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Lengths of Projections in Rings of Operators

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Introduction

In the theory of rings of operators, Dye [1, §4] introduced in a σ-finite, essentially finite ring the concept of the length of the identity by the cyclic dissection, and proved that such a ring is expressible as a direct sum of rings such that every summand is of the uniform length, that is, all central projections in each summand have the same length ([1], Theorem 3).

In an arbitrary ring, Ogasawara [5, p. 257] defined the lengths of σ-finite projections, in connection with the lengths of normal states. In this paper, for any (not necessarily σ-finite) projection P whose central envelope is σ-finite relative to the center, we define the length by the same way as in [5], that is, the length of P is the least cardinal of the cyclic projections by whose sum P may be represented. Our definition coincides with that of [1] in the σ-finite, essentially finite case. Generalizing the result of Theorem 3 of [1], we get the decomposition theorem for length (Theorem 2), which also includes Theorem 1 of [3] as a special case.

Next we define the l-function of a ring $\boldsymbol{M}$ which is closely related to the lengths of the central σ-finite projections, and prove that the quotient of the l-function of $\boldsymbol{M}'$ divided by that of $\boldsymbol{M}$ is a unitary invariant of $\boldsymbol{M}$, which, in the semi-finite case, coincides with the unitary invariant which was introduced by Pallu de La Barrière [6], and, in the case of type III, coincides with the inverse of the coupling operator which was introduced by Griffin [3].

§ 1. Lengths of projections

A projection P in a ring $\boldsymbol{M}$ of operators on a Hilbert space $\mathfrak{H}$ is said to be *cyclic* relative to $\boldsymbol{M}$ if there exists a vector $x \in \mathfrak{H}$ such that $[\boldsymbol{M}'x] = P\mathfrak{H}$ (see Dye [1], Notation 2.1). P is said to be *σ-finite* (=countably decomposable) relative to $\boldsymbol{M}$ if any orthogonal family of non-zero projections in $\boldsymbol{M}$ and less than P is at most countable. P is σ-finite if and only if P is a sum of countable cyclic projections. If $\boldsymbol{M}$ is commutative, then any

σ-finite projection in $\boldsymbol{M}$ is cyclic. A ring $\boldsymbol{M}$ is said to be *essentially finite* if every cyclic projection in $\boldsymbol{M}$ is finite.

$\boldsymbol{M}_P$ stands for the set of projections in $\boldsymbol{M}$. For any $P \in \boldsymbol{M}_P$ we denote by $[P]$ the class of all projections $Q \in \boldsymbol{M}_P$ such that $Q \sim P$, and define the ordering $[P_1] \leqq [P_2]$ by $P_1 \precsim P_2$. Then, by virtue of the comparability theorem we see that the set $\{[P]\,;\, P \in \boldsymbol{M}_P\}$ forms a lattice (see [4, p. 225]). $[P]$ is said to be cyclic (resp. σ-finite, finite, properly infinite) if P is so. If there is an orthogonal family $\{P_\alpha\,;\, \alpha \in \mathfrak{A}\}$ such that $P_\alpha \in [P]$ for all $\alpha \in \mathfrak{A}$, then we denote the class $[\sum_{\alpha \in \mathfrak{A}} P_\alpha]$ by $n[P]$, n being the cardinal of $\mathfrak{A}$. In the sequel, the central envelope of P is denoted by $e(P)$.

Now we prepare the following lemmas.

LEMMA 1. *Let $P \in \boldsymbol{M}_P$ be σ-finite and $Q \in \boldsymbol{M}_P$ be properly infinite. If $e(P) \leqq e(Q)$, then $P \precsim Q$.*

PROOF. Remark that for any projections $R, S \in \boldsymbol{M}_P$ with $e(R) \cap e(S) \neq 0$ there exist $R_1, S_1 \in \boldsymbol{M}_P$ such that $0 \neq R_1 \leqq R$, $0 \neq S_1 \leqq S$ and $R_1 \sim S_1$. It follows from this fact that there exists an orthogonal family $\{P_\alpha\,;\, \alpha \in \mathfrak{A}\}$ such that $P = \sum_{\alpha \in \mathfrak{A}} P_\alpha$ and $0 \neq P_\alpha \precsim Q$ for all $\alpha \in \mathfrak{A}$. Since P is σ-finite, $\mathfrak{A}$ is countable, so we may write $P = \sum_{i=1}^{\infty} P_i$, $P_i \precsim Q$. Since Q is properly infinite, we can write $Q = \sum_{i=1}^{\infty} Q_i$ where $Q_i \sim Q$ for all i. Then $P_i \precsim Q_i$, so that $P \precsim Q$, as desired.

LEMMA 2. *Let $P \in \boldsymbol{M}_P$ be properly infinite and $e(P)$ be σ-finite relative to the center $\boldsymbol{M}^\natural$ of $\boldsymbol{M}$. Then there exists a properly infinite, σ-finite projection Q in $\boldsymbol{M}$ such that $Q \leqq P$ and $e(Q) = e(P)$.*

PROOF. Using Zorn's lemma we get $e(P) = \sum_{\alpha \in \mathfrak{A}} e(P_\alpha)$ where $0 \neq P_\alpha \leqq P$ and P_α are all cyclic. Since $e(P)$ is σ-finite, $\mathfrak{A}$ is countable, so that $Q_1 = \sum_{\alpha \in \mathfrak{A}} P_\alpha$ is σ-finite. Since P is properly infinite, we have $[P] = \aleph_0[P] \geqq \aleph_0[Q_1]$. Hence there is a projection Q such that $Q \in \aleph_0[Q_1]$ and $Q \leqq P$. Since any projection in $\aleph_0[R]$, $R \neq 0$, is properly infinite, it follows that Q is properly infinite. Since Q_1 is σ-finite, so is Q. We have $e(Q) = e(Q_1) = \bigcup_{\alpha \in \mathfrak{A}} e(P_\alpha) = e(P)$. This completes the proof.

REMARK 1. We have the decompositions $I = Z^f + Z^i = Z^{f'} + Z^{i'}$ (Z^f, Z^i, $Z^{f'}$, $Z^{i'} \in \boldsymbol{M}_P^\natural$) such that $\boldsymbol{M}_{Z^f}$, $\boldsymbol{M}'_{Z^{f'}}$ are finite and $\boldsymbol{M}_{Z^i}$, $\boldsymbol{M}'_{Z^{i'}}$ are properly infinite. $\boldsymbol{M}_{Z^f \cup Z^{f'}}$ is clearly essentially finite. Next, we can prove that $\boldsymbol{M}_{Z^i \cap Z^{i'}}$ has no non-zero essentially finite part. It follows from Lemma 3 of [5] that if $\boldsymbol{M}'$ is properly infinite, then every σ-finite projection in $\boldsymbol{M}$ is cyclic. Hence, if $\boldsymbol{M}$ and $\boldsymbol{M}'$ are properly infinite, then by Lemma 2 there exists a properly infinite, cyclic projection in $\boldsymbol{M}$, so that $\boldsymbol{M}$ is not essentially finite. Therefore $\boldsymbol{M}_{Z^i \cap Z^{i'}}$ has no non-zero essentially finite part.

For any $P \in \boldsymbol{M}_P$ we denote by $[P]^c$ the largest one (if it exists) among cyclic classes $[Q]$ with $[Q] \leqq [P]$. Remark that if $[ZP]^c$ and $[Z^\perp P]^c$ exist for some $Z \in \boldsymbol{M}_P^\natural$ then $[P]^c$ exists. We can show that $[P]^c$ exists for any σ-finite $P \in \boldsymbol{M}_P$. For, Lemma 4.1 of [1] shows the existence of $[Z'P]^c$ since $\boldsymbol{M}_{Z^{f'}}$ is essentially finite, and $Z^{i'}P$ is cyclic since $\boldsymbol{M}'_{Z^{i'}}$ is properly

infinite (Remark 1). Therefore $[Z^f P]^c$ and $[Z^i P]^c$ exist, so that $[P]^c$ exists.

THEOREM 1. *The following four conditions on $P \in \boldsymbol{M}_P$ are equivalent:*

(i) *There exists a cyclic projection Q in $\boldsymbol{M}$ such that $e(P)=e(Q)$;*

(ii) *There exists a σ-finite projection Q in $\boldsymbol{M}$ such that $e(P)=e(Q)$;*

(iii) *$e(P)$ is σ-finite relative to $\boldsymbol{M}^\natural$ (=cyclic relative to $\boldsymbol{M}^\natural$);*

(iv) *$[P]^c$ exists.*

PROOF. It is obvious that (i)→(ii).

(ii)→(iii): Since Q is σ-finite, it follows that $e(Q)$ is σ-finite relative to $\boldsymbol{M}^\natural$, and therefore so is $e(P)$.

(iii)→(iv): Remark that if $[e(P)]^c$ exists then $[P]^c$ exists, for $[e(P)]^c \cap [P]$ is $[P]^c$. From this, it suffices to prove the existence of $[e(P)]^c$. We may assume that $e(P)=I$. Since I is σ-finite relative to $\boldsymbol{M}^\natural$, it follows from Lemma 1.2.7 of [2] that Z^f is σ-finite relative to $\boldsymbol{M}$, so that $[Z^f]^c$ exists. Thus it remains to show the existence of $[Z^i]^c$. By Lemma 2 there exists a properly infinite, σ-finite projection $Q_1 \in \boldsymbol{M}$ such that $e(Q_1)=Z^i$. For any cyclic class $[Q]$ with $[Q] \leqq [Z^i]$, we have $[Q] \leqq [Q_1]$ by Lemma 1, so that $[Q] \leqq [Q_1]^c$. This implies that $[Q_1]^c$ is $[Z^i]^c$.

(iv)→(i): Let $Q \in [P]^c$, then Q is cyclic and $e(Q) \leqq e(P)$. We can prove that $e(Q)=e(P)$. Because, we may write $P=\sum_\alpha P_\alpha$ where P_α are all cyclic, and then we have $P_\alpha \lesssim Q$, so that $e(P)=\bigcup_\alpha e(P_\alpha) \leqq e(Q) \leqq e(P)$. This completes the proof of the theorem.

We shall define the *length* of a non-zero projection P such that $e(P)$ is σ-finite relative to $\boldsymbol{M}^\natural$, by the least cardinal (not necessarily countable) of non-zero cyclic projections P_α in $\boldsymbol{M}$ such that $\sum_\alpha P_\alpha = P$.

In the case that $\boldsymbol{M}$ is essentially finite and σ-finite, Dye [1, p. 260] defined the length of I by the cyclic dissection. It is easily seen that in this case our definition coincides with that of Dye.

LEMMA 3. *There exists the l. u. b. $[P_0]$ of all cyclic classes. Then, $e(P_0)=I$, and if $e(P)$ is σ-finite relative to $\boldsymbol{M}^\natural$ then $[P]^c=[P_0] \cap [P]$.*

PROOF. We have a decomposition $I=\sum_{\alpha \in \mathfrak{A}} Z_\alpha$, $Z_\alpha \in \boldsymbol{M}_P^\natural$ such that Z_α are all σ-finite relative to $\boldsymbol{M}^\natural$. Since $[Z_\alpha]^c$ exists for all $\alpha \in \mathfrak{A}$ by Theorem 1, there exists a projection $P_0=\sum_{\alpha \in \mathfrak{A}} Q_\alpha$ where $Q_\alpha \in [Z_\alpha]^c$. Then $[P_0]=\bigcup_{\alpha \in \mathfrak{A}} [Z_\alpha]^c$, and if $[Q]$ is cyclic then $[Z_\alpha Q] \leqq [Z_\alpha]^c$ for all $\alpha \in \mathfrak{A}$, which imply $[Q] \leqq [P_0]$. Therefore $[P_0]$ is the l. u. b. of all cyclic classes. It is clear that $e(P_0)=I$. Next, if Z is a central σ-finite projection, then $ZZ_\alpha=0$ except a countable subset $\{\alpha_i\}$ of $\mathfrak{A}$, and hence $ZP_0=\sum_{\alpha_i} ZQ_{\alpha_i}$. Since ZQ_{α_i} are all cyclic and $e(Q_\alpha)$ are mutually orthogonal, ZP_0 is also cyclic. From this, if $e(P)$ is σ-finite relative to $\boldsymbol{M}^\natural$, then $e(P)P_0$ is cyclic, so that $[P_0] \cap [P]$ is cyclic. Hence $[P]^c=[P_0] \cap [P]$.

We remark that $\boldsymbol{M}$ is essentially finite if and only if $[P_0]$ is finite. Because, if $[P_0]$ is finite, then all cyclic classes are finite, and conversely, if all cyclic classes are finite, then Q_α are all finite, and then $P_0=\sum_{\alpha \in \mathfrak{A}} Q_\alpha$ is finite since $e(Q_\alpha)$ are mutually orthogonal. From this fact and Remark

1, it is easily seen that $(Z^f \cup Z^{f'})P_0$ and $(Z^i \cap Z^{i'})P_0$ are the finite part and the properly infinite part of P_0 respectively.

LEMMA 4. *Let $\boldsymbol{M}$ be properly infinite. For any non-zero projection $P \in \boldsymbol{M}$, there exist a non-zero central projection Z and an infinite cardinal n such that $[Z]=n[ZP]$.*

PROOF. By Zorn's Lemma, there exists a maximal orthogonal family $\{P_\alpha ; \alpha \in \mathfrak{A}\}$ such that $P_\alpha \sim P$ for every $\alpha \in \mathfrak{A}$. By the comparability theorem there exists a central projection Z such that $Z(I-\sum_{\alpha\in\mathfrak{A}} P_\alpha) \lesssim ZP$, $Z^\perp(I-\sum_{\alpha\in\mathfrak{A}} P_\alpha) \gtrsim Z^\perp P$. From the maximality of our family, we have $Z \neq 0$. Since $\boldsymbol{M}$ is properly infinite, the cardinal n of $\mathfrak{A}$ is infinite. Hence $n[ZP] \leqq [Z] \leqq (n+1)[ZP]=n[ZP]$. This completes the proof.

LEMMA 5. *Let P, Q be non-zero cyclic projections in $\boldsymbol{M}$. If n, n' are infinite cardinals and $n[P] \geqq n'[Q]$, then $n \geqq n'$.*

PROOF. This lemma can be proved by the same method as in the proof of Lemma 1.2 of [3], and the details are omitted.

In the σ-finite, essentially finite case, Dye [1, p. 261] introduced the concept of "uniform length." We generalize this concept in a general ring $\boldsymbol{M}$ as follows: A non-zero central projection Z in $\boldsymbol{M}$ is said to be *uniformly of length n* (n is a cardinal) if any non-zero σ-finite central projection Z' with $Z' \leqq Z$ has length n.

Now we shall prove the following decomposition theorem for length which includes Theorem 3 of [1] as a special case.

THEOREM 2. *Let n_0 be the cardinal of $\mathfrak{H}$. There exists a unique decomposition $I=\sum_{1\leqq n\leqq n_0} Z_n$ (n: cardinals), $Z_n \in \boldsymbol{M}_P^\natural$ such that if $Z_n \neq 0$, then Z_n is uniformly of length n. Then*
for $n<\aleph_0$, $[Z_n]=(n-1)[Z_nP_0]+[R_n]$, $[R_n] \leqq [Z_nP_0]$, $e(R_n)=Z_n$;
for $n \geqq \aleph_0$, $[Z_n]=n[Z_nP_0]$.

PROOF. If such a decomposition exists, then the uniqueness is obvious. We can reduce the proof of the existence to each of the following three cases: (i) $\boldsymbol{M}$ is finite; (ii) $\boldsymbol{M}$ is properly infinite and P_0 is finite; (iii) $\boldsymbol{M}$ and P_0 are properly infinite.

Case (i): We use the similar technics as in the proof of Theorem 4.1 of [4]. For $1 \leqq n < \aleph_0$ we put

$$S_n = \bigcup (Z \in \boldsymbol{M}_P^\natural ;\ [Z]=(n-1)[ZP_0]+[R],\ e(R)=Z).$$

Then it is easy to show that $S_1=I$, $S_n \geqq S_{n+1}$ and that $[S_n]=(n-1)[S_nP_0]+[R]$, $e(R)=S_n$. Put $Z_n=S_n-S_{n+1}$. Since $\bigcap_n S_n=0$ by Lemma 4.1 of [4], we have $\sum_n Z_n=I$. Clearly $[Z_n]=(n-1)[Z_nP_0]+[R_n]$, $e(R_n)=Z_n$. We shall show that $[R_n] \leqq [Z_nP_0]$. By Theorem 3.2 of [4] there exists $Z \in \boldsymbol{M}_P^\natural$ such that $[ZR_n] \leqq [ZP_0]$, $[Z^\perp R_n]$ $[Z^\perp P_0]$ (this means that $[Z^\perp R_n]=[Z^\perp P_0]+[Q]$, $e(Q)=Z^\perp$). Then we have $Z_n \leqq Z$. Because, if $Z'=Z_nZ^\perp$ is not zero, then $[Z']=n[Z'P_0]+[R]$, $e[R]=Z'$, so that $Z' \leqq S_{n+1}$, which contradicts the fact that $Z' \leqq Z_n$. Hence $Z_nZ^\perp=0$, which implies $Z_n \leqq Z$. Since $[ZR_n] \leqq [ZP_0]$, we have $[R_n]=[Z_nR_n] \leqq [Z_nP_0]$. Note that $Z_1=Z_1P_0$, for $[Z_1]=[R_1] \leqq [Z_1P_0]$.

Let Z be a non-zero central σ-finite projection and $Z \leqq Z_n$. It is clear that the length of $Z \leqq n$. Assume that $Z = \sum_{i=1}^{m} Q_i$ where Q_i are all cyclic and $m \leqq n-1$. Then, since $[Q_i] \leqq [Z]^c \leqq [ZP_0]$, we have $[Z] \leqq (n-1)[ZP_0]$, which contradicts the fact that $Z \leqq Z_n$. Therefore Z has length n.

Case (ii): We may assume that I is σ-finite relative to $\boldsymbol{M}^{\natural}$. Then P_0 is cyclic. For any infinite cardinal n ($\leqq n_0$), we put

$$Z_n = \bigcup (Z \in \boldsymbol{M}_P^{\natural} \; ; \; [Z] = n[ZP_0]) .$$

It is easy to show that $[Z_n] = n[Z_n P_0]$. Z_n are mutually orthogonal, for, if $Z_n Z_{n'} \neq 0$, then $n[Z_n Z_{n'} P_0] = n'[Z_n Z_{n'} P_0]$ and by Lemma 5 we have $n = n'$. Next we have $\sum_{\aleph_0 \leqq n \leqq n_0} Z_n = I$. Because, if $I - \sum_n Z_n \neq 0$, then by Lemma 4 there exists a non-zero central projection Z such that $Z \leqq I - \sum_n Z_n$, $[Z] = n[ZP_0]$ where $\aleph_0 \leqq n \leqq n_0$. This is a contradiction.

Let Z be a non-zero central σ-finite projection and $Z \leqq Z_n$. It is clear that the length of $Z \leqq n$. If $Z = \sum_{\alpha \in \mathfrak{A}} Q_\alpha$ where Q_α are all cyclic, then the cardinal n' of $\mathfrak{A}$ is infinite since Q_α are all finite and Z is properly infinite. We have $[Q_\alpha] \leqq [ZP_0]$ for all $\alpha \in \mathfrak{A}$, so that if $n' \leqq n$ then $n'[ZP_0] \geqq [Z] = n[ZP_0] \geqq n'[ZP_0]$. By Lemma 5 we have $n' = n$ and hence Z has length n.

Case (iii): We determine Z_n, $\aleph_0 \leqq n \leqq n_0$, as in Case (ii), and we have $I = \sum_n Z_n$, $[Z_n] = n[Z_n P_0]$. But, since P_0 is properly infinite, we have $[Z_{\aleph_0}] = \aleph_0 [Z_{\aleph_0} P_0] = [Z_{\aleph_0} P_0]$, which implies that $Z_{\aleph_0}$ is uniformly of length 1. Hence we ought to replace $Z_{\aleph_0}$ by Z_1. Thus we get $I = Z_1 + \sum_{n > \aleph_0} Z_n$.

Next, we shall show that, when $n > \aleph_0$, Z_n is uniformly of length n. Let Z be a non-zero central σ-finite projection and $Z \leqq Z_n$. If $Z = \sum_{\alpha \in \mathfrak{A}} Q_\alpha$ where Q_α are all cyclic, then since Z is not σ-finite and Q_α are all σ-finite, the cardinal of $\mathfrak{A}$ is not countable, so that it is infinite. Therefore, by the same method as in Case (ii) we conclude that Z has length n. This completes the proof of the theorem.

REMARK 2. From the proof of Theorem 2 and Remark 1, we have $Z^f \cap Z^{f'} \leqq \sum_{n < \aleph_0} Z_n$, $Z^f \cap Z^{i'} \leqq Z_1$, $Z^i \cap Z^{f'} \leqq \sum_{n \geqq \aleph_0} Z_n$, $Z^i \cap Z^{i'} \leqq Z_1 + \sum_{n > \aleph_0} Z_n$.

REMARK 3. Griffin [3] defined that a non-zero central projection Z has dimension n, where n is an infinite cardinal, if Z may be a sum of σ-finite central projections Z_α such that $[Z_\alpha] = n[P_\alpha]$ for some cyclic projections P_α ([3], Definition 1.2). In the case that $\boldsymbol{M}$ is properly infinite, from Theorem 2 we have the decomposition $I = Z_1 + \sum_{n \geqq \aleph_0} Z_n$ where $[Z_n] = n[Z_n P_0]$ and $[Z_1] = [Z_1 P_0] = \aleph_0 [Z_1 P_0]$. By the proof of Lemma 3, for any central σ-finite projection Z, ZP_0 is cyclic. Therefore, Z_n has dimension n and Z_1 has dimension $\aleph_0$. Thus in the case of type III, Theorem 2 implies Theorem 1 of [3].

REMARK 4. We consider the special case that $\boldsymbol{M} \supset \boldsymbol{M}'$. Then every cyclic projection in $\boldsymbol{M}$ is irreducible since $\boldsymbol{M}'$ is commutative (see Ogasawara [6], Lemma 2). Therefore it is easily seen that P_0 is a maximal irreducible projection. By Theorem 2 we have $I = \sum_{1 \leqq n \leqq n_0} Z_n$, $[Z_n] = n[Z_n P_0]$

for all n. Thus this decomposition is equal to that of Theorem 2 of Segal [7], for $\boldsymbol{M}_{Z_n P_0 \mathfrak{H}}$ is maximal abelian.

§ 2. l-functions and unitary invariants

Now we shall define the l-function of $\boldsymbol{M}$ which is closely related to the lengths of central σ-finite projections in $\boldsymbol{M}$. Let Ω be the spectre of $\boldsymbol{M}^{\natural}$ and $E(Z)$ be the compact, open subset of Ω which corresponds to $Z \in \boldsymbol{M}_P^{\natural}$. Since $(Z^f \cup Z^{f'})P_0$ is finite, there exists a dimension function d_1 on $\boldsymbol{M}_P$ such that $d_1(P_0, \omega) \equiv 1$ on $E(Z^f \cup Z^{f'})$ (see [4, §5]). Then $d_1(I)$ is finite valued on $E(Z^f)$ except on a non-dense subset. We put $d_1(I, \omega) \equiv \aleph_0$ when $d_1(I, \omega)$ is infinite. We define a function $l(\omega)$ on a open, dense subset of Ω as follows (see Remark 2);

$$
\begin{aligned}
&l(\omega) \equiv d_1(I, \omega) && \text{on} \quad E(Z^f)\,; \\
&l(\omega) \equiv n && \text{on} \quad E(Z_n Z^i) \quad \text{where} \quad n \geqq \aleph_0\,; \\
&l(\omega) \equiv \aleph_0 && \text{on} \quad E(Z_1 Z^i)\,.
\end{aligned}
$$

The range of $l(\omega)$ is a subset of the totally ordered set $\mathfrak{S}$ of non-negative reals and infinite cardinals $\leqq n_0$, n_0 being the cardinal of $\mathfrak{H}$. A topology of $\mathfrak{S}$ is determined by the requirement that the family of all open intervals is a base. Then, $l(\omega)$ is continuous, so that it can be uniquely extended (continuously) on the whole Ω. We shall call this extension the *l-function* of $\boldsymbol{M}$, and denote it by $l(\omega)$ also.

$l(\omega)$ is finite valued ($\geqq 1$) on $E(Z^f)$ except on a non-dense subset and is infinite valued ($\leqq n_0$) on $E(Z^i)$. By Theorem 2, it is not difficult to show that, for any non-zero central σ-finite projection Z with $Z \not\leqq Z_1 Z^i$, the length of Z is equal to the least cardinal n such that $l(\omega) \leqq n$ for all $\omega \in E(Z) - E(Z_1 Z^i)$. If $Z \leqq Z_1 Z^i$, then Z has length 1. Moreover we have the following property of the l-function.

LEMMA 6. *Let $\boldsymbol{M}$ be properly infinite. Then the l-function $l(\omega)$ of $\boldsymbol{M}$ is an algebraic invariant. That is, if $\boldsymbol{M}$ is $*$-isomorphic to $\tilde{\boldsymbol{M}}$, then $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}$ have the same l-function by the identification of the spectres of $\boldsymbol{M}^{\natural}$ and $\tilde{\boldsymbol{M}}^{\natural}$.*

PROOF. We may assume that I is σ-finite relative to $\boldsymbol{M}^{\natural}$. Let $Q_0 \in \aleph_0[P_0]$. Then Q_0 is properly infinite, σ-finite and $e(Q_0)=1$. If Q is properly infinite, σ-finite and $e(Q)=1$, then by Lemma 1 we have $Q \sim Q_0$. This shows that the class $[Q_0] = \aleph_0[P_0]$ is uniquely determined by the algebraic structure of $\boldsymbol{M}$. By the proof of Theorem 2 we get the decomposition $I = \sum_{n \geqq \aleph_0} Z_n$ where $[Z_n] = n[Z_n P_0]$. Since we have $[Z_n] = n[Z_n Q_0]$, this decomposition is purely algebraic. By the definition, $l(\omega) \equiv n$ on $E(Z_n)$. Therefore $l(\omega)$ is an algebraic invariant.

REMARK 5. In the case that $\boldsymbol{M}$ is properly infinite and semi-finite, Pallu de La Barrière introduced the concept of the uniform order as follows ([6, p. 19], Definition 7): $Z\mathfrak{H}(Z \in \boldsymbol{M}_P^{\natural})$ is uniformly of order n ($\geqq \aleph_0$) if there is a finite projection $P \in \boldsymbol{M}$ such that $[Z] = n[P]$. In this case,

by Theorem 2 we have the decomposition $I=\sum_{n\geq\aleph_0} Z_n$ where $[Z_n]=n[Z_nP_0]$. Then $Z_n\mathfrak{H}$ is uniformly of order n. Because, using Lemma 1 it is easy to show that there is a finite projection P such that $\aleph_0[P]=\aleph_0[P_0]$, whence $n[Z_nP]=n[Z_nP_0]=[Z_n]$. Therefore $l(\omega)$ is equal to the algebraic invariant introduced by Pallu de La Barrière ([6, p. 21], Definition 8).

In the sequel, the l-function of $\boldsymbol{M}'$ is denoted by $l'(\omega)$.

In the case that $\boldsymbol{M}$ and $\boldsymbol{M}'$ are finite, let d_0 (resp. d_0') be the dimension function of $\boldsymbol{M}_P$ (resp. $\boldsymbol{M}_P'$) such that $d_0(I)=1$ (resp. $d'(I)=1$). In this case, the unitary invarient $C_1(\omega)$ of $\boldsymbol{M}$ was introduced by the equation $C_1d_0'([\boldsymbol{M}x])=d_0([\boldsymbol{M}'x])$ for every $x\in\mathfrak{H}$. Now we have the following lemma.

LEMMA 7. *Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be finite. Then $l'/l=C_1$.*

PROOF. Let $[P_0]$ (resp. $[P_0']$) be the l. u. b. of all cyclic classes relative to $\boldsymbol{M}$ (resp. $\boldsymbol{M}'$). We shall show that $C_1=d_0(P_0)/d_0'(P_0')$. It suffices to prove the equation when I is σ-finite relative to $\boldsymbol{M}^\natural$. Then, there exist $x, y\in\mathfrak{H}$ such that $P_0\mathfrak{H}=[\boldsymbol{M}'x]$, $P_0'\mathfrak{H}=[\boldsymbol{M}y]$. By the properties of P_0 and P_0' we have $[\boldsymbol{M}'y]\precsim[\boldsymbol{M}'x]$, $[\boldsymbol{M}x]\precsim[\boldsymbol{M}y]$, which imply $[\boldsymbol{M}x]\sim[\boldsymbol{M}y]$. Hence $C_1=d_0([\boldsymbol{M}'x])/d_0'([\boldsymbol{M}x])=d_0(P_0)/d_0'(P_0')$. Next, by the properties of dimension functions we have $d_1(I)/d_1(P_0)=d_0(I)/d_0(P_0)$, which implies $l=1/d_0(P_0)$. Similarly $l'=1/d_0'(P_0')$. Therefore $C_1=l'/l$.

REMARK 6. In the case that $\boldsymbol{M}$ and $\boldsymbol{M}'$ are finite, it follows from Lemma 1.2.8 of [2] that $l\cap l'=1$. Thus we have $d_0(P_0)=1/l=C_1\cap 1$. Moreover, in the general case, we have $l(\omega)\equiv 1$ on $E(Z^f\cap Z^{i'})$, and $l'(\omega)\equiv 1$ on $E(Z^i\cap Z^{f'})$. Hence we have the following result: $\boldsymbol{M}$ is essentially finite if and only if $l\cap l'=1$.

In an arbitrary ring $\boldsymbol{M}$, we define the unitary invariant $C(\omega)$ of $\boldsymbol{M}$ by the quotient $l'(\omega)/l(\omega)$. In the semi-finite case, according to Lemma 7 and Remark 5, C coincides with the unitary invariant in [6]. In the case of type III, according to Remark 3, C is the inverse of the coupling operator in [3].

References

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