $\mathfrak{N}$, etc.

PROPOSITION 3. *If M, M' are spaces and f is a continuous mapping of M into M', then f induces a simplicial mapping of the A-complex of M into the one of M'.*

For $f(\bigcap_{j=0}^{n} \overline{N}_{i_j}) \subseteq \bigcap_{j=0}^{n} f(\overline{N}_{i_j}) \subseteq \bigcap_{j=0}^{n} \overline{f(N_{i_j})}$. Thus $\bigcap_{j=0}^{n} \overline{f(N_{i_j})} \neq \phi$.

LEMMA 6. *Let M be a connected space, such that $M=N_1 \cup N_2 \cup \cdots \cup N_k$, where N_i are arcwise components, and $N_1 \cup \cdots \cup N_l$ (where $0 \leq l \leq k$ and $l=0$ means $N_1 \cup \cdots \cup N_l=\phi$) is closed, and let f be a continuous mapping of M into itself. Then there exists a pair of integers, j_1, j_2, such that $0 \leq j_1$, $j_2 \leq l$ or $l+1 \leq j_1$, $j_2 \leq k$ and $f(N_{j_1}) \subseteq N_{j_2}$.*

Proof. This is true for $l=0$. For $l\geq 1$, suppose, on the contrary that $f(N_1\cup N_2\cup\cdots\cup N_l)\subseteq N_{l+1}\cup\cdots\cup N_k$ and $f(N_{l+1}\cup\cdots\cup N_k)\subseteq N_1\cup\cdots\cup N_l$. Since M is connected, there exists a point, $p\in(N_1\cup\cdots\cup N_l)\cap(\bar{N}_{l+1}\cup\cdots\cup\bar{N}_k)$. Hence $f(p)\in f(N_1\cup\cdots\cup N_l)\subseteq N_{l+1}\cup\cdots\cup N_k$. On the other hand $f(p)\in f(\bar{N}_{l+1}\cup\cdots\cup\bar{N}_k)\subseteq f(\overline{N_{l+1}\cup\cdots\cup N_k})\subseteq N_1\cup\cdots\cup N_l$. Thus $(N_1\cup\cdots\cup N_l)\cap(N_{l+1}\cup\cdots\cup N_k)\neq\phi$, which is impossible. Q.E.D.

Assume the same condition as in Lemma 6. Let f, f' be continuous mappings of M into itself. By Proposition 2, $f(N_i)\subseteq N_{j(i)}$ and $f'(N_i)\subseteq N_{j'(i)}$ for each i. We define that f and f' are equivalent if and only if $j(i)=j'(i)$ for each i. Hence the equivalency divides all the continuous mappings of M in itself, whose arcwise components are finite in number, into classess. And each class is represented by a permutation, $(1, 2,\cdots, k)\rightarrow(j(1), j(2),\cdots, j(k))$. By Proposition 3 and Lemma 6 the number of the classes of mappings is restricted.

Theorem 3. *Let M be a continuum such that $M=N_1\cup N_2$, where N_i are arcwise components having a.f.p.p. and $N_1=\bar{N}_1$. Then M has f.p.p.*

Proof. By virtue of Lemma 6, for a continuous mapping, f, of M into itself, we have $f(N_1)\subseteq N_1$ or $f(N_2)\subseteq N_2$, say $f(N_1)\subseteq N_1$. Since N_1 has a.f.p.p., $\bar{N}_1$ has f.p.p., and there exists a point, $p\in\bar{N}_1$, such that $f(p)=p$. Q.E.D.

M in Example 2 satisfies the condition in Theorem 3.

Example 3. In R^2, let $N_1=\{(\xi_1, \xi_2)\,|\,(0<\xi_1\leq 1/\pi)\ \&\ \xi_2=(1+\frac{1}{2}\sin 1/\xi_1)\}\cup\{(\xi_1, \xi_2)\,|\,(\xi_1=1/\pi)\ \&\ (1\geq\xi_2\geq -1)\}\cup\{(\xi_1, \xi_2)\,|\,(0<\xi_1\leq 1/\pi)\ \&\ (\xi_2=-1)\}\cup\{(\xi_1, \xi_2)\,|\,(\xi_1=0)\ \&\ (-3/2\leq\xi_2\leq -1/2)\}$ and $N_2=\{(\xi_1, \xi_2)\,|\,(0>\xi_1\geq -1/\pi)\ \&\ (\xi_2=-1+\frac{1}{2}\sin 1/\xi_1)\}\cup\{(\xi_1, \xi_2)\,|\,(\xi_1=-1/\pi)\ \&\ (1\geq\xi_2\geq -1)\}\cup\{(\xi_1, \xi_2)\,|\,(0>\xi_1\geq -1/\pi)\ \&\ (\xi_2=1)\}\cup\{(\xi_1, \xi_2)\,|\,(\xi_1=0)\ \&\ (3/2\geq\xi_2\geq 1/2)\}$.

Neither N_1 nor N_2 are closed and $M=N_1\cup N_2$ has not f.p.p.

Example 4. In R^2, let $N_1=\{(\xi_1, \xi_2)\,|\,(\xi_1=0)\ \&\ (-3/2\leq\xi_2\leq 3/2)\}$ and $N_2=\{(\xi_1, \xi_2)\,|\,(0<\xi_1\leq 1/\pi)\ \&\ (\xi_2=1+\frac{1}{2}\sin 1/\xi_1)\}\cup\{\xi_1, \xi_2)\,|\,(\xi_1=1/\pi)\ \&\ (-1\leq\xi_2\leq 1)\}\cup\{(\xi_1, \xi_2)\,|\,(0<\xi_1\leq 1/\pi)\ \&\ (\xi_2=-1+\frac{1}{2}\sin 1/\xi_1)\}$.

Since N_1 is closed, $M=N_1\cup N_2$ has f.p.p. Furthermore M separates R^2 and is not acyclic, i.e. the 1-Betti number for Cêch homology is equal to 1. Although for locally connected continua the property to have f.p.p. is equivalent to "not separating R^2", this is not true for *continua not being locally connected.*

Lemma 7. *Let M be a continuum such that $M=N_1\cup N_2\cup\cdots\cup N_k$ where N_i are arcwise components having a.f.p.p., and let l $(0\leq l<k)$ be the number of N_i which are closed. If f is a continuous [topological] mapping of M into [onto] itself, there exists a point, $p\in M$ such that $f^m(p)=p$, where $1\leq m\leq k$ [$m\leq min\,(k-l, l)$ for $l>0$, and $m\leq k$ for $l=0$]. If $l=1$ or $k-1$, each topological mapping of M onto itself has a fixed point.*

Proof. Since arcwise components of M are finite in number, there exists an integer $m \leq k$ and N_i, such that $f^m(N_i) \subseteq N_i$. Since N_i has a.f.p.p., $\overline{N}_i$ has f.p.p. Thus there exists a point $p \in \overline{N}_i$, such that $f^m(p)=p$. If f is a topological mapping, the image under f of a closed set is closed and hence we have the conclusion.

THEOREM 4. *Let $M=N_1 \cup N_2 \cup \cdots \cup N_k$ be a continuum whose A-complex is acyclic, and let k be the dimension of the A-complex of M. If N_i's have a.f.p.p. then for each continuous [topological] mapping, f, of into [onto] itself, there exists a point, $p \in M$, such that $f^m(p)=p$ for some $m \leq k$ [$m \leq \max(\min \alpha(s), \beta(s))$ where $\alpha(s)$ [$\beta(s)$] means the number of arcwise components of M contained in the simplex, s, and [not] being closed].*

PROOF. Each simplicial mapping of a finite complex being acyclic has a fixed simplex. Q.E.D.

THEOREM 5. *If N is an arcwise component of a continuum, M, which has a.f.p.p. and such that $M=\overline{N}$ and if each connected component of $\overline{M-N}$ has f.p.p., M has also f.p.p.*

PROOF. Let f be a continuous mapping of M into itself. If $f(N) \subseteq N$, there exists a point, $p \in \overline{N}=M$, such that $f(p)=p$. If $f(N) \subset \overline{M-N}$, $f(N)$ is contained in a connected component, C, of $\overline{M-N}$. Since $f(M)=f(\overline{N}) \subseteq \overline{f(N)} \subseteq C$, we have $f(C) \subseteq C$. Thus there exists a point $p \in C$, such that $f(p)=p$. Q.E.D.

In conclusion I wish to express my hearty thanks to Prof. K. Morinaga for his kind guidance.

References

[1] Bing, R. H., Snake-like continua, Duke Math. J. Vol. 18 (1951).

[2] Cartwright, M. L. and Littlewood, J. E., Some fixed point theorems, Ann. of Math., Vol. 54 (1951), 1–37.

[3] Fort, M. K. Jr., Open topological disks in the plane, J. Indian Math. Soc. 18 (1954), 23–26.

[4] Newmann, M. H. A., (Unpublished, reviewed by Hamilton in [6]).

[5] ———, Elements of the topology of plane sets of points.

[6] Hamilton, O. H., A short proof of the Cartwright-Littlewood fixed point theorem, Canadian J. Math., Vol. 6 (1954), 522–524.

[7] Reifenberg, E. R., On the Cartwright-Littlewood fixed point theorem, Ann. of Math. Vol. 61 (1955).

[8] Strother, W. L., On an open question concerning fixed points, Proc. Amer. Math. Soc., Vol. 4 (1953), 988–993.

[9] Ward, L. E., Fixed point theorems for special types of continua, Bull. Amer. Math. Soc. Vol. 61 (1955), Abst.

Mathematical Institute,
Hiroshima University