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On Prime Operations in the Theory of Ideals

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Introduction. This paper contains some contributions to the theory of prime operations. As is well-known, Prüfer and Krull introduced two important *prime operations*, namely the so-called a- and b-*operations*, in an integrally closed integral domain, and Krull proved that these two operations coincide on finitely generated ideals. But his proof seems to be neither simple nor straightforward. And the question, whether they coincide or not on arbitrary ideals, has been left open.

The main purpose of this paper is to prove that the a-operation is nothing but the b-operation. Our proof depends entirely on the existence theorem of valuations and is very simple.

In this paper, we always denote by $\mathfrak{o}$ an integral domain, and by K its field of quotients. By an $\mathfrak{o}$-ideal, we mean a fractional ideal of $\mathfrak{o}$.

1. Axioms and examples of prime operations. In his book [2, p. 118], Krull gave a system of axioms of prime operations. But, as we shall see below, his axioms are not independent. Therefore we shall begin with modifying his axioms.

Let $\mathfrak{a}$ be an $\mathfrak{o}$-ideal. A mapping $\mathfrak{a}\rightarrow\mathfrak{a}'$ ($\mathfrak{a}'$ is also an $\mathfrak{o}$-ideal) is a prime operation if it satisfies the following conditions.

P_1. $\mathfrak{a}\subseteq\mathfrak{a}'$.

P_2. $(\mathfrak{a}')'=\mathfrak{a}'$.

P_3. $\mathfrak{a}\subseteq\mathfrak{b}$ implies $\mathfrak{a}'\subseteq\mathfrak{b}'$.

P_4. $\mathfrak{o}=\mathfrak{o}'$.

P_5. $((\alpha)\mathfrak{a})'=(\alpha)\mathfrak{a}'$ for any $\alpha\in K$.

From these axioms, we deduce the following relations.

1) $(\mathfrak{a}'+\mathfrak{b}')'=(\mathfrak{a}+\mathfrak{b})'$, 2) $(\mathfrak{a}'\mathfrak{b}')'=(\mathfrak{a}\mathfrak{b})'$, 3) $(\mathfrak{a}'\cap\mathfrak{b}')'=\mathfrak{a}'\cap\mathfrak{b}'$.

In fact, 1) and 3) are immediate consequences of P_1, P_2 and P_3. As for 2), we have $\alpha\mathfrak{b}\subseteq\mathfrak{a}\mathfrak{b}$ for any $\alpha\in\mathfrak{a}$, hence $\alpha\mathfrak{b}'=((\alpha)\mathfrak{b})'\subseteq(\mathfrak{a}\mathfrak{b})'$ by P_3 and P_5, therefore $\mathfrak{a}\mathfrak{b}'\subseteq(\mathfrak{a}\mathfrak{b})'$. From this, in the same way, $\mathfrak{a}'\mathfrak{b}'\subseteq(\mathfrak{a}\mathfrak{b}')'\subseteq(\mathfrak{a}\mathfrak{b})''=(\mathfrak{a}\mathfrak{b})'$. Therefore $(\mathfrak{a}'\mathfrak{b}')'\subseteq(\mathfrak{a}\mathfrak{b})''=(\mathfrak{a}\mathfrak{b})'$ by P_1 and P_2. The converse inclusion is obvious.

Next we shall give some examples of prime operations.

v-operation. The most familiar prime operation is the mapping $\mathfrak{a}\rightarrow\mathfrak{a}_v=(\mathfrak{a}^{-1})^{-1}$ and is termed v-operation. From the definition of $\mathfrak{a}_v$, $x\in K$ belongs to $\mathfrak{a}_v$ if and only if x is divisible (with respect to $\mathfrak{o}$) by any common divisor of all elements of $\mathfrak{a}$. Hence

$$\mathfrak{a}_v=\bigcap_{\alpha\mathfrak{o}\supseteq\mathfrak{a}}\alpha\mathfrak{o}$$

where $\alpha\mathfrak{o}$ runs over all principal ideals which contain $\mathfrak{a}$.

This remark enables us to see that the v-operation has the special position among prime operations, namely

$$(*)\qquad \mathfrak{a}\subseteq\mathfrak{a}'\subseteq\mathfrak{a}_v$$

for any prime operation: $\mathfrak{a}\rightarrow\mathfrak{a}'$.

In fact, $\mathfrak{a}\subseteq\alpha\mathfrak{o}$ implies $\mathfrak{a}'\subseteq(\alpha\mathfrak{o})'=\alpha\mathfrak{o}$ by P_1, P_4 and P_5, consequently $\mathfrak{a}'\subseteq\mathfrak{a}_v$.

From $(*)$ follow the equations:

$$\mathfrak{a}^{-1}=(\mathfrak{a}')^{-1}=(\mathfrak{a}^{-1})' \quad\text{and}\quad (\mathfrak{a}')_v=\mathfrak{a}_v.$$

d- and b-operations. Consider a family of rings $\{R_\iota;\iota\in M\}$ such that $\mathfrak{o}\subseteq R_\iota\subseteq K$ and $\mathfrak{o}=\bigcap_{\iota\in M}R_\iota$. Let $\mathfrak{a}$ be an $\mathfrak{o}$-ideal, then, as is easily seen, the mapping

$$\mathfrak{a}\rightarrow\mathfrak{a}_d=\bigcap_{\iota\in M}\mathfrak{a}R_\iota$$

is a prime operation. A prime operation of this kind shall be called d-operation. As for $\{R_\iota;\iota\in M\}$ we can take various kinds of families of rings [3, §7].

In the case when $\mathfrak{o}$ is integrally closed, it is known that $\mathfrak{o}$ can be represented as the intersection of valuation rings of K, therefore the mapping

$$\mathfrak{a}\rightarrow\mathfrak{a}_b=\bigcap_{\iota\in I}\mathfrak{a}R_\iota$$

where $R_\iota(\iota\in I)$ runs over all valuation rings of K containing $\mathfrak{o}$, is one of d-operations. This prime operation is called b-operation.

a-operation. We say that $x\in K$ is integral over an $\mathfrak{o}$-ideal $\mathfrak{a}$, if it satisfies an equation of the form

$$x^n+\alpha_1x^{n-1}+\cdots+\alpha_n=0,\ \text{with}\ \alpha_i\in\mathfrak{a}^i\ (i=1,\cdots,n).$$

For this, it is necessary and sufficient that there exists a finitely generated $\mathfrak{o}$-ideal $\mathfrak{b}$ such that $x\mathfrak{b}\subseteq\mathfrak{a}\mathfrak{b}$. If $\mathfrak{o}$ is integrally closed, the set $\mathfrak{a}_a$ of all elements $x\in K$ which are integral over $\mathfrak{a}$ forms an $\mathfrak{o}$-ideal and the mapping: $\mathfrak{a}\rightarrow\mathfrak{a}_a$ is also a prime operation. This mapping was considered first by Prüfer in [6] and is known as a-operation.

2. Main theorem. In the preceding section, a- and b-operations were introduced in an integrally closed integral domain. For these operations, as was mentioned in the introduction, Krull proved that $\mathfrak{a}_a=\mathfrak{a}_b$ for any finitely generated ideal $\mathfrak{a}$.

But we can prove further the following

THEOREM. *a- and b-operations are the same.*

PROOF. Suppose first $x \in \mathfrak{a}_a$, then x satisfies an equation

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n = 0, \text{ with } \alpha_i \in \mathfrak{a}^i \ (i=1, \cdots, n).$$

Then, for any valuation v of K whose valuation ring contains $\mathfrak{o}$,

$$v(x^n) = v(\alpha_1 x^{n-1} + \cdots + \alpha_n) \geqq \min_{i=1,\cdots,n} \{v(\alpha_i) + v(x^{n-i})\}.$$

We assume the minimum value is taken at, say, $i = i_0$. Then $i_0 v(x) \geqq v(\alpha_{i_0})$. Since $\alpha_{i_0} \in \mathfrak{a}^{i_0}$, we have $v(\alpha_{i_0}) \geqq i_0 v(\alpha)$ for some $\alpha \in \mathfrak{a}$, therefore $v(x) \geqq v(\alpha)$, which shows $x \in \mathfrak{a}R$, where R is the valuation ring of v. Hence $x \in \mathfrak{a}_b$.

Conversely, suppose $x \notin \mathfrak{a}_a$. Let $\{\beta_\lambda;\ \lambda \in \Lambda\}$ be a system of generators of $\mathfrak{a}$. Form a ring $A = \mathfrak{o}\left[\cdots, \frac{\beta_\lambda}{x}, \cdots\right]$ and consider an ideal $M = \left(\cdots, \frac{\beta_\lambda}{x}, \cdots\right)A$. We first assert that $1 \notin M$. In fact, if, $1 \in M$, then there exists a relation of the form

$$1 = \sum_{i=1}^{t} \frac{\beta_i}{x} f_i\left(\frac{\beta_1}{x}, \cdots, \frac{\beta_t}{x}\right),$$

where f's are polynomials with coefficients in $\mathfrak{o}$. Rearranging the terms, this relation may be written in the following form:

$$1 = \sum_{i=1}^{s} g_i\left(\frac{\beta_1}{x}, \cdots, \frac{\beta_t}{x}\right),$$

where g_i is a form of degree i. By multiplying this equation by x^s, we have

$$x^s = g_1(\beta_1, \cdots, \beta_t)x^{s-1} + \cdots + g_s(\beta_1, \cdots, \beta_t).$$

This shows $x \in \mathfrak{a}_a$, which is a contradiction.

Now we know that $M \neq A$, then by virtue of the existence theorem of valuations [5, Lemma 3, p. 95], there exists a valuation v of K such that $R_v \supseteq A$ and $M_v \supseteq M$ where R_v and M_v mean the valuation ring and the valuation ideal of v respectively. Hence $v(\beta_\lambda / x) > 0$ for any λ. Whence follows, as we shall see below, $x \notin \mathfrak{a}R_v$, consequently $x \notin \mathfrak{a}_b$. If $x \in \mathfrak{a}R_v$, x can be written in the form

$$x = \beta_1 x_1 + \cdots + \beta_n x_n, \text{ with } x_i \in R_v,$$

for suitable $\beta_1, \cdots, \beta_n$. Therefore $v(x) \geqq \min_{i=1,\cdots,n} \{v(\beta_i) + v(x_i)\}$, hence $v(x) \geqq v(\beta_{i_0})$ for some i_0 $(1 \leqq i_0 \leqq n)$, which is a contradiction, q.e.d.

Notice that, in the case when $\mathfrak{o}$ is Noetherian, we can take a discrete valuation ring of rank 1 [1, Lemma 1, p. 85], in place of R_v in the proof of the theorem. Therefore

$$\mathfrak{a}_b = \bigcap_{\iota \in I} \mathfrak{a}R_\iota$$

where $R_\iota (\iota \in I)$ runs over all discrete valuation rings of rank 1 which contain $\mathfrak{o}$.

On the other hand, in an "Endliche diskrete Hauptordnung A", it is known that

$$\mathfrak{a}_v = \bigcap_{\lambda \in J} \mathfrak{a} R_\lambda$$

where R_λ runs over all discrete rank 1 valuation rings which can be obtained by forming a quotient ring of A with respect to a minimal prime ideal [2, p. 119].

Combining these two facts, we have the following

PROPOSITION. *Let $\mathfrak{o}$ be an integrally closed Noetherian domain, then for any $\mathfrak{o}$-ideal $\mathfrak{a}$,*

$$\mathfrak{a}_a = \mathfrak{a}_b = \bigcap_{\iota \in I} \mathfrak{a} R_\iota \subseteq \mathfrak{a}_v = \bigcap_{\lambda \in J} \mathfrak{a} R_\lambda$$

where the former intersection is taken over all discrete rank 1 valuation rings of K containing $\mathfrak{o}$, and the latter one is over all discrete rank 1 valuation rings which are quotient rings of $\mathfrak{o}$.

Now, let $f(X)$ be a polynomial of a variable X over K. By the coefficient ideal of $f(X)$, we mean the $\mathfrak{o}$-ideal generated by the coefficients of $f(X)$. With the aid of the preceding theorem, we can prove the following Gauss' theorem for prime operations.

PROPOSITION. *Suppose $\mathfrak{o}$ is integrally closed, and let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be coefficient ideals of $f(X)$, $g(X)$ and $h(X)=f(X)g(X)$ respectively. Then we have $(\mathfrak{a}\mathfrak{b})' = \mathfrak{c}'$ for any prime operation: $\mathfrak{a} \to \mathfrak{a}'$ which satisfies the following condition*

$$\mathfrak{d}_b \subseteq \mathfrak{d}$$

for any $\mathfrak{o}$-ideal $\mathfrak{d}$.

PROOF. As is easily seen, it is sufficient to show that $(\mathfrak{a}\mathfrak{b})_b = \mathfrak{c}_b$. Set

$$f(X) = a_0 X^n + \cdots + a_n, \quad g(X) = b_0 X^m + \cdots + b_m \text{ and}$$
$$h(X) = f(X)g(X) = c_0 X^{n+m} + \cdots + c_{n+m}.$$

Since $(\mathfrak{a}\mathfrak{b})_b = \bigcap \mathfrak{a}\mathfrak{b} R_\iota$ and $\mathfrak{c}_b = \bigcap \mathfrak{c} R_\iota$, it is enough to show that $\mathfrak{a}\mathfrak{b} R_\iota = \mathfrak{c} R_\iota$ for any ι. Put $R = R_\iota$ and $v = v_\iota$. And let a_{i_0} be the first a_i whose value is equal to $\min_{i=0,\dots,n} v(a_i)$ and define b_{j_0} in the same way. Then

$$\min v(a_i b_j) = v(a_{i_0} b_{j_0}) = v(c_{i_0+j_0}) = \min v(c_{i+j}),$$

hence $\mathfrak{a}\mathfrak{b} R = \mathfrak{c} R$, which proves our assertion.

Remark: In the section 1, a- and b-operations were introduced only in an integrally closed integral domain. However, if we restrict our considerations to integral ideals, similar operations may be considered even in the case when $\mathfrak{o}$ is not integrally closed. Namely, we define the mappings $\mathfrak{a}\to\mathfrak{a}_a$ and $\mathfrak{a}\to\mathfrak{a}_b$ for integral ideals as follows:

$\mathfrak{a}_a=\{$the set of all elements x in $\mathfrak{o}$ which are integral over $\mathfrak{a}\}$

$\mathfrak{a}_b=(\bigcap_{\iota\in I}\mathfrak{a}R_\iota)\cap\mathfrak{o}$, where R_ι runs over all valuation rings of K containing $\mathfrak{o}$.

Even in this case, the proof of our theorem shows that $\mathfrak{a}_a=\mathfrak{a}_b$, and consequently $(\mathfrak{a}\bar{\mathfrak{o}})_a\cap\mathfrak{o}=\mathfrak{a}_a$, where $\bar{\mathfrak{o}}$ is the integral closure of $\mathfrak{o}$ in K.

Next, we shall consider a relationship between these operations and the asymptotic closure operation. For this, we must recall some definitions.

A *pseudo-valuation* v of $\mathfrak{o}$ is a mapping: $x\to v(x)$ of $\mathfrak{o}$ into (R, ∞) satisfying

i) $v(0)=\infty, \quad v(1)=0,$

ii) $v(x-y)\geqq\min\{v(x), v(y)\},$

iii) $v(xy)\geqq v(x)+v(y),$

where R is the ordered additive group of real numbers.

A pseudo-valuation v is called *homogeneous* if it satisfies the further condition

iv) $v(x^n)=nv(x)$ for any positive integer n.

Next, we shall define a pseudo-valuation $v_\mathfrak{a}$ of $\mathfrak{o}$ by defining $v_\mathfrak{a}(x)=n$ if $x\in\mathfrak{a}^n$ and $x\notin\mathfrak{a}^{n+1}$, and $v_\mathfrak{a}(x)=\infty$ if $x\in\bigcap_{n=1}^{\infty}\mathfrak{a}^n$. Then, we see that $\bar{v}_\mathfrak{a}(x)=\lim_{n\to\infty}v_\mathfrak{a}(x^n)/n$ is a homogeneous pseudo-valuation [7, Theorem 1.3, p. 109] and we shall denote by $\mathfrak{a}_s$ the set consisting of all elements x in $\mathfrak{o}$ such that $\bar{v}_\mathfrak{a}(x)\geqq 1$ and call it the *asymptotic closure* of $\mathfrak{a}$ in $\mathfrak{o}$.

In the case when $\mathfrak{o}$ is Noetherian, this definition of the asymptotic closure is equivalent to the one given by Samuel [9] and in this case Muhly proved that $\mathfrak{a}_b=\mathfrak{a}_s$ [4]. But, even in the case when $\mathfrak{o}$ is not Noetherian, we can also prove this fact for finitely generated ideals.

In fact, if $x\in\mathfrak{a}_a$, then by a computation similar to the one, which we gave in the first part of the proof of our theorem, we have $x\in\mathfrak{a}_s$.

Conversely, for an element x of $\mathfrak{a}_s$, suppose that there exists a valuation w of K such that whose valuation ring R_w contains $\mathfrak{o}$ and $x\notin\mathfrak{a}R_w$. Since $\mathfrak{a}$ is finitely generated, $\mathfrak{a}R_w=aR_w$ for some element $a\in\mathfrak{a}$, hence $w(x)<w(a)$. On the other hand, by the definition of $\bar{v}_\mathfrak{a}(x)$, for every real number $\varepsilon>0$, we have $v_\mathfrak{a}(x^n)/n>1-\varepsilon$ for sufficiently large n, hence $v_\mathfrak{a}(x^n)>n(1-\varepsilon)$, whence $nw(x)=w(x^n)>n(1-\varepsilon)w(a)$. Therefore, these relations show that the subgroup, generated by $w(x)$ and $w(a)$, of the value group of w is archimedian. Hence $w(x)>(1-\varepsilon)w(a)$, which is a contradiction.

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