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On the Singularity of General Linear Groups

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§1. Introduction

Let G be a Lie group, $\mathfrak{G}$ its Lie algebra, then there exists the exponential mapping from $\mathfrak{G}$ into $G: X \rightarrow \exp X$, and this mapping is locally homeomorphic at the zero element O of $\mathfrak{G}$. When the exponential mapping: $X \rightarrow \exp X$ is not locally homeomorphic at $X_0 \in \mathfrak{G}$, X_0 is called a singular point of $\mathfrak{G}$. And a set $\{\exp tX;\ t \text{ real}\}$ is called a path through the unit element E of G.

In this paper we shall investigate the path-structure and its singularity of $\exp \mathfrak{G}$, where $\exp \mathfrak{G}$ means the image of the exponential mapping: $\exp \mathfrak{G} = \{\exp X;\ X \in \mathfrak{G}\}$. Let R and C be the fields of real numbers and complex numbers respectively. In §2, we have a general consideration concerning the singularity of Lie groups, and in §§3 and 4, from our standpoint we shall consider the path-structure and its singularity of the complex general linear group $GL(n, C)$ and the real general linear group $GL(n, R)$ respectively.

§2. The singularity of Lie groups

Let G be a Lie group, $\mathfrak{G}$ its Lie algebra, and $X_i (i=1,2,\cdots,r)$ be a base of $\mathfrak{G}$. Then any element x_0 of $\exp \mathfrak{G}$ is expressed by $x_0 = \exp \sum x_0^i X_i$, and any element x of $\mathfrak{G}$ in a sufficiently small neighborhood of x_0 is expressed by $x = \exp \sum v^i X_i \exp \sum x_0^i X_i$, where $|v^i| (i=1,2,\cdots,r)$ are sufficiently small. The exponential mapping: $\sum x^i X_i \rightarrow \exp \sum x^i X_i$ is locally homeomorphic at $X_0 = \sum x_0^i X_i$, if and only if there exist two neighborhoods $\mathfrak{U}$ and $\mathfrak{V}$ of O in $\mathfrak{G}$ which are homeomorphic by the correspondence $\sum u^i X_i \in \mathfrak{U} \leftrightarrow \sum v^i X_i \in \mathfrak{V}$ such that

$$\exp \sum (x_0^i + u^i) X_i = \exp \sum v^i X_i \exp \sum x_0^i X_i. \tag{2.1}$$

(2.1) is written as

$$\exp \sum (x_0^i + u^i) x_i \exp (-\sum x_0^i X_i) = \exp \sum v^i X_i. \tag{2.2}$$

From (2.2) we have ([1], p. 156)[1)]

$$v^i = \sum J(x_0)^i_j u^j \equiv \sum ((\exp C(x_0) - E)/C(x_0))^i_j u^j, \tag{2.3}$$

1) Numbers in brackets refer to the references at the end of the paper.

where $C(x)=||\sum x^i c_{ij}^k||$, the c_{ij}^k are the constants of structure of $\mathfrak{G}$ such that $[X_i, X_j]=\sum c_{ij}^k X_k$, and by $(\exp A-E)/A$ we denote $\sum_1^\infty \frac{1}{m!} A^{m-1}$ for any matrix A. Thus, from (2.3) we have

THEOREM 1. *The exponential mapping:* $\sum x^i X_i \to \exp \sum x^i X_i$ *is locally homeomorphic at* $X_0=\sum x_0^i X_i \in \mathfrak{G}$, *if and only if the matrix* $C(x_0)$ *has no characteristic root such as* $\lambda=2l\pi\sqrt{-1}$ *(l is a non-zero integer).*

$X_0=\sum x_0^i X_i$ is said to be regular, if the matrix $C(x_0)$ has no characteristic root such as $\lambda=2l\pi\sqrt{-1}$ (l is non-zero integer); and X_0 is said to be singular, if X_0 is not regular.

We assume that G is a linear Lie group, and $\mathfrak{G}$ is its linear Lie algebra. Let $\mathfrak{C}(M)$ be the commutator set of M, $\mathfrak{C}_{\mathfrak{G}}(M)=\mathfrak{C}(M)\cap\mathfrak{G}$, and $\rho(M)$ the rank of M. Then we have

THEOREM 2. X_0 *is singular, if and only if* $\mathfrak{C}_{\mathfrak{G}}(\exp X_0)\supsetneqq\mathfrak{C}_{\mathfrak{G}}(X_0)$.

PROOF. It is clear that $\mathfrak{C}_{\mathfrak{G}}(\exp X_0)\supset\mathfrak{C}_{\mathfrak{G}}(X_0)$. Assume that $Y\in\mathfrak{C}_{\mathfrak{G}}(\exp X_0)$, i.e., $(\exp X_0)Y(\exp X_0)^{-1}=Y$, where $Y=\sum y^i X_i$; since $(\exp X_0)Y(\exp X_0)^{-1}$ $\sum_i(\sum_j(\exp C(x_0))_j^i y^j)X_i$, then we have $\sum(\exp C(x_0)-E)_j^i y^j=0$, where E is the unit matrix of degree r. Hence, $Y\in\mathfrak{C}_{\mathfrak{G}}(\exp X_0)$, if and only if $\sum(\exp C(x_0)-E)_j^i y^j=0$. And also it is clear that $Z\in\mathfrak{C}_{\mathfrak{G}}(X_0)$, if and only if $\sum C(x_0)_j^i z^j=0$, where $Z=\sum z^i X_i$. If and only if X_0 is singular, $C(x_0)$ has a characteristic root $\lambda=2l\pi\sqrt{-1}$ (l is a non-zero integer). By considering the canonical form of $C(x_0)$, we have

$$\rho(\exp C(x_0)-E)=\rho(C(x_0))-p,$$

where p is the number of blocks belonging to the characteristic roots such as $2l\pi\sqrt{-1}$ in the canonical form of $C(x_0)$. Therefore, X_0 is singular, if and only if $\mathfrak{C}_{\mathfrak{G}}(\exp X_0)\supsetneqq\mathfrak{C}_{\mathfrak{G}}(X_0)$.

REMARK 1. $\dim\mathfrak{C}_{\mathfrak{G}}(\exp X_0)=\dim\mathfrak{C}_{\mathfrak{G}}(X_0)+p$, where p is the number of blocks belonging to the characteristic roots such as $\lambda=2\pi l\sqrt{-1}$ (l is a non-zero integer) in the canonical form of $C(x_0)$.

§3. The complex general linear group

Let $GL(n, C)$ be the complex general linear group, $\mathfrak{Gl}(n, C)$ its Lie algebra, then it is well know that $GL(n, C)=\exp\mathfrak{Gl}(n, C)$. In this section we shall consider the singular points of $\mathfrak{Gl}(n, C)$. From the consideration in §2, we see that

$$\exp V\exp A=\exp(A+U), \tag{3.1}$$

where U and V are infinitesimal matrices, if and only if

$$V=((\exp \boldsymbol{A}-I)/\boldsymbol{A})U\equiv\sum_1^\infty\frac{1}{m!}\boldsymbol{A}^{m-1}U, \tag{3.2}$$

where $\quad \boldsymbol{A}^0U=U,\quad \boldsymbol{A}U=[A, U]=AU-UA,\quad \boldsymbol{A}^kU=\boldsymbol{A}(\boldsymbol{A}^{k-1}U).$

Here if, in place of $U=\|u_{ij}\|$ we take $\boldsymbol{u}={}^t(u_{11}, u_{21}, \cdots, u_{n1}, u_{12}, \cdots, u_{n2}, \cdots, u_{nn})$, then $U'=\boldsymbol{A}U$ is placed by $\boldsymbol{u}'=\tilde{A}\boldsymbol{u}\equiv(E\times A-{}^tA\times E)\boldsymbol{u}$, where $\tilde{A}=E\times A-{}^tA\times E$, $\times$ means the Kronecker's product and ${}^tA$ means the transposed matrix of A. And then (3.2) is written as

$$\boldsymbol{v}=J(A)\boldsymbol{u}\equiv((\exp\tilde{A}-E)/\tilde{A})\boldsymbol{u}, \tag{3.3}$$

where E is the unit matrix of degree n^2. Since

$$\det J(A)=\prod_{i,j=1}^{n}(\exp(\lambda_i-\lambda_j)-1)/(\lambda_i-\lambda_j), \tag{3.4}$$

where λ_i are the characteristic roots of A, if we denote by $\mathfrak{Gl}_0(n, C)$ the set of all the matrices whose characteristic roots do not satisfy the condition: $\lambda_i-\lambda_j=2l\pi\sqrt{-1}$ (l is a non-zero integer). Then we have

THEOREM 3. *The mapping: $A\to\exp A$ from $\mathfrak{Gl}(n, C)$ into $GL(n, C)$ is locally homeomorphic at A_0, if and only if $A_0\in\mathfrak{Gl}_0(n, C)$.*

REMARK 2. Let $\mathring{A}$ be the canonical form of A:

$$A\sim\mathring{A}=\dot{\sum}+\overbrace{\begin{pmatrix}\lambda_i & 1 & & 0\\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \ddots\ 1\\ 0 & & & \lambda_i\end{pmatrix}}^{n_i},$$

where $A\sim B$ means that A is similar to B, then we can calculate the rank of $J(A)$:

$$\rho(J(A))=n^2-\sum\nolimits_{(i,j)}^{*}\min(n_i, n_j),$$

where $\sum_{(i,j)}^{*}$ means the summation over the pairs (i, j) such that $\lambda_i-\lambda_j=2m_{ij}\pi\sqrt{-1}\neq0$ (m_{ij}: integers). If $\mathfrak{U}_A$ is a neighborhood of A in $\mathfrak{Gl}(n, C)$, then we have

$$\dim(\exp\mathfrak{U}_A)=\rho(J(A))=n^2-\nu,$$
$$\nu=\sum\nolimits_{(i,j)}^{*}\min(n_i, n_j).$$

This ν coincides with the ν in Theorem VI in the previous paper ([2]). Moreover, if A is singular, i.e., $\rho(J(A))<n^2$, then $\rho(J(A))\leqq n^2-2$.

By definition of the singular point in §2, A is singular, if and only if $A\in\mathfrak{Gl}_*(n, C)\equiv\mathfrak{Gl}(n, C)-\mathfrak{Gl}_0(n, C)$. A path: $M(t)=\exp tA$ ($M=\exp A$, $0\leqq t\leqq1$) is called to be singular, if $A\in\mathfrak{Gl}_*(n, C)$, and the path is called to be regular, if $A\in\mathfrak{Gl}_0(n, C)$. In the previous paper ([2]) we have obtained that $GL(n, C)=\exp\mathfrak{Gl}_0(n, C)=\exp\mathfrak{Gl}_{(-\pi,\pi]}(n, C)$, where $\mathfrak{Gl}_{(-\pi,\pi]}(n, C)$ is the set of all the matrices whose characteristic roots have their imaginary parts in a half closed interval $(-\pi, \pi]$. Hence, for any point $M\in GL(n, C)$, there exists always a countable number of regular paths from E to M.

THEOREM 4. *There exists a singular path from E to M, if and only if $\dim\mathfrak{C}(M)>n$.*

PROOF. From the results concerning the logarithmic function of matrix in the previous paper ([2]), it follows that $M=\exp A$ and $A\in\mathfrak{Gl}_*(n, C)$, if and only if the canonical form of M contains at least two blocks belonging to the same characteristic root, that is

$$M\sim\mathring{M}=\overbrace{\begin{pmatrix}\lambda & 1 & & & 0\\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1\\ 0 & & & & \lambda\end{pmatrix}}^{n_1}\dot{+}\overbrace{\begin{pmatrix}\lambda & 1 & & & 0\\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1\\ 0 & & & & \lambda\end{pmatrix}}^{n_2}\dot{+}\cdots.$$

This condition is satisfied, if and only if $\dim\mathfrak{C}(M)\equiv\dim\mathfrak{C}(\mathring{M})>n$. Thus, this theorem is proved.

Moreover the condition in Theorem 4 is equivalent to the condition that the minimal polynomial of M is of degree less than n.

THEOREM 5. *A path*: $M(t)=\exp tA$ $(M=\exp A,\ 0\leqq t\leqq 1)$ *from* E *to* M *is singular, if and only if* $\mathfrak{C}(A)\neq\mathfrak{C}(M)$.

PROOF. It is obvious from Theorem 2.

REMARK 3. By Theorem 3, we set that $\mathfrak{Gl}_0(n, C)$ is the maximal set for which the mapping: $A\to\exp A$ from $\mathfrak{Gl}(n, C)$ into $GL(n, C)$ is locally homeomorphic. And $\mathfrak{Gl}_0(n, C)$ is open and dense in $\mathfrak{Gl}(n, C)$, and is arc-wise connected, but is not simply connected.

In fact, clearly, $\mathfrak{Gl}_0(n, C)$ is open and dense in $\mathfrak{Gl}(n, C)$. For any $A\in\mathfrak{Gl}_0(n, C)$, we can take a sufficiently small positive number θ_0 such that $e^{i\theta}A\in\mathfrak{Gl}_0(n, C)$ for all $\theta: 0\leqq\theta\leqq\theta_0$, $(i=\sqrt{-1})$, and $t(e^{i\theta_0}A)\in\mathfrak{Gl}_0(n, C)$ for all $t: 0\leqq t\leqq 1$. Then we can connect A and $e^{i\theta_0}A$ in $\mathfrak{Gl}_0(n, C)$ by an arc $A(\theta)=e^{i\theta}A$ $(0\leqq\theta\leqq\theta_0)$, and $e^{i\theta_0}A$ and O in $\mathfrak{Gl}_0(n, C)$ by an arc $B(t)=t(e^{i\theta_0}A)$, $(0\leqq t\leqq 1)$. Thus, $\mathfrak{Gl}_0(n, C)$ is connected. But $\mathfrak{Gl}_0(n, C)$ is not simply connected. To show this, it is sufficient to consider the case $n=2$. In $\mathfrak{Gl}_0(2, C)$, we consider a curve: $A(\theta)=\begin{pmatrix}0 & 0\\ 0 & r_0e^{i\theta}\end{pmatrix}$, $(0\leqq\theta\leqq 2\pi)$, where r_0 is a fixed real number such that $2l\pi<r_0<(2l+1)\pi$. If the curve is deformable in $\mathfrak{Gl}_0(2, C)$ to a point, then, corresponding to the deformation, the curve: $z(\theta)=r_0e^{i\theta}$ $(0\leqq\theta\leqq 2\pi)$ in the complex plane must be deformable in the complex plane to a point. Since the curve: $z(\theta)=r_0e^{i\theta}$ contains a point $2l\pi\sqrt{-1}$ as an inner point. In the process of deformation of the curve $A(\theta)$, the corresponding curve $z(\theta)$ passes through the point $2l\pi\sqrt{-1}$. Then, the corresponding point is not contained in $\mathfrak{Gl}_0(2, C)$. Therefore, $\mathfrak{Gl}_0(2, C)$ is not simply connected.

REMARK 4. A path $M(t)=\exp tA$ $(0\leqq t\leqq 1)$ is regular and closed, if and only if $A=2m\pi\sqrt{-1}\,E$, where m is an integer.

In fact, if the path is closed, then A is given by

$$A=2\pi\sqrt{-1}\cdot S(m_1E_1\dot{+}m_2E_2\dot{+}\cdots\dot{+}m_pE_p)S^{-1},$$

where $m_1,\cdots, m_p$ are integers and S is an arbitrary regular matrix. Furthermore, if and only if $m_i=m_j$ $(=m)$ for all i, j, then $A\in\mathfrak{Gl}_0(n, C)$. Thus, the assertion is proved.

§4. The real general linear group

Let $GL(n, R)$ be a real general linear group, $GL^{+}(n, R)$ the connected component of the unit element of $GL(n, R)$, and $\mathfrak{Gl}(n, R)$ its Lie algebra. In this section we shall investigate the relation between $GL^{+}(n, R)$ and $\exp \mathfrak{Gl}(n, R)$, and the path-structure and its singularity of $\exp \mathfrak{Gl}(n, R)$. To do this, by using the real canonical form of a real matrix, we shall first determine the real matrices such that $\exp A = M$ for a given real matrix M. Any real matrix A is transformed to the real canonical form $\overset{\circ}{A}$ by a real regular matrix P, that is,

$$(4.1)\qquad A = P\overset{\circ}{A}P^{-1},\quad \overset{\circ}{A} = \sum\dot{+}\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix} \dot{+} \sum\dot{+}\begin{pmatrix} L & E_2 & & 0 \\ & L & \ddots & \\ & & \ddots & E_2 \\ 0 & & & L \end{pmatrix},$$

where the first summation does not contain the same blocks, λ is real, $L = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and μ and ν are real.

From (4.1) we have

$$(4.2)\qquad \exp A = P(\exp \overset{\circ}{A})P^{-1},$$

$$\exp \overset{\circ}{A} \sim \sum\dot{+}\begin{pmatrix} e^{\lambda} & 1 & & 0 \\ & e^{\lambda} & \ddots & \\ & & \ddots & 1 \\ 0 & & & e^{\lambda} \end{pmatrix} \dot{+} \sum\dot{+}\begin{pmatrix} \exp L & E_2 & & 0 \\ & \exp L & \ddots & \\ & & \ddots & E_2 \\ 0 & & & \exp L \end{pmatrix},$$

where $\exp L = \begin{pmatrix} e^{\mu}\cos\nu & e^{\mu}\sin\nu \\ -e^{\mu}\sin\nu & e^{\mu}\cos\nu \end{pmatrix}$, $M \sim N$ means that $M = PNP^{-1}$ and P is real. Hence the real matrix M such that $M = \exp A$ for a real matrix A has the real canonical form as follows:

$$(4.3)\qquad M \sim \overset{\circ}{M} = \sum\dot{+}\begin{pmatrix} \kappa & 1 & & & 0 \\ & \kappa & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \kappa \end{pmatrix} \dot{+} \sum\dot{+}\begin{pmatrix} K & E_2 & & 0 \\ & K & \ddots & \\ & & \ddots & E_2 \\ 0 & & & K \end{pmatrix},$$

where $\kappa > 0$, $K = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, and α and β are real. That is, if the real matrix $M = \exp A$ has real negative characteristic roots, then the real canonical form of M must contain in the form of pairs the blocks belonging to the real negative characteristic roots. For, then $\overset{\circ}{M}$ must contain the blocks of type:

$$\begin{pmatrix} K & E_2 & & 0 \\ & K & \ddots & \\ & & \ddots & E_2 \\ 0 & & & K \end{pmatrix},\quad K = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},\quad \alpha < 0,$$

and this block is similar to (by a real transformation)

$$\begin{pmatrix} \alpha & 1 & & 0 \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix} \dot{+} \begin{pmatrix} \alpha & 1 & & 0 \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix}.$$

Conversely, let M be a real matrix satisfying this condition, and write it in the canonical form (4.3), then a real matrix A such that $\exp A = M$ must be similar to A_0:

$$A_0 = \sum \dot{+} \begin{pmatrix} \lambda & \kappa^{-1} & -\frac{1}{2}\kappa^{-2} & \cdots & \\ & \lambda & \kappa^{-1} & \ddots & \vdots \\ & & \ddots & \ddots & -\frac{1}{2}\kappa^{-2} \\ & & & \ddots & \kappa^{-1} \\ 0 & & & & \lambda \end{pmatrix} \dot{+} \sum \dot{+} \begin{pmatrix} L & K^{-1} & -\frac{1}{2}K^{-2} & \cdots & \\ & L & K^{-1} & \ddots & \vdots \\ & & \ddots & \ddots & -\frac{1}{2}K^{-2} \\ & & & \ddots & K^{-1} \\ 0 & & & & L \end{pmatrix}, \tag{4.4}$$

where $\kappa = e^{\lambda}$ and $K = \exp L$. Therefore λ, μ and ν are determined as follows:

$$\lambda = \log \kappa, \quad \mu = \log \sqrt{\alpha^2+\beta^2} \quad \text{and} \quad \nu = \theta + 2m\pi, \tag{4.5}$$

where m is an integer and $\theta = \arg(\alpha + \sqrt{-1}\,\beta)$, $(-\pi < \theta \leqq \pi)$. From (4.4) it follows that $\exp A_0 = M_0$. And moreover we have

$$\exp A = M, \quad A \sim A_0 \quad \text{and} \quad M \sim M_0. \tag{4.6}$$

Hence we can write $M_0 = P_0^{-1} M P_0$ and $A_0 = Q^{-1} A Q$, where P_0 is a fixed matrix. From these conditions we have

$$P_0^{-1} M P_0 = \exp(Q^{-1} A Q) = Q^{-1} (\exp A) Q = Q^{-1} M Q,$$

from which $Q P_0^{-1} = S \in \mathfrak{C}(M)$, i.e., $Q \in \mathfrak{C}(M) P_0$. For another matrix P_1 such that $P_1^{-1} M P_1 = M_0$, clearly we have $\mathfrak{C}(M) P_0 = \mathfrak{C}(M) P_1$. Thus we obtain

THEOREM 6. *M is an element of $\exp \mathfrak{Gl}(n, R)$, if and only if the real canonical form of M contains in the form of pairs the blocks belonging to the real negative characteristic roots, whenever there exist the real negative characteristic roots of M. For such a matrix M, any matrix A such that $\exp A = M$ is given by $A = Q A_0 Q^{-1}$, $Q \in \mathfrak{C}(M) P_0$, where P_0 is an arbitrarily chosen and fixed matrix such that $P_0^{-1} M P_0 = M_0$, M_0 is given by* (4.3), *and A_0 is given by* (4.4) *corresponding to M_0 in* (4.3).

If $M \in GL^+(n, R)$, then M is expressed as ([6])

$$M = \exp A_1 \exp A_2 \cdots \exp A_k, \tag{4.7}$$

where $A_1, A_2, \cdots, A_k \in \mathfrak{Gl}(n, R)$. Since $\det \exp A = \exp \operatorname{tr} A > 0$ for any $A \in \mathfrak{Gl}(n, R)$, we have $\det M > 0$. Conversely, if $M \in GL(n, R)$ and $\det M > 0$,

$$M = \exp B, \quad B \in \mathfrak{Gl}_{(-\pi, \pi]}(n, C) \tag{4.8}$$

Since M is real, $\exp B = \exp \overline{B}$; hence we have $\exp B \exp(-\overline{B}) = \exp(-\overline{B}) \exp B = E$, and also $B, -\overline{B} \in \mathfrak{Gl}_{(-\pi, \pi]}(n, C)$. By the results of the previous paper ([2]), from these fact it follows that B and $\overline{B}$ are commutative. So we have

$$(4.9)\qquad M=\exp((B+\overline{B})/2)\exp((B-\overline{B})/2),$$

where $B_1=(B+\overline{B})/2$ is real. If we put $N=\exp((B-\overline{B})/2)$, then N is real and $N^2=E$. By considering the real canonical form of N, we have $N=P(-E_{r_1}\dotplus E_{r_2})P^{-1}$, where E_r is the unit matrix of degree r. Since $\det M>0$ and $\det\exp((B+\overline{B})/2)>0$, we have $\det N>0$; hence r_1 must be even, so we have

$$N=\exp B_2,\quad B_2=\pi P\left\{\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\dotplus O\right\}P^{-1}.$$

Therefore, we obtain

$$(4.10)\qquad M=\exp B_1\exp B_2,\quad B_1, B_2\in\mathfrak{Gl}(n, R),$$

and hence M and E are connected in $GL(n, R)$ by a curve: $M(t)=\exp tB_1 \exp tB_2$, $(0\leqq t\leqq 1)$; so that $M\in GL^+(n, R)$. Thus, $GL^+(n, R)$ is the set of all the real matrices whose determinants are positive, and moreover

$$(4.11)\qquad GL^+(n, R)=\exp\mathfrak{Gl}(n, R)\exp\mathfrak{Gl}(n, R).$$

On the other hand, by Theorem 6 we see that

$$(4.12)\qquad \overline{\exp\mathfrak{Gl}(n, R)}\subsetneqq GL^+(n, R),$$

where $\overline{\mathfrak{A}}$ means the closure of $\mathfrak{A}$ with respect to the usual topology of $GL(n, R)$. In fact, we shall consider $M\in GL^+(n, R)$ such that $M\sim M_0=\mathrm{diag}\{\alpha_1, \alpha_2,\cdots, \alpha_n\}$, where $0>\alpha_1>\cdots>\alpha_n$, if n is even, $\alpha_1>0>\alpha_2>\cdots>\alpha_n$ if n is odd. By Theorem 6, $M\notin\exp\mathfrak{Gl}(n, R)$. And the set of all the matrices satisfying this condition is open in $GL^+(n, R)$, so that $\overline{\exp\mathfrak{Gl}(n, R)}\subsetneqq GL^+(n, R)$. By Theorem 6, it is also clear that $\overline{\exp\mathfrak{Gl}(n, R)}=\exp\mathfrak{Gl}(n, R)$.

Let $GL_0(n, R)$ be the set of all the real regular matrices without real negative characteristic roots, then from the results of the previous paper ([3]), it follows that $GL_0(n, R)=\exp\mathfrak{Gl}_{(\pi,\pi)}(n, R)$ and in this correspondence, the exponential mapping is homeomorphic, where $\mathfrak{Gl}_{(-\pi,\pi)}(n, R)$ is the set of all the real matrices whose characteristic roots have their imaginary parts in an open interval $(-\pi, \pi)$. From this fact it follows that $GL_0(n, R)$ is a simply connected domain. Moreover $GL_0(n, R)$ is a maximal simply connected domain of $\exp\mathfrak{Gl}(n, R)$. In fact, if $M\in\exp\mathfrak{Gl}(n, R)-GL_0(n, R)$, then we have

$$(4.13)\qquad M=PM_0P^{-1},\quad M_0=\sum_{(\lambda<0)}\dotplus\begin{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} & E_2 & & 0\\ & \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} & E_2 & \\ & & \ddots & \\ 0 & & & \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\end{pmatrix}\dotplus H.$$

Here we consider the curve: $M(\theta)$, $(0\leqq\theta\leqq 2\pi)$ in $\exp\mathfrak{Gl}(n, R)$, where $M(\theta)$ is given by

$$(4.14)\quad M(\theta)=PM_0(\theta)P^{-1},\ M_0(\theta)=\textstyle\sum_{(\lambda<0)}\dot{+}\begin{pmatrix} \begin{pmatrix} \lambda\cos\theta & \lambda\sin\theta \\ -\lambda\sin\theta & \lambda\cos\theta \end{pmatrix} & E_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & E_2 \\ 0 & & & \begin{pmatrix} \lambda\cos\theta & \lambda\sin\theta \\ -\lambda\sin\theta & \lambda\cos\theta \end{pmatrix} \end{pmatrix}\dot{+}H.$$

The locus of the characteristic root $\lambda e^{i\theta}$ $(0\leqq\theta\leqq 2\pi)$ is a circle with center 0 in the complex plane. This circle can not be deformable to a point without passing through 0. Hence, the curve: $M(\theta)$, $(0\leqq\theta\leqq 2\pi)$ is not deformable in $\exp\mathfrak{Gl}(n, R)$ to a point. Therefore, $GL_0(n, R)$ is a maximal simply connected domain of $\exp\mathfrak{Gl}(n, R)$.

Furthermore, $\overline{GL_0(n, R)}=\exp\mathfrak{Gl}(n, R)$. In fact, if $M\in\exp\mathfrak{Gl}(n, R)-GL_0(n, R)$, then M is written as (4.13). And if $M(\theta)$ is given by (4.14), then $M(\theta)\to M$ for $\theta\to 0$; and moreover $M(\theta)\in GL_0(n, R)$ for $\theta\neq 0$, so that $\overline{GL_0(n, R)}=\exp\mathfrak{Gl}(n, R)$. (By theorem 6, it is clear that $GL_0(n, R)\subsetneqq\exp\mathfrak{Gl}(n, R)$).

Next, for $M\in\exp\mathfrak{Gl}(n, R)-GL_0(n, R)$, ($M$ is written as (4.13)), if we take

$$(4.15)\quad M_1(\varepsilon)=PM_*(\varepsilon)P^{-1},\ M_*(\varepsilon)=\textstyle\sum_{(\lambda<0)}\dot{+}\begin{pmatrix} \begin{pmatrix} \lambda+\varepsilon & 0 \\ 0 & \lambda \end{pmatrix} & E_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & E_2 \\ 0 & & & \begin{pmatrix} \lambda+\varepsilon & 0 \\ 0 & \lambda \end{pmatrix} \end{pmatrix}\dot{+}H,$$

then $M_1(\varepsilon)\in GL^+(n, R)-\exp\mathfrak{Gl}(n, R)$ for sufficiently small $\varepsilon>0$, and $M_1(\varepsilon)\to M$ for $\varepsilon\to 0$. And also $GL_0(n, R)$ is open in $\exp\mathfrak{Gl}(n, R)$ and $\overline{GL_0(n, R)}=\exp\mathfrak{Gl}(n, R)$. Hence $\mathrm{Bdry}(\exp\mathfrak{Gl}(n, R))=\exp\mathfrak{Gl}(n, R)-GL_0(n, R)$.

Summarizing these results we have

THEOREM 7. $\overline{GL_0(n,R)}\subsetneqq GL_0(n,R)=\exp\mathfrak{Gl}(n,R)=\overline{\exp\mathfrak{Gl}(n,R)}\subsetneqq GL^+(n,R)=\exp\mathfrak{Gl}(n, R)\exp\mathfrak{Gl}(n, R)$. *And* $GL_0(n, R)$ *is a maximal simply connected domain of* $\exp\mathfrak{Gl}(n, R)$, *and* $\mathrm{Bdry}(\exp\mathfrak{Gl}(n, R))=\exp\mathfrak{Gl}(n, R)-GL_0(n, R)$.

REMARK 5. K. Schröder ([7]), by showing that $G^0=\overline{\exp\mathfrak{G}}$ for the connected component G^0 of the unit element of a linear Lie group G and its Lie algebra $\mathfrak{G}$, has proved that $G^0=\exp\mathfrak{G}\exp\mathfrak{G}$. But it does not seem to be true that $G^0=\overline{\exp\mathfrak{G}}$. For example, by Theorem 7, for $G=GL(n, R)$ we have $\overline{\exp\mathfrak{Gl}(n, R)}\subsetneqq GL^+(n, R)=G^0$. Moreover, a proof of that $G^0=\exp\mathfrak{G}\cdot\exp\mathfrak{G}$ for a semi-simple Lie group is obtained from the results of G. D. Mostow ([5]).

Finally we shall consider the path-structure of $GL^+(n, R)$ (precisely, of $\exp\mathfrak{Gl}(n, R)$). By using Theorem 6 we obtain the following theorems. Here we denote by $\mathfrak{S}$ the set of $M\in\exp\mathfrak{Gl}(n, R)$ such that

$$M \sim \overset{\circ}{M} = \sum_{(\kappa>0)} \dot{+} \begin{pmatrix} \kappa & 1 & & 0 \\ & \kappa & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 0 \\ & & & & \kappa \end{pmatrix},$$

where the blocks are all distict.

THEOREM 8. *There exists one and only one path through E and M ($M \in \exp \mathfrak{Gl}(n, R)$), if and only if $M \in \mathfrak{S}$. If $M \notin \mathfrak{S}$, there exist at least a countable number of paths through E and M. And $\mathfrak{S}$ is open in* $\exp \mathfrak{Gl}(n, R)$.

THEOREM 9. *If and only if $M \in$* Bdry ($\exp \mathfrak{Gl}(n, R)$), *the paths through E and M are always singular.*

In the other word, if and only if $M \in GL_0(n, R)$, there exists a regular path from E to M.

REMARK 6. By the similar argument as in the previous paper ([4]), we can prove the following propositions:

For an element M of $GL_0(n, R)$, there exists one and only one path through E and M which is entirely contained in $GL_0(n, R)$. And for an element M of Bdry ($\exp \mathfrak{Gl}(n, R)$), there exist, at least, two paths from E to M which are contained in $GL_0(n, R)$ except for M.

Any path from E to M intersects Bdry ($\exp \mathfrak{Gl}(n, R)$), at most, in a finite number of points.

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