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Spin-orbit Interaction Energy of an Electron based on the New Fundamental Group of Transformations

By

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§ 1. Introduction

Spin-orbit interaction energy of an electron was first introduced by Uhlenbeck and Goudsmit [1]*) showing how the idea of spin of an electron can be used to explain the anomalous Zeeman effect. However, the assumption they had to make seemed to lead to the value twice as large as the experimental value. It was due to the fact that the calculation was referred to the observer moving instantaneously with the same velocity as that of an electron. But, in fact, the velocity of an electron in atom changes successively for the observer fixed to the nucleus. Thereupon, taking it into account, L. H. Thomas [2] and J. Frenkel [3] calculated the value of spin-orbit interaction energy. Their calculations led to the same value as observed. In this way, it was shown that Uhlenbeck and Goudsmit's assumptions really led to the correct doublet separation at the same time as the anomalous Zeeman effect when the problem was treated by the quantum mechanics [4]. Nowadays, the effect of the spin-orbit interaction on the energy levels of hydrogen is deduced from the Dirac's theory of electron. However, in this paper, we shall describe our theory comparing it with that of L. H. Thomas.

The main fact used by L. H. Thomas in obtaining the correct result is that the combination of two "Lorentz transformations without rotation" in general is not of the same form but is equivalent to a Lorentz transformation with a rotation. The "Lorentz transformations without rotation" are defined by the following equations [2, 5, 6, 7]:

$$\left.\begin{aligned} x'^i &= x^i + u^i\left[\frac{(ux)}{u^2}\left\{\frac{1}{\sqrt{1-u^2/c^2}}-1\right\}-\frac{t}{\sqrt{1-u^2/c^2}}\right] \quad (i=1,2,3) \\ t' &= [t-(ux)/c^2]/\sqrt{1-u^2/c^2} \end{aligned}\right\} \tag{1.1}$$

where x^i, t and x'^i, t' denote the space-time coordinates in the systems K and K', the components of the uniform velocity of K' relative to K being u^h $(h=1, 2, 3)$. The round parenthesis of u and x: (ux) denotes the inner product of u^i and x^i. The above-mentioned fact means that the trans-

*) Numbers in brackets refer to the references at the end of the paper.

formations (1.1) do not form a 3-parameter group regarding u^h $(h=1, 2, 3)$ as parameters. This fact does not take place for the Galilean transformations in non-relativistic kinematics. Obviously the Galilean transformations without rotation of the Cartesian axes:

$$x'^i=x^i+u^it \quad (i=1, 2, 3),\ t'=t$$

form a 3-parameter group regarding u^h as parameters. So it may be natural to expect that even in relativistic kinematics the transformations between the space-time coordinates in two inertial systems moving with uniform velocity to each other, should have group property (regarding the three components of the velocity as parameters of the transformations). As shown in the previous paper [7], the fact that the transformations defined by (1.1) do not form a group is due to the adoption of the following assumption: the component of the position vector x^i in the direction perpendicular to the velocity u^i undergoes no change. However, we think that this assumption has no necessary reason. So, abandoning the above assumption, we modify the transformations defined by (1.1) such that they form a 3-parameter group (regarding u^h $(h=1, 2, 3)$ as parameters) and leave the form of $ds^2=(dx^1)^2+(dx^2)^2+(dx^3)^2-c^2(dt)^2$ invariant. In the previous papers [7, 8], we have described the equations of such modified transformations in the following form:

$$\left.\begin{aligned} x'^i&=x^j\left[\delta^i_j-\frac{d^i-u^i/c}{1-(du)/c}d_j-d^i\left\{\frac{u_j/c}{\sqrt{1-u^2/c^2}}-\frac{d_j\sqrt{1-u^2/c^2}}{1-(du)/c}\right\}\right]\\ &\quad+t\left[d^i\frac{u^2/c-(du)\{1-\sqrt{1-u^2/c^2}\}}{\{1-(du)/c\}\sqrt{1-u^2/c^2}}-\frac{u^i}{1-(du)/c}\right]\quad (i,j=1,2,3)\\ t'&=[t-(ux)/c^2]/\sqrt{1-u^2/c^2}\end{aligned}\right\}\qquad (1.2)$$

which are the equations (1.2) in the previous paper [7]. Here $d^h=d_h$ $(h=1, 2, 3)$ are any constants satisfying the condition $(d\cdot d)=1$. We have called the transformations (1.2) "the new fundamental group of transformations" and proposed to replace (1.1) by (1.2), as representing the relations between the coordinates in two inertial systems one of which moves with uniform velocity to the other. The essential difference between the transformations (1.1) and (1.2) is that the transformations (1.2) introduce automatically a certain direction whose direction cosines are d_1, d_2, d_3. In the foregoing papers [6, 7, 8], we have intended to develope the theory based on the new fundamental group of transformations.

Now, we must show how the spin-orbit interaction energy can be evaluated based on (1.2). For this aim, corresponding to the Thomas precession [2, 5], we have calculated the precession caused by the successive transformations (1.2) of the new group as follows [7]. Let us consider three inertial systems K, K' and K'' of which K' moves with velocity u^i relative to K while K'' moves with infinitesimal velocity v^i relative to K'. As the connection between the coordinates (x, t) in K and (x', t') in K' we take (1.2). In the same way, the connection between (x', t') and the co-

ordinates (x'', t'') in K'' is represented by the equations obtained from (1.2) after replacing (x, t, u) by (x', t', v) and (x', t') by (x'', t''). By the group property of (1.2), the relation between (x, t) and (x'', t'') is expressed by the equations obtained from (1.2) after replacing (x', t') by (x'', t'') and u^i by certain parameters w^i (the components of the velocity of K'' relative to K). Since v^i are infinitesimal quantities, we can put

$$w_i = u_i + \delta u_i$$

regarding δu_i as infinitesimal quantities. We shall obtain a method to represent the transition from K' to K'' measured in K. For this purpose, by considering certain spatial rotation in K, we introduce for K' the system (y') such that the systems (y') and K' are connected by Lorentz transformation without rotation. Further, applying the same procedure to K'', we introduce the system (y'') in K. Then the transition from (y') to (y'') in K corresponds to the transition from K' to K''. The transition from (y') to (y'') is expressed by an infinitesimal rotation which we denote by the vector Ω. The actual form of Ω has been obtained by (2.8) and (2.9) in the previous paper [7]. Namely

$$\Omega = \frac{1}{1-(du)/c} \begin{Bmatrix} \left\{2(1-\sqrt{1-u^2/c^2})\frac{(d\cdot u)}{u^2} - \frac{1}{c}\right\}[\boldsymbol{d}\times\delta\boldsymbol{u}] \\ -2(1-\sqrt{1-u^2/c^2})\frac{(d\cdot\delta u)}{u^2}[\boldsymbol{d}\times\boldsymbol{u}] \\ -\left(\frac{1}{\sqrt{1-u^2/c^2}}-1\right)\frac{(u\cdot\delta u)}{cu^2}[\boldsymbol{d}\times\boldsymbol{u}] \\ -(1-\sqrt{1-u^2/c^2})\frac{1}{u^2}[\boldsymbol{u}\times\delta\boldsymbol{u}] \end{Bmatrix} \tag{1.3}$$

where $\boldsymbol{u}=(u^1, u^2, u^3)$, $\boldsymbol{d}=(d^1, d^2, d^3)$, $\delta\boldsymbol{u}=(\delta u^1, \delta u^2, \delta u^3)$, and $[\boldsymbol{d}\times\delta\boldsymbol{u}]$ denotes the ordinary vector product of $\boldsymbol{d}$ and $\delta\boldsymbol{u}$.

In order to calculate the spin-orbit interaction energy of an electron based on (1.3), we now consider a point compass, i. e. a material particle which in some way or other defines a direction. If the velocity of the particle relative to K is $\boldsymbol{u}=\boldsymbol{u}(t)$, and if we put $\delta\boldsymbol{u}=\dot{\boldsymbol{u}}(t)\cdot dt$ in the above consideration, the systems K' and K'' will be momentary rest systems of inertia for the particle at the times t and $t+dt$ respectively. And the rotation vector defined by (1.3) represents the rotation corresponding to the transition from K' to K'' measured in K. Since, furthermore, the direction of the compass relative to the rest system is constant, this means that the direction of the compass relative to K is turned through an angle corresponding to the rotation vector Ω. In other words, the compass performs a precession relative to K with the velocity of precession

$$\boldsymbol{\omega}=\frac{1}{\sqrt{1-(d\cdot u)/c}}\left[\begin{array}{l}\left\{2(1-\sqrt{1-u^2/c^2})\dfrac{(d\cdot u)}{u^2}-\dfrac{1}{c}\right\}[\boldsymbol{d}\times\dot{\boldsymbol{u}}]\\ -2(1-\sqrt{1-u^2/c^2})\dfrac{(d\cdot\dot{u})}{u^2}[\boldsymbol{d}\times\boldsymbol{u}]\\ -\left(\dfrac{1}{\sqrt{1-u^2/c^2}}-1\right)\dfrac{(u\cdot\dot{u})}{cu^2}[\boldsymbol{d}\times\boldsymbol{u}]\\ -(1-\sqrt{1-u^2/c^2})\dfrac{1}{u^2}[\boldsymbol{u}\times\dot{\boldsymbol{u}}]\end{array}\right] \tag{1.4}$$

where $\dot{\boldsymbol{u}}=d\boldsymbol{u}/dt$ is the acceleration of the point compass.

In the following sections §2, §3, using (1.4), we shall calculate spin-orbit interaction energy of an electron in hydrogenic atom. Then comparing the result with the experimental value, we conclude that d^h appeared in the transformation (1.2) should be interpreted as representing the direction of spin vector of an electron.

§2. Estimation of spin-orbit interaction energy

From the consideration of §1, the spin vector $\boldsymbol{s}$ of the electron in hydrogenic atom, as seen by an observer fixed to the nucleus, precesses with angular velocity ω:

$$\boldsymbol{\omega}=-\frac{1}{c}[(\boldsymbol{d}+\boldsymbol{u}/2c)\times\dot{\boldsymbol{u}}]-\frac{1}{c^2}(d\cdot\dot{u})[\boldsymbol{d}\times\boldsymbol{u}]$$

which is obtained from (1.4) by neglecting all terms of higher than the first order in u/c.

Following the usual method, this precession gives rise to an instantaneous energy change, in addition to the term $(\boldsymbol{\omega}_L\cdot\boldsymbol{s})$ due to Larmor precession,

$$H'=(\boldsymbol{\omega}_L\cdot\boldsymbol{s})+(\boldsymbol{\omega}\cdot\boldsymbol{s}) \tag{2.1}$$

Now, we shall evaluate this additional spin-orbit interaction term. For this purpose, we must first set up, in our formulation based on the new fundamental group of transformations, the Hamiltonian without spin-orbit interaction and the eigen function for this unperturbed Hamiltonian.

In our theory, as shown in the previous paper [8], we have defined energy-momentum vector p_i, E as an invariant vector of the new fundamental group of transformations. Specially, when the energy-momentum vector is invariant under the rotations around the axis $\boldsymbol{d}$, we have

$$\left.\begin{aligned}p_i&=\frac{m_0u_i}{\sqrt{1-u^2/c^2}}+\frac{n_0\sqrt{1-u^2/c^2}}{1-(d\cdot u)/c}cd_i\\ E/c&=\frac{m_0c}{\sqrt{1-u^2/c^2}}+\frac{n_0\sqrt{1-u^2/c^2}}{1-(d\cdot u)/c}c\end{aligned}\right\} \tag{2.2}$$

where m_0 and n_0 are arbitrary constants.

In this paper, we start with the energy-momentum vector defined by

(2.2). From (2.2), we have the relation between energy and momentum as follows:

$$E^2/c^2=(p_i p^i)+c^2(m_0^2+2m_0n_0)\ .$$

This form is the same as in the ordinary theory except the rest-mass is replaced by $\sqrt{m_0^2+2m_0n_0}$. Hence the Hamiltonian H of an electron in the Coulomb potential $V(r)=-Ze/r$ of the nucleus with charge Ze can be expressed in the following form:

$$\begin{aligned}H&=E-\kappa c^2+V(r)=c\sqrt{\kappa^2c^2+\boldsymbol{p}^2}-\kappa c^2+V(r)\\&=\frac{1}{2\kappa}\boldsymbol{p}^2-\frac{1}{8\kappa^3c^2}\boldsymbol{p}^4+\cdots+V(r)\end{aligned}\tag{2.3}$$

where

$$\kappa^2=m_0^2+2m_0n_0\ .\tag{2.4}$$

As in the ordinary theory, the Schrödinger equation*)

$$H_0\psi=\left[\frac{1}{2\kappa}\boldsymbol{p}^2+V(r)\right]\psi=\left[-\frac{\hbar^2}{2\kappa}\nabla^2+V(r)\right]\psi=W\psi\tag{2.5}$$

gives eigen values of energy W_n:

$$W_n=-\frac{\kappa e^4Z^2}{2\hbar^2n^2}\quad(n=1,\,2,\,\cdots)\tag{2.6}$$

and eigen function

$$\psi_{n,l}=\frac{R_{n,l}(r)}{r}\Theta_{l,m}(\theta)\Phi_m(\varphi)\quad\begin{pmatrix}l=0,\,1,\,2,\,\cdots,\,(n-1)\\m=-l,\,-l+1,\,\cdots,\,l\end{pmatrix}\tag{2.7}$$

in which

$$R_{n,l}(r)=\sqrt{\frac{Z(n-l-1)!}{n^2a[(n+l)!]^3}}\,e^{-\rho/2}\rho^{l+1}L_{n+l}^{2l+1}(\rho)$$

$$\Theta_{l,m}(\theta)=(-1)^l\sqrt{\frac{2l+1}{2}\frac{(l+m)!}{(l-m)!}}\,\frac{1}{2^ll!}\,\frac{1}{\sin^m\theta}\,\frac{d^{l-m}}{(d\cos\theta)^{l-m}}\sin^{2l}\theta$$

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}\exp(im\varphi)$$

where $\rho=2Zr/na$ and $a=\hbar^2/\kappa e^2$. [9]

Taking into account the energy correction due to the second term of (2.3), the Hamiltonian (2.3) gives the eigen values of energy [9]

$$W_{nl}=-\frac{\kappa c^2}{2}\,\frac{\alpha^2Z^2}{n^2}\left\{1+\frac{\alpha^2Z^2}{n^2}\left(\frac{n}{l+1/2}-\frac{3}{4}\right)+\cdots\right\},\quad\alpha=e^2/\hbar c=1/137.\tag{2.8}$$

In our system with the eigen function (2.7), $\boldsymbol{p}$ is of order $\hbar/a$. On the other hand, we treat the problem under the assumption that $u/c\ll1$. Hence it may be reasonable to put $|n_0/m_0|\ll1$.

Next, we must express spin-orbit interaction part (2.1) in terms of the canonical momentum $\boldsymbol{p}$ and the coordinates $\boldsymbol{r}$ of the electron. Using the relation (2.2) and neglecting all terms of higher than the first order in u/c, we can express $\boldsymbol{u}$ in terms of $\boldsymbol{p}$ as follows:

*) In our theory $\boldsymbol{p}$ and κ are not the same as the momentum and mass in ordinary theory. However, for a while, we assume that $\boldsymbol{p}$ and κ play the same role as the momentum and mass in the ordinary theory.

$$\boldsymbol{u}=\frac{1}{m_0}\boldsymbol{p}-\frac{n_0}{m_0+n_0}c\boldsymbol{d}-\frac{n_0}{m_0(m_0+n_0)}(\boldsymbol{p}\cdot\boldsymbol{d})\boldsymbol{d} \tag{2.9}$$

Therefore, adopting (2.9), $(\boldsymbol{\omega}\cdot\boldsymbol{s})$ in (2.1) can be rewritten in the form:

$$(\boldsymbol{\omega}\cdot\boldsymbol{s})=\frac{1}{c}\left(1-\frac{n_0}{2(m_0+n_0)}\right)([\boldsymbol{d}\times\boldsymbol{s}]\cdot\dot{\boldsymbol{u}})-\frac{1}{2m_0c^2}([\boldsymbol{p}\times\dot{\boldsymbol{u}}]\cdot\boldsymbol{s})$$
$$+\frac{1}{m_0c^2}\left\{-\frac{n_0}{2(m_0+n_0)}(\boldsymbol{d}\cdot\boldsymbol{p})([\boldsymbol{d}\times\boldsymbol{s}]\cdot\dot{\boldsymbol{u}})+(\boldsymbol{d}\cdot\dot{\boldsymbol{u}})([\boldsymbol{d}\times\boldsymbol{s}]\cdot\boldsymbol{p})\right\}. \tag{2.10}$$

Now, we shall assume that $\dot{\boldsymbol{u}}$ is given by the equation

$$\dot{\mathfrak{u}}=-\frac{e}{\kappa}\boldsymbol{E} \quad \text{with} \quad \boldsymbol{E}=\frac{\partial}{\partial r}\left(-\frac{Ze}{r}\right)=Ze\frac{\boldsymbol{r}}{r^3} \tag{2.11}$$

in which we take κ in the place of mass in the ordinary theory. We insert (2.11) into (2.10) and substitute the resulting expression for $(\boldsymbol{\omega}\cdot\boldsymbol{s})$ in (2.1). Then, assuming temporarily that $(\boldsymbol{\omega}_L\cdot\boldsymbol{s})=\frac{Ze^2}{\kappa m_0c^2r^3}([\boldsymbol{r}\times\boldsymbol{p}]\cdot\boldsymbol{s})$ as in ordinary theory, (2.1) is expressed as

$$H'=-\frac{Ze^2}{\kappa c}\left(1-\frac{n_0}{2(m_0+n_0)}\right)([\boldsymbol{d}\times\boldsymbol{s}]\cdot\boldsymbol{r})/r^3+\frac{Ze^2}{2\kappa m_0c^2}([\boldsymbol{r}\times\boldsymbol{p}]\cdot\boldsymbol{s})/r^3$$
$$+\frac{Ze^2}{\kappa m_0c^2}\left\{\frac{n_0}{2(m_0+n_0)}(\boldsymbol{d}\cdot\boldsymbol{p})([\boldsymbol{d}\times\boldsymbol{s}]\cdot\boldsymbol{r})-(\boldsymbol{d}\cdot\boldsymbol{r})([\boldsymbol{d}\times\boldsymbol{s}]\cdot\boldsymbol{p})\right\}\Big/r^3 \tag{2.12}$$

The first order perturbation arising from (2.12) is

$$(H')_{n,l;\,n,l'}=\int\tilde{\psi}_{n,l}H'\psi_{n,l'}\,dv \tag{2.13}$$

where $\psi_{n,l}$ is the eigen function given by (2.7) and $\tilde{\psi}_{n,l}$ is complex conjugate to $\psi_{n,l}$. The second term in (2.12) is nothing but the one deduced from the usual spin-orbit interaction except m_0^3 is replaced by κm_0. We shall denote this term by W_T'. Namely,

$$W_T'=\int\tilde{\psi}_{n,l}(r,\theta,\varphi)\left[\frac{Ze^2}{2\kappa m_0c^2}\frac{1}{r^3}([\boldsymbol{r}\times\boldsymbol{p}]\cdot\boldsymbol{s})\right]\psi_{n,l'}(r,\theta,\varphi)r^2\sin\theta\cdot dr\cdot d\theta\cdot d\varphi\,. \tag{2.14}$$

Now, we shall estimate the first term, say H_1', of (2.12). When H_1' does not vanish, for the sake of convenience, we take the direction of $[\boldsymbol{d}\times\boldsymbol{s}]$ as that of z axis. Then,

$$H_1'=-\frac{Ze^2}{\kappa c}\left(1-\frac{n_0}{2(m_0+n_0)}\right)|\boldsymbol{d}\times\boldsymbol{s}|\frac{1}{r^2}\cos\theta=\lambda\frac{1}{r^2}\cos\theta \tag{2.15}$$

putting $\lambda=-Ze^2|\boldsymbol{d}\times\boldsymbol{s}|\{1-n_0/2(m_0+n_0)\}/\kappa c$. The first order perturbation due to (2.15) is expressed, from (2.13), as

$$(H_1')_{n,l;n,l'}=\lambda\int_0^\infty R_{n,l}R_{n,l'}\frac{1}{r^2}dr\int_0^\pi\Theta_{l,m}\Theta_{l',m}\cdot\cos\theta\cdot\sin\theta\,d\theta$$

The integral over θ has a value only when $l'=l\pm1$. For $l'=l-1$

$$(H_1')_{n,l;n,l-1}=\lambda\left(\frac{2Z}{na}\right)^3\frac{a}{4Z}\frac{(n+l)!\,(n+l-1)!}{[(n+l)!\,]^3}\sqrt{n^2-l^2}\sqrt{\frac{(l-m)(l+m)}{(2l+1)(2l-1)}}$$
$$\times\int_0^\infty e^{-\rho}\rho^{2l-1}L_{n+l}^{2l+1}(\rho)L_{n+l-1}^{2l-1}(\rho)d\rho$$

Using the properties of associated Laguerre polynomial, we can show that

$(H_1')_{n,l;n,l-1}=0$. Similarly we have $(H_1')_{n,l;n,l+1}=0$. Hence we have

$$(H_1')_{n,l;n,l'}=0\ . \tag{2.16}$$

Next, by the usual method, we shall estimate the second order perturbation arising from (2.15), as follows

$$W_1''=\sum_{n'\neq n}\frac{(H_1')_{n,l;n',l'}(H_1')_{n',l';n,l}}{W_n-W_{n'}} \tag{2.17}$$

where $(H_1')_{n,l;n',l'}=\int\tilde{\psi}_{n,l}H_1'\psi_{n',l'}dv$, $(H_1')_{n'l';nl}=\int\tilde{\psi}_{n',l'}H_1'\psi_{nl}dv$, and W_n' is the expression obtained from (2.6) after replacing n by n'. For the estimation, the calculation of the integral $\int_0^\infty R_{nl}(r)R_{n'l'}(r)\frac{1}{r^2}dr$ $(n\neq n')$ is necessary. Since this integral has complicated form, we shall, here, estimate only the numerical order. We can easily show that

$$(H_1')_{n,l;n',l'}/W_T'\simeq ca\kappa/\hbar=c\hbar/e^2=137\ . \tag{2.18}$$

On the other hand,

$$W_T'/W_n\simeq Z^2\alpha^2/n^2\simeq Z^2/(137n^2)\ .$$

Hence $(H_1')_{n,l;n',l'}$ is the same order as W_n. Therefore, from (2.17), W_1'' is, at least, the same order as $(H_1')_{n',l';n,l}$. Accordingly, by (2.18), W_1'' becomes far larger than W_T'. In other words, our spin-orbit interaction term gives not only values that contradict with experimental energy values, but also breaks down the usual perturbation method. Therefore, in order that we may have adequate theory, we must search for such a case that the first term of (2.12) contributes nothing to spin-orbit interaction energy. The case fitted for our aim is that $(1-n_0/2(m_0+n_0))[\boldsymbol{d}\times\boldsymbol{s}]=0$, namely $[\boldsymbol{d}\times\boldsymbol{s}]=0$ since $1-n_0/2(m_0+n_0)\neq 0$. In the following section we shall investigate the case $[\boldsymbol{d}\times\boldsymbol{s}]=0$.

§ 3. Case where $\boldsymbol{d}$ is parallel to $\boldsymbol{s}$

We shall now consider the case where

$$[\boldsymbol{d}\times\boldsymbol{s}]=0\ , \tag{3.1}$$

i. e. $\boldsymbol{d}$ is parallel to $\boldsymbol{s}$. In this case (2.12) becomes

$$H'=\frac{Ze^2}{2\kappa m_0c^2}([\boldsymbol{r}\times\boldsymbol{p}]\cdot\boldsymbol{s})/r^3\ . \tag{3.2}$$

In order that this gives the value coincides with the experimental value it must be that $\kappa m_0=m^2$, accordingly

$$n_0=0\ ,\qquad \kappa=m_0\ . \tag{3.3}$$

Then we have

$$p_i=m_0u_i/\sqrt{1-u^2/c^2}$$
$$E/c=m_0c/\sqrt{1-u^2/c^2}$$

which coincide with the usual special relativistic term. Thus, (3.2) gives the same eigen values of energy as the one deduced by the usual formalism.

Now, putting

$$[\boldsymbol{r}\times\boldsymbol{p}]=\boldsymbol{l}$$

(the eigen value of $\boldsymbol{l}$ being $0, 1, 2, \cdots$), we consider the total angular momentum: $\boldsymbol{J}=\boldsymbol{l}+\boldsymbol{s}$. In usual way, by quantizing $\boldsymbol{l}$, $\boldsymbol{s}$ and $\boldsymbol{J}$ such that they satisfy the commutation relations: $J_1J_2-J_2J_1=i\hbar J_3$ etc., we see that $\boldsymbol{l}^2$, $\boldsymbol{s}^2$ and $\boldsymbol{J}^2$ has eigen values $l(l+1)\hbar^2$, $s(s+1)\hbar^2$ and $j(j+1)\hbar^2$ respectively. Here $j=|l-s|, |l-s|+1, \cdots, l+s$. For $s=1/2$, j takes the values $l+1/2$ and $l-1/2$ unless $l=0$. (for $l=0$, $j=1/2$) From (2, 14), the first order perturbation gives the following eigen values of energy [9]:

$$\left.\begin{aligned} W' &= W_{nl}+\frac{\kappa c^2}{2}\left(\frac{\alpha^2Z^2}{n^2}\right)^2\frac{n}{(2l+1)(l+1)} \qquad \left(j=l+\frac{1}{2}\right) \\ &= W_{nl}-\frac{\kappa c^2}{2}\left(\frac{\alpha^2Z^2}{n^2}\right)^2\frac{n}{(2l+1)l} \qquad \left(j=l-\frac{1}{2}\right) \end{aligned}\right\} \tag{3.5}$$

To conclude, we shall compare our procedures with the usual one introduced by L. H. Thomas. Let K' be a frame of reference in which an electron is momentarily at the origin, and which moves with the instantaneous velocity of the electron at time t, measured in the frame of reference K fixed to nucleus. Furthermore, let K'' be a frame of reference in which the electron is instantaneously fixed at the origin at time $t+\Delta t$ measured in K. We are able to consider that the components of the velocity $\Delta\boldsymbol{v}$ of K'' relative to K' are infinitesimal quantities. In our formalism based on the new fundamental group of transformations, as shown in the previous paper [7], the direction of compass fixed to K'' performs a precession with the velocity of precession

$$\tilde{\boldsymbol{\omega}}=-\frac{1}{c}[\boldsymbol{d}\times\dot{\boldsymbol{v}}]$$

measured in K'. On the contrary, in the ordinary procedure based on the usual Lorentz transformations without rotation, we have $\tilde{\boldsymbol{\omega}}=0$. Thus, in order to preserve the parallelism between our method and the usual one, $\tilde{\boldsymbol{\omega}}$ must be only superficial one and contributes nothing to the physical effect. Accordingly, it will be reasonable to consider $\boldsymbol{d}$ be parallel to $\boldsymbol{s}$. When $\boldsymbol{d}$ is parallel to $\boldsymbol{s}$, the additional energy due to $\tilde{\boldsymbol{\omega}}$ becomes to be zero measured in K', and we can obtain $(\boldsymbol{\omega}\cdot\boldsymbol{s})$ in a concise form. In this case, by means of (1.4), only the term $-\frac{1}{2c^2}[\boldsymbol{u}\times\dot{\boldsymbol{u}}]$ in $\boldsymbol{\omega}$ has physical effect in the frame of reference K. From the above consideration it may be natural to interprete d^h appeared in the transformations (1.2), as representing spin axis of an electron.

§ 4. Concluding remarks

In the preceding sections we have treated the spin-orbit interaction energy based on the new group of transformations (1.2). However, the

transformations considered are characterized by the property that the transformations leave the equation of plane wave front advancing in the direction d^n with the velocity of light: $(d\cdot x)-ct=0$ invariant [10]. Hence our theory seems to be in connection with Dirac's theory concerning the forms of relativistic dynamics [11]. Therefore it is necessary to investigate the problem concerning the so-called front form of relativistic dynamics. This problem will be treated in our future paper.

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