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On partial Summability aud Convolutions in the Theory of Vector Valued Distributions

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In his theory of vecter valued distributions ([11], [12]) L. Schwartz introduced the concept of partial summability of a kernel distribution, which makes it possible to give the precise meaning to the Fourier integral written formally by $\int e^{-2\pi i\hat{x}y}\, T(y)\, dy$ in his work [8]. For any tempered distribution T, the distribution $e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y})$ is partially summable with respect to y and the partial integral $\int e^{-2\pi i\hat{x}y}\, T(y)dy$ is the Fourier transform of T. We show in Sec. 1 below that, for a distribution T, if $e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y})$ is partially summable with respect to y, T must be tempered. Consequently the set of distributions T for which $e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y})$ is partially summable with respect to y is exactly $\mathcal{S}'$, the space of tempered distributions. We show that the same is also true for vector valued distributions.

One of the present authors proved [13] that various definitions concerning the convolution of two distributions which are available in the literature are equivalent to each other. The results established there will be generalized for kernel distributions $K(\hat{x}, \hat{y})$ on $R^n \times R^n$ [see Sec. 2]: $K(\hat{x}-\hat{y}, \hat{y})$ is partially summable with respect to y if and only if $\varphi(\hat{x}+\hat{y})\, K(\hat{x}, \hat{y})$ is summable for any $\varphi \in \mathcal{D}$. The convolution $\overset{*}{K}$ is defined by $<\overset{*}{K}, \varphi> = \iint \varphi(x+y)\, K(x, y)\, dxdy$, $\varphi \in \mathcal{D}$ or by $\int K(\hat{x}-y, y)\, dy$. The analogous considerations on $\mathcal{S}'$-convolutions are given. The concept of $\mathcal{S}'$-convolution of two distributions was first introduced in [5] and its further investigation was carried out in [13]. We introduce the space (noted by $\mathcal{O}'_{x,y}$) of kernel distributions for which the convolution is defined. If we $\mathcal{O}'_{x,y}$ take the topology introduced in a natural manner, then $\mathcal{O}'_{x,y}$ will be a permitted, ultra-bornological, complete space of distributions on $R^n \times R^n$.

Finally Sec. 3 is concerned with the convolution defined by starting with the tensor product of vector valued distributions. The results obtained for ordinary distributions [13] will be extended to vector valued distributions, especially we show that $\vec{S}(\hat{x}-\hat{y}) \otimes\otimes_\iota \vec{T}(\hat{y})$ is partially summable with respect to y if and only if $\varphi(\hat{x}+\hat{y})\, (\vec{S}_x \otimes\otimes_\iota \vec{T}_y)$ is summable. We believe that this result will be of use for further investigation of Schwartz' theory of the convolutions of vector valued distributions.

For the most part of this paper we use the notations of L. Schwartz' papers [11], [12] often without any special mention about them.

§ 1. Partial summability of a kernel distribution

Let R^n be an n-dimensional Euclidean space. Let T be a summable distribution on R^n, that is, $T \in \mathcal{D}'_{L^1}$, then T is considered as a continuous linear form on $(\mathcal{B}_c)$. Following L. Schwartz [11] the integral $\int T(x)dx$ is defined as $<T, 1>$, where 1 is a function on R^n identically equal to 1 which belongs to $(\mathcal{B}_c)$.

A bistribution $K(\dot{x}, \dot{y})$ on $R^n \times R^m$ is a continuous linear form on $\mathcal{D}_{x,y}$ (sometimes called kernel distribution or simply kernel), where x and y denote the canonical variables of R^n and R^m respectively, and $\mathcal{D}_{x,y}$ denotes the space of indefinitely differentiable functions with compact supports on $R^n \times R^m$. L. Schwartz' kernel theorem ([7], [11]) states that $\mathcal{D}'_{x,y}$, the space of distributions on $R^n \times R^m$, is canonically isomorphic to $\mathcal{D}'_x \varepsilon \mathcal{D}'_y$. The canonical correspondence of $K(\dot{x}, \dot{y})$ and a continuous linear application $\mathcal{L}_K: \mathcal{D}_x \to \mathcal{D}'_y$ (resp. $\mathcal{L}'_K: \mathcal{D}_y \to \mathcal{D}'_x$) is given by the relations:

$$<\mathcal{L}_K(\varphi), \psi> = <K, \varphi \otimes \psi> = <\mathcal{L}'_K(\psi), \varphi>$$

for every $\varphi \in \mathcal{D}_x$ and for every $\psi \in \mathcal{D}_y$.

$\mathcal{L}_K(\varphi)$ is also denoted by $\varphi \cdot K$ or formally by an integral $\int \varphi(x)K(x,\dot{y})dx$, where $\varphi(\dot{x})K(\dot{x}, \dot{y})$ is the multiplicative product of $\varphi \in \mathcal{D}_x$ and K. Similarly, $\mathcal{L}'_K(\psi)$ is denoted by $K \cdot \psi$ or formally by an integral $\int K(\dot{x}, y)\psi(y)dy$. The precise meaning of the above integrals is given by the following

DEFINITION (L. Schwartz [11]). *A distribution $K(\dot{x}, \dot{y})$ on $R^n \times R^m$ is said to be partially summable with respect to y, if $K(\dot{x}, \dot{y}) \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ or $\varphi \cdot K \in (\mathcal{D}'_{L^1})_y$ for every $\varphi \in \mathcal{D}$. And the partial integral with respect to y $\int_{R^m} K(\dot{x},y)\,dy$ is defined by the relation:*

$$<\int_{R^m} K(\dot{x}, y)\,dy, \varphi(\dot{x})> = \int_{R^m} (\varphi \cdot K)(y)dy$$

for every $\varphi \in \mathcal{D}_x$.

In a similar manner we can define the concept of the partially summability of $K(\dot{x}, \dot{y})$ with respect to x.

Examples. 1°. For any distribution $K(\dot{x}, \dot{y})$ and $\varphi \in \mathcal{D}_x$, the multiplicative product $\varphi(\dot{x})K(\dot{x}, \dot{y})$ is partially summable with respect to x, and the partial integral $\int \varphi(x)K(x, \dot{y})dx$ is precisely $\varphi \cdot K$ ([11], *p.* 91).

2°. For any distribution $K(\dot{x}, \dot{y})$ on $R^n \times R^n$ and any $\varphi \in \mathcal{D}_x$, $\varphi(\dot{x}+\dot{y})K(\dot{x},\dot{y})$

$\in \mathcal{E}'_x(\mathcal{D}'_y) \subset (\mathcal{D}'_{L^1})_x(\mathcal{D}'_y)$. For if we put $K_1(\hat{x}, \hat{y}) = \varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$, $\psi(\hat{y})K_1(\hat{x}, \hat{y}) = \varphi(\hat{x}+\hat{y})\psi(\hat{y})K(\hat{x}, \hat{y})$ for any $\psi \in \mathcal{D}_y$. $\varphi(\hat{x}+\hat{y})\psi(\hat{y})$ is a function of $\mathcal{D}_{x,y}$, therefore $\psi(\hat{y})K_1(\hat{x}, \hat{y}) \in \mathcal{E}'_{xy}$. Hence $K_1 \cdot \psi \in \mathcal{E}'_x$. This implies that $K_1 \in \mathcal{E}'_x(\mathcal{D}'_y)$ and *a fortiori* $K_1 \in (\mathcal{D}'_{L^1})_x(\mathcal{D}'_y)$.

3° For any tempered distribution $K(\hat{x}, \hat{y})$ on $R^n \times R^n$ and any $\varphi \in \mathcal{S}$, $\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y}) \in (\mathcal{O}'_c)_x(\mathcal{D}'_y) \subset (\mathcal{D}'_{L^1})_x(\mathcal{D}'_y)$. This can be proved as in 2°.

L. Schwartz proved ([10], p. 133) that for any $T \in \mathcal{S}'$, the space of tempered distributions on R^n, the multiplicative product $e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y})$ is partially summable with respect to y and its integral $\int e^{-2\pi i\hat{x}y}\, T(y)dy$ coincides with the Fourier-transform $\mathcal{F}(T)$.

Conversely, we show

PROPOSITION 1. *Let T be any distribution on R^n. If the multiplicative product $e^{-2\pi i\hat{x}\hat{y}}T(\hat{y})$ is partially summable with respect to y, then $T \in \mathcal{S}'$.*

PROOF. From our hypothesis, we have

$\int_{R^n} \varphi(x)e^{-2\pi ix\hat{y}}\, T(\hat{y})\,dx = \mathcal{F}(\varphi)(\hat{y})T(\hat{y}) \in \mathcal{D}'_{L^1}$ for any $\varphi \in \mathcal{D}$. The application $\mathcal{L}: \varphi \to \mathcal{F}(\varphi)T$ of $\mathcal{D}$ to $\mathcal{D}'_{L^1}$ is continuous by Theorem 2 of [14], since the application $\varphi \to \mathcal{F}(\varphi)T$ is continuous from $\mathcal{D}$ to $\mathcal{D}'$. Let Ω be a relatively compact open subset of R^n containing the origin of R^n. Since the space $\mathcal{D}_{\bar{\Omega}}$ is of type (**F**) and since the space $\mathcal{D}'_{L^1}$ is of type (**DF**), the restriction $\mathcal{L} \mid \mathcal{D}_{\bar{\Omega}}$ is a bounded application ([11], p. 62) i.e. it transforms a suitable neighbourhood of zero in $\mathcal{D}_{\bar{\Omega}}$ into a bounded subset A in $\mathcal{D}'_{L^1}$. We may assume that A is an absolutely convex closed bounded set. Then the application of $\mathcal{D}_{\bar{\Omega}}$ to $(\mathcal{D}'_{L^1})_A$, the subspace of $\mathcal{D}'_{L^1}$ generated by A, is continuous in the topology induced by $\mathcal{D}^m_{\bar{\Omega}}$ for some positive integer m. We can take a positive integer p such that a $u \in \mathcal{D}^m_{\bar{\Omega}}$ is a parametrix of an iterated Laplacian Δ^p ([8], p. 47):

$$\delta = \Delta^p u + \eta, \quad \eta \in \mathcal{D}_{\bar{\Omega}}. \tag{1}$$

Since $u \in \mathcal{D}^m_{\bar{\Omega}}$, we can choose a sequence $\{u_j\}$ such that $u_j \in \mathcal{D}_{\bar{\Omega}}$ and $u_j \to u$ in $\mathcal{D}^m_{\bar{\Omega}}$ as $i \to \infty$. Hence $\mathcal{F}(u)T \in \mathcal{D}'_{L^1}$. From (1) we have

$$T = (-4\pi^2)^p r^{2p} \mathcal{F}(u)T + \mathcal{F}(\eta)T, \tag{2}$$

where r denotes the length of x. Since $\mathcal{F}(u)T$, $\mathcal{F}(\eta)T \in \mathcal{D}'_{L^1} \subset \mathcal{S}'$ and $r^{2p} \in \mathcal{O}_M$, it follows from (2) that $T \in \mathcal{S}'$. This completes the proof.

REMARK 1°. If for a fixed x, $e^{-2\pi ix\hat{y}}\, T(\hat{y}) \in (\mathcal{D}'_{L^1})_y$, then $T \in \mathcal{D}'_{L^1}$ and the application $x \to e^{-2\pi ix\hat{y}}\, T(\hat{y})$ of R^n into $\mathcal{D}'_{L^1}$ is continuous. In this case $\mathcal{F}(T)$ is a continuous function and for each $x \in R^n$

$$\mathcal{F}(T)(x) = \int_{R^n} e^{-2\pi ixy}\, T(y)dy.$$

2°. We can show that an analogous statement of Prop. 1 holds for a kernal $K(\hat{x}, \hat{y})=2\pi J_{n/2-1}(2\pi|\hat{x}||\hat{y}|)/(S_{n-1}|\hat{x}|^{n/2-1}|\hat{y}|^{n/2-1})$, which is the kernel of the Fourier transformation of tempered distributions invariant under rotation. Here $J_{n/2-1}$ denotes a Bessel function of order $\frac{n}{2}-1$ and S_{n-1} the surface area of unit sphere of R^n. To begin with, we shall give a brief account of distributions invariant under rotation. Let $O(n)$ be the Lie group of rotations ρ around the origin of R^n. For any distribution T, ρT stands for the distribution defined by the relation:

$$<\rho T, \varphi>=<T, \rho^{-1}\varphi> \quad \text{for every } \varphi \in \mathcal{D}.$$

We put $T^\natural=\int \rho T d\rho$, where $d\rho$ denotes the invariant measure of $O(n)$ with total mass 1. $T^\natural$ is called the spherical mean of T. Since the application $\rho \rightarrow \rho T$ of $O(n)$ into $\mathcal{D}'$ is continuous, $T^\natural$ is a distribution. It is easy to verify that $T^\natural$ is tempered if T is also. We call T to be invariant under rotation if $T=T^\natural$. When T is a function f, then $T^\natural$ is also expressed as

$$\frac{1}{S_{n-1}}\int f(r\sigma)d\sigma$$

where $d\sigma$ denotes the volume element of the unit sphere of R^n. (cf. [10]. exposé 7). As for the convolution, if S is invariant under rotation, it is not difficult to see that

$$(S*T)^\natural=S*T^\natural,$$

so that $S*T$ is invariant under rotation if both S and T are also.

Let φ be any element of $\mathcal{S}$. It follows from the expression of $\varphi^\natural$ that $\varphi^\natural \in \mathcal{S}$ and $\varphi \rightarrow \varphi^\natural$ is a continuous endomorphism of $\mathcal{S}$. Hence, for any element T of $\mathcal{S}'$ the relation $<T^\natural, \varphi>=<T, \varphi^\natural>$ shows us that $T^\natural \in \mathcal{S}'$ and that the application $\natural$ is a continuous endomorphism of $\mathcal{S}'$. We can also verify that $\mathcal{F}(T)^\natural=\mathcal{F}(T^\natural)$ since this equation holds by direct calculations for any $\varphi \in \mathcal{S}$ and the application $\natural$ is continuous as just mentioned. The same is also true for the inverse Fourier transformation, so that $\mathcal{F}(T)$ is invariant under rotation if and only if T is also.

Now we consider the endomorphism $\natural \otimes I$ of $\mathcal{S}'_x((\mathcal{D}'_{L^1})_y)$. For any $T \in \mathcal{S}'$, we know that $e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y}) \in \mathcal{S}'_x((\mathcal{D}'_{L^1})_y)$. Then

$$(3) \qquad (\natural \otimes I)(e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y}))=K(\hat{x}, \hat{y})T(\hat{y}),$$

where $K(\hat{x}, \hat{y})$ is the kernel stated in the beginning.
Indeed, for any T in $\mathcal{S}'$, the relation (3) is a direct consequence of calculations and $T(\hat{y}) \rightarrow e^{-2\pi i\hat{x}\hat{y}}\, T(\hat{y})$ is continuous from $\mathcal{S}'$ to $\mathcal{S}'_x((\mathcal{D}'_{L^1})_y)$, so that (3) holds for any $T \in \mathcal{S}'$. Therefore, for any $T \in \mathcal{S}'$

$$K(\hat{x}, \hat{y})T(\hat{y}) \in \mathcal{S}'_x((\mathcal{D}'_{L^1})_y)$$

and

$$\mathscr{F}(T)^{\natural}(\dot{x})=\int_{R^n} K(\dot{x}, y)T(y)dy.$$

Conversely, for a given distribution T, we assume that $K(\dot{x}, \dot{y})T(\dot{y})$ is partially summable with respect to y. We shall show that $T \in \mathscr{S}'$. We put $K_1(\dot{x}, \dot{y})=K(\dot{x}, \dot{y})T(\dot{y})$, then, for any $\varphi \in \mathscr{D}$

$$\varphi \cdot K_1=(\varphi \cdot K)T$$

and

$$\varphi \cdot K=\mathscr{F}(\varphi)^{\natural}=\mathscr{F}(\varphi^{\natural}).$$

While in the Prop. 1, a parametrix u of iterated Laplacian Δ^p can be taken as invariant under rotation. Then we can go along the same line as in the proof of Prop. 1 to conclude that $T \in \mathscr{S}'$.

Before stating the next proposition we shall give a short discussion on the notation $\delta(\dot{z}-\dot{x})K(\dot{x}, \dot{y})$, where δ denotes Dirac mesure on R^n and $K(\dot{x}, \dot{y})$ denotes the distribution on $R^n \times R^m$. This is the image of the tensor $\delta(\dot{\zeta})\otimes K(\dot{\xi}, \dot{\eta})$ by change of variables: $\zeta=z-x$, $\xi=x$, $\eta=y$. Hence $\delta(\dot{z}-\dot{x})K(\dot{x}, \dot{y}) \in \mathscr{D}'_z(\mathscr{D}'_{x,y})$. It is to be noted that

$$(4) \qquad \varphi(\dot{z}) \cdot \delta(\dot{z}-\dot{x})K(\dot{x}, \dot{y})=\varphi(\dot{x})K(\dot{x}, \dot{y}) \quad \text{for every } \varphi \in \mathscr{D}_z.$$

In fact, for every $\psi \in \mathscr{D}_{x,y}$, we have

$$\begin{aligned}
&<\varphi(\dot{z}) \cdot \delta(\dot{z}-\dot{x})K(\dot{x}, \dot{y}), \psi(\dot{x}, \dot{y})> \\
&=\iiint_{R^n\times R^m\times R^n} \delta(z-x)K(x, y)\varphi(z)\psi(x, y)dxdydz \\
&=\iiint_{R^n\times R^m\times R^n} (\delta(\zeta)\otimes K(\xi, \eta))\varphi(\zeta+\xi)\psi(\xi, \eta)d\xi d\eta d\zeta \\
&=\iint_{R^n\times R^m} K(\xi, \eta)\psi(\xi, \eta) \left\{ \int_{R^n} \delta(\zeta)\varphi(\zeta+\xi)d\zeta\right\} d\xi d\eta \\
&=\iint_{R^n\times R^m} K(\xi, \eta)\psi(\xi, \eta)\varphi(\xi)d\xi d\eta \\
&=<\varphi(\dot{x})K(\dot{x}, \dot{y}), \psi(\dot{x}, \dot{y})>.
\end{aligned}$$

Hence we have (4).

PROPOSITION 2. *For any distribution $K(\dot{x}, \dot{y})$ on $R^n \times R^m$, the following conditions are equivalent:*

$$(1) \qquad K(\dot{x}, \dot{y}) \in \mathscr{D}'_x((\mathscr{D}'_{L^1})_y),$$

$$(2) \qquad \delta(\dot{z}-\dot{x})K(\dot{x}, \dot{y}) \in \mathscr{D}'_z((\mathscr{D}'_{L^1})_{x,\, y})$$

And, if $K(\dot{x}, \dot{y})$ satisfies (1) *or* (2), *then we have*

$$\int_{R^m} K(\hat{z}, y)dy = \iint_{R^n \times R^m} \delta(\hat{z}-x)K(x, y)dxdy.$$

PROOF. By definition, (1) and (2) are equivalent to

(1′) $\int \varphi(x)K(x, \hat{y})dx \in (\mathcal{D}'_{L^1})_y$ for every $\varphi \in \mathcal{D}_x$

and

(2′) $\varphi(\hat{x})K(\hat{x}, \hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}$ for every $\varphi \in \mathcal{D}_x$

respectively.

(1)→(2′). (1) implies that $\varphi(\hat{x})K(\hat{x}, \hat{y}) \in \mathcal{E}'_x((\mathcal{D}'_{L^1})_y)$ for every $\varphi \in \mathcal{D}_x$. Then it suffices to see that $\mathcal{E}'_x((\mathcal{D}'_{L^1})_y) \subset (\mathcal{D}'_{L^1})_{x,y}$. Since $\mathcal{E}'_x$ is nuclear and the injection $\mathcal{E}'_x \to (\mathcal{D}'_{L^1})_x$ is continuous, this is obvious.

(2′)→(1′). Fubini's theorem [11] shows us that (2′) implies (1′). Finally, for any $\varphi \in \mathcal{D}_z$ we have

$$\begin{aligned} &< \varphi(\hat{z}), \iint_{R^n \times R^m} \delta(\hat{z}-x)K(x, y)dxdy > \\ &= \iint_{R^n \times R^m} \varphi(\hat{z}) \cdot \delta(\hat{z}-x)K(x, y)dxdy \\ &= \iint_{R^n \times R^m} \varphi(x)K(x, y)dxdy \\ &= < \varphi(\hat{z}), \int_{R^m} K(\hat{z}, y)dy > \end{aligned}$$

Hence $\int_{R^m} K(\hat{z}, y)dy = \iint_{R^n \times R^m} \delta(\hat{z}-x)K(x, y)dxdy$. This completes the proof.

Now we turn to the investigations of the space $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$. We begin with the following

LEMMA 1. *Let E be a permitted barrelled space of distributions with admissible (normal in Schwartz' terminologies) strong dual E'. Then E' is also a permitted barrelled space of distributions.*

PROOF. Let $\{\alpha_k\}_{k=1,2,3,\dots}$ be any sequence of multiplicators and $\{\rho_k\}_{k=1,2,3,\dots}$ any sequence of regularizations. That E is permitted means that $(\alpha_k e)*\rho_k \to e$ in E (resp. $\alpha_k(e*\rho_k) \to e$ in E) for every e as $k \to \infty$. Since E is a barrelled permitted space and $\mathcal{D}$ is dense in E', it follows from Lemma 1 of Y. Hirata [4] that E' is permitted. Then it suffices to show that E is distinguished, that is, any $\sigma(E'', E')$-bounded subset of E'' is contained in the $\sigma(E'', E')$-closure of a bounded subset of E. Let B be any $\sigma(E'', E')$-bounded subset of E''. B is also a bounded subset of E'' since E' is quasi-complete as a dual of a barrelled space. Set

$$A = \bigcup_{k=1}^{\infty} \alpha_k (B * \rho_k).$$

A is considered as a subset of E. We show A is bounded in E. For this end it suffices to see that A is $\sigma(E, E')$-bounded. Let e' be any element of E'. As $(\alpha_k e')*\rho_k \to e'$ in E', $\{(\alpha_k e')*\rho_k\}$ is a bounded subset of E'. Now since B is bounded in E'' and since

$$\langle A, e'\rangle = \bigcup_{k=1}^{\infty} \langle B, (\alpha_k e')*\check{\rho}_k\rangle,$$

it follows that $\langle A, e'\rangle$ is bounded for every $e' \in E'$, so that A is bounded. It is not difficult to see that B is contained in the $\sigma(E'', E')$-closure of B. This completes the proof.

As an exemple, $\dot{\mathcal{B}}$, the space of indefinitely differentiable functions tending to zero at infinity together with derivatives of every order, satisfies the condition of the lemma, so that its strong dual is barrelled, therefore bornological since $\mathcal{D}'_{L^1}$ is the strong dual of a space of type (**F**) [11]. This is also concluded from the fact that $\dot{\mathcal{B}}$ is a quasi-normable space of type (**F**), the proof of which is carried out by the verification of a criterion due to Grothendieck ([2], p. 107) concerning the quasi-normability of a locally convex space and is not so difficult, so that the proof is omitted.

Now we show

PROPOSITION 3. (1) *$\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is the strong dual of $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$ and is a permitted, ultra-bornological complete space of distributions.*

(2) *The strong dual of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is $\bar{\mathcal{D}}_x((\dot{\mathcal{B}})_y)$.*

PROOF. Ad (1). Owing to a result of L. Schwartz [12; p. 104], it follows since $\dot{\mathcal{B}}$ is a space of type (**F**) that the strong dual of $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$ is $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ which is the strict inductive limit of an increasing sequence of space $\mathcal{D}_{x,B_p}(\dot{\mathcal{B}})$ of type (**F**), where B_p stands for the ball with radius p and center at O in R^n. $\mathcal{D}_{x,B_p}(\dot{\mathcal{B}}_y)$ is a topological subspace of $\dot{\mathcal{B}}_{x,y}$ such that the supports of functions in $\mathcal{D}_{x,B_p}(\dot{\mathcal{B}}_y)$ are contained in $B_p \times R^m$. The duality between $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ and $\mathcal{D}_x(\dot{\mathcal{B}}_y)$ is given by

$$\langle T, \varphi\rangle = \iint_{R^{n\times m}} T(x, y)\, \varphi(x, y) dx dy,$$

$T \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ and $\varphi \in \bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$.

Since every space of type (**LF**) is barrelled, we see that $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$ is barrelled. Moreover it is permitted. This is immediately verified by direct calculations. And it is easy to see that $\mathcal{D}_x \otimes \mathcal{D}_y$ is dense in the dual $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$. Hence we can apply Lemma 1 to $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$ to conclude that $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is permitted. The dual of any distinguished space of type (**LF**) in the strict sense, i. e. in the sense of Dieudonné and Schwartz [1] is ultra-bornogical [3]. It follows that $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is a permitted ultra-bornological space of distributions.

Ad (2). Let E be the strong dual of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$. E consists of all the bilinear continuous forms β on $\mathcal{D}'_x \times (\mathcal{D}'_{L^1})_y$. The associated linear application $\tilde{\beta}$: $(\mathcal{D}'_{L^1})_y \to \mathcal{D}_x$ defined by

$$<\tilde{\beta}(T), S> = \beta(S, T)$$

for every $S \in \mathcal{D}'_x$ and $T \in (\mathcal{D}'_{L^1})_y$ is bounded. We can put for each $\tilde{\beta}$

$$\text{(i)} \qquad \tilde{\beta}(T)(x) = \int_{R^m} T(y)\beta(x, y)dy,$$

where $\beta(x, y)$ is a member of $\mathcal{B}_y$ for every fixed x. $\beta(x, y)$ is also defined by

$$\text{(ii)} \qquad \beta(x, y) = \beta(\delta_x, \delta_y)$$

where δ_x and δ_y denote the point measures located at x and y respectively. $\beta(x, y)$ vanishes outside a $B_p \times R^m$. Since

$$\text{(iii)} \qquad D_x^p D_y^q \beta(x, y) = \beta((-1)^{|p|} D_x^p \delta_x, (-1)^{|q|} D_y^q \delta_y),$$

it follows by usual reasonings that the function β is a member of $\bar{\mathcal{D}}_x(\mathcal{B}_y)$. Conversely any $\beta \in \bar{\mathcal{D}}_x(\mathcal{B}_y)$ defines an element of E. This is almost clear from (i). Therefore E is algebraically $\bar{\mathcal{D}}_x(\mathcal{B}_y)$.

We shall compare the topologies of E and $\bar{\mathcal{D}}_x(\mathcal{B}_y)$. The latter is, by definition, the strict inductive limit of an increasing sequence of spaces $\mathcal{D}'_{x,B_p}(\mathcal{B}_y)$ of type $(\boldsymbol{F})$. That every bounded subset of $\bar{\mathcal{D}}_x(\mathcal{B}_y)$ is a bounded subset of E is clear from (i). Hence the injection $\bar{\mathcal{D}}_x(\mathcal{B}_y) \to E$ is continuous since $\bar{\mathcal{D}}_x(\mathcal{B}_y)$ is bornological. On the other hand, a fundamental system of neighbourhoods of 0 in E is obtained by taking the $\sigma(E, \mathcal{D}'_x((\mathcal{D}'_{L^1})_y))$-closure of that of $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$. A fundamental system of neighourhood of 0 in $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$ is given by the family of the subsets $U(\{\varepsilon_k\}, \{m_k\})$ of $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$ defined by the following conditions ([9], p. 95):

$$(\alpha) \qquad \varepsilon_k \downarrow 0 \text{ and } m_k \uparrow \infty \text{ as } k \to \infty.$$

$$(\beta) \qquad \varphi \in U(\{\varepsilon_k\})\{m_k\}) \text{ if and only if}$$

$$\sup_{|p| \leqq m_k} \sup_{x \notin B_k} |D_{x,y}^p \varphi(x, y)| \leqq \varepsilon_k.$$

Then the $\sigma(E, \mathcal{D}'_x((\mathcal{D}'_{L^1})_y))$-closure of a $U(\{\varepsilon_k\}, \{m_k\})$ is contained in the set defined similarly for $\bar{\mathcal{D}}_x(\mathcal{B}_y)$. This is clear from (ii) and (iii). Hence the injection $E \to \bar{\mathcal{D}}_x(\mathcal{B}_y)$ is continuous. Therefore E is topologically $\bar{\mathcal{D}}_x(\mathcal{B}_y)$. Thus we see that the strong dual of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is $\bar{\mathcal{D}}_x(\mathcal{B}_y)$. This completes the proof.

Since $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is barrelled, it is easy to see that $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ has the approximation properties by truncature and regularization ([11], p. 8 Remarks).

PROPOSITION 4. *The application* $K(\hat{x}, \hat{y}) \to \delta(\hat{z} - \hat{x})K(\hat{x}, \hat{y})$ *of* $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ *into* $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$ *is monomorphic.*

PROOF. The application is injective. In fact, suppose $\delta(\hat{z}-\hat{x})K(\hat{x}, \hat{y})=0$. Then $\delta(\hat{z})\otimes K(\hat{x}, \hat{y})=0$. Hence $K(\hat{x}, \hat{y})=0$. $\{K\}\to 0$ in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ if and only if, for any bounded subset A of $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$, $\iint K(x, y)\varphi(x, y)dxdy\to 0$ uniformly when φ runs through A. $\{\delta(\hat{z}-\hat{x})K(\hat{x}, \hat{y})\}\to 0$ in $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$ if and only if, for any bounded subset B of $\mathcal{D}_x$ and any bounded subset C of $\dot{\mathcal{B}}_{x,y}$, $\{\iint K(x, y)\alpha(x)\beta(x, y)dxdy\}\to 0$ uniformly when α, β run through B and C respectively. Therefore if we can show that the sets BC form the set of bounded subsets of $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$, the proof will be completed. It is clear that any BC is bounded in $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$. Conversely, any A is a bounded subset of $\dot{\mathcal{B}}_{x,y}$, with supports contained in a $B_p\times R^m$. If we take $\alpha\in\mathcal{D}_x$ such that $\alpha(x)=1$ on a neighbourhood of B_p, then $A=\alpha A$, and therefore A is a set BC, as desired.

By making use of the category theorem of Baire we show

LEMMA 2. *Let E and F be spaces of type* (**F**), *and let G be a space of type* (**DF**). *Let $\{u_\alpha\}_{\alpha\in A}$ be a set of separately continuous bilinear applications of $E\times F$ into G. If, for every $x\in E$ and $y\in F$, the set $\bigcup_{\alpha\in A} u_\alpha(x, y)$ is bounded, then $\{u_\alpha\}_{\alpha\in A}$ is equibounded.*

PROOF. Let $\{B_k\}$ be a fundamental sequence of bounded subsets of G. We may take B_k to be a bounded absolutely convex closed subset of G. For every $x\in E$, we put $F_k(x)=\{y;\ u_\alpha(x, y)\in B_k$ for every $\alpha\in A\}$. Then $F_k(x)$ is an absolutely closed convex subset of F and $F=\bigcup_k F_k(x)$ since $\bigcup_\alpha u_\alpha(x, y)$ is contained in some B_k. Owing to the category theorem of Baire we see that there exists an $F_k(x)$ such that $F_k(x)$ is a neighbourhood of zero of F. Now let $\{V_n\}$ be a fundamental system of neighbourhoods of zero in F. We may assume that $V_n\supset V_{n+1}$. If we put $E_k=\{x;\ u_\alpha(x, V_k)\subset B_k\}$, then E_k is an absolutely convex closed subset of E and $E=\bigcup_k E_k$. Then as before we can apply the category theorem to conclude that some E_k is a neighbourhood of zero in E. Hence there exists a neighbourhood U(resp. V) of zero in E (resp. F) such that $\bigcup_{\alpha\in A} u_\alpha(U, V)$ is contained in a B_k. This completes the proof.

PROPOSITION 5. *For any subset A of $\mathcal{D}'_{x,y}$, the following properties are equivalent to each other:*

(1) *A is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$,*

(2) *$A*\psi$ is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ for every $\psi\in\mathcal{D}_{x,y}$,*

(3) *$\varphi(A*\psi)$ is relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$ for every $\psi\in\mathcal{D}_{x,y}$ and $\varphi\in\mathcal{D}_x$,*

(4) *φA is relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$ for every $\varphi\in\mathcal{D}_x$.*

PROOF. Ad (1)→(2). The endomorphism $T\to T*\psi$ of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is continuous for every $\psi\in\mathcal{D}_{x,y}$([14], Theorem 2). Hence $A*\psi$ is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ for every $\psi \in \mathcal{D}_{x,y}$.

Ad (2)→(3). Since for any $\varphi \in \mathcal{D}_x$, the application $T \to \varphi T$ of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ to $(\mathcal{D}'_{L^1})_{x,y}$ is continuous ([14], Theorem 2), $\varphi(A*\psi)$ is also relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$ for every $\varphi \in \mathcal{D}_x$ and $\psi \in \mathcal{D}_{x,y}$.

Ad (3)→(4). Consider the application $\mathcal{L}_T$: $(\varphi, \psi) \to \varphi(T*\psi)$ of $(\mathcal{D}_H)_x \times (\mathcal{D}_K)_{x,y}$ to $(\mathcal{D}'_{L^1})_{x,y}$, where H (resp. K) is the unit ball with the center at zero in R^n(resp. $R^n \times R^m$). This application is separately continuous [14; Theorem 2]. $(\mathcal{D}_H)$, $(\mathcal{D}_K)_{x,y}$ are spaces of type (**F**) and $(\mathcal{D}'_{L^1})_{x,y}$ is a space of type (**DF**). We can now apply Lemma 2 to conclude that the applications $\{\mathcal{L}_T\}_{T \in A}$: $(\mathcal{D}_H)_x \times (\mathcal{D}_K)_{x,y} \to ((\mathcal{D}'_{L^1})_{x,y})_B$ are equicontinuous in the topology induced by $(\mathcal{D}_H)_x \times (\mathcal{D}^m_K)_{x,y}$ for some positive integer m, where $((\mathcal{D}'_{L^1})_{x,y})_B$ is the subspace of $(\mathcal{D}'_{L^1})_{x,y}$ generated by B. $(\mathcal{D}'_{L^1})$ satisfies the strict Mackey's condition ([2], p. 103) since $\dot{\mathcal{B}}_{x,y}$ is quasi-normable, hence we may assume that the set $\varphi(A*\psi)$ is relatively compact in $((\mathcal{D}'_{L^1})_{x,y})_B$ for every $\varphi \in (\mathcal{D}_H)_x$ and $\psi \in (\mathcal{D}_K)_{x,y}$.

Choose a parametrix $u \in (\mathcal{D}^m_K)_{x,y}$ for an iterated Laplacian Δ^p. Then we have $\varphi T = \varphi(T*\Delta^p u) + \varphi(T*\eta)$, where $\eta \in (\mathcal{D}_K)_{x,y}$. Since $u \in (\mathcal{D}^m_K)_{x,y}$, we can choose a sequence $\{u_j\}$ such that $u_j \in (\mathcal{D}_K)_{x,y}$ and $u_j \to u$ in $(\mathcal{D}^m_K)$. Then $\varphi(A*u_j)$ and $\varphi(A*\eta)$ are relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$. $\varphi(A*u)$ is also relatively compact because $\varphi(T*u_j) \to \varphi(T*u)$ uniformly when T runs through A. Since $\varphi\left(T*\frac{\partial}{\partial x_i}u\right) = \frac{\partial}{\partial x_i}\{\varphi(T*u)\} - \left(\frac{\partial}{\partial x_i}\varphi\right)(T*u)$, it follows that $\varphi\left(A*\frac{\partial}{\partial x_i}u\right)$ is relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$. Repeating this process, we can see that $\varphi(A*\Delta^p u)$ is relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$. Therefore φA is relatively compact in $(\mathcal{D}'_{L^1})_{x,y}$.

Ad (4)→(1). If T runs through A, then $\dot{\varphi} \cdot T = \int \varphi\, T\, dx$ falls in a bounded set of $(\mathcal{D}'_{L^1})_y$ for any $\varphi \in \mathcal{D}_x$. This means that A is bounded in $\mathcal{L}(\mathcal{D}_x; (\mathcal{D}'_{L^1})_y)$ in the topology of simple convergence, hence A is a bounded subset of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$. The application $(\varphi, T) \in \mathcal{E}_x \times \mathcal{D}'_x((\mathcal{D}'_{L^1})_y) \to \varphi T \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ is hypocontinuous, since it is separately continuous and $\mathcal{E}_x$, $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ are barrelled. Hence for any sequence of multiplicators $\{\alpha_n\}$, $\{\alpha_n T\}$ converges to T as $n \to \infty$. Since $\alpha_n \in \mathcal{D}_x$, each $\alpha_n A$ is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$, therefore A is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_L)_y)$. This completes the proof.

As a special case of Prop. 5 we mention the following

COROLLARY. *Let T be any distribution on $R^n \times R^m$. If $T*\psi \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ for every $\psi \in \mathcal{D}_{x,y}$ then also $T \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$.*

Let E be a locally convex Hausdorff topological vector space. A linear continuous application $\vec{T}$ of $\mathcal{D}$ into E is defined to be an E-valued distribution or a distribution with values in E. We denote by $\mathcal{D}'(E)$ the space of E-valued distributions. On $\mathcal{D}'(E)$ we put the topology of uniform convergence on bounded sets of $\mathcal{D}$. If u: $E'_c \to \mathcal{D}'$ is a continuous linear application, it is the transpose of a uniquely determined E-valued distribution $\vec{T}$, i.e., $u = {}^t\vec{T}$. Let $\mathcal{H}$ be a space of distributions. The space $\mathcal{H}(E)$ consists of all E-valued distributions $\vec{T}$ which have the following property, ${}^t\vec{T}: E'_c \to \mathcal{D}'$ maps actually E'_c into $\mathcal{H}$ and is a continuous application of E'_c into $\mathcal{H}$. $\mathcal{H}(E)$ takes the topology of uniform convergence on equicontinuous subset of E'_c. Namely $\mathcal{H}(E)=\mathcal{L}_\varepsilon(E'_c, \mathcal{H})$. It is well known that $\mathcal{L}_\varepsilon(E'_c, \mathcal{H}) \approx \mathcal{L}_\varepsilon(\mathcal{H}'_c, E)$. According to L. Schwartz ([11], p. 130) we state the following

Definition. *A distribution $\vec{K} \in \mathcal{D}'_{x,y}(E)$ is said to be partially summable with respect to y if $\vec{K} \in (\mathcal{D}'_x(\mathcal{D}'_{L^1})_y)(E) = \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(E))$. The partial integral, noted by $\int \vec{K}(\dot{x}, y)\, dy$, is the image of $\vec{K}$ by the continuous application $I_x \otimes \int_{R^n} \otimes I_E$ of $(\mathcal{D}'_x(\mathcal{D}'_{L^1})_y)(E)$ into $\mathcal{D}'_x(E)$, where I_x and I_E are the identities of $\mathcal{D}'_x$ and E respectrively.*

L. Schwartz ([11], p. 133) defined the Fourier transform of vector valued tempered distribution $\vec{T} \in \mathcal{S}'_x(E)$: $e^{-2\pi i \dot{x}\dot{\xi}}\vec{T}(\dot{x})$ is partially summable with respect to x and

$$<\int e^{-2\pi i x \dot{\xi}}\, \vec{T}(x)dx,\, \vec{e}'> = \mathcal{F}(<\vec{T}, \vec{e}'>) \text{ for every } \vec{e}' \in E'.$$

We can show that if E is quasi-complete and $e^{-2\pi i \dot{x}\dot{\xi}}\, \vec{T}(\dot{x})$ is partially summable with respect to x (more generally if $e^{-2\pi i \dot{x}\dot{\xi}} <\vec{T}(x), \vec{e}'>$ is partially summable with respect to x for every $\vec{e}' \in E'$), then $\vec{T} \in \mathcal{S}'(E)$. The proof may be carried out with obvious modifications along a similar line as in Prop. 1.

Finally we conclude this section with the following

Proposition 6. *Let E be a quasi-complete locally convex Hausdorff topological vector space. For any E-valued distribution $\vec{K}$ on $R^n \times R^m$, $\vec{K}(\dot{x}, \dot{y})$ is partially summable with respect to y if and only if $\vec{K} * \psi$ is also for every $\psi \in \mathcal{D}_{x,y}$.*

Proof. The "only if" part is obvious since the application $T \to T * \psi$ of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ into itself is continuous. To prove the "if" part, we first assume that E is complete. Any linear application of E'_c into a second locally convex space F is continuous if the application is continuous on any equicontinuous subset of E'_c ([11], p. 41). Hence it suffices to show that $\vec{e}' \to \langle \vec{K}, \vec{e}' \rangle$ is continuous from any equicontinuons subset of E'_c into $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$. Let A be any absolutely convex equicontinuous subset of E'_c. Since $\vec{K} * \psi \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(E))$ for everv $\psi \in \mathcal{D}_{x,y}$, so the set $<\vec{K}, A>*\psi$ is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$ for every $\psi \in \mathcal{D}_{x,y}$. It follows from Prop. 5 that $<\vec{K}, A>$ is relatively compact in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y)$. Let $\vec{e}' \to 0$ in A. If T is any limiting element of $\{<\vec{K}, \vec{e}'>\}$, whose existence is assured by the relative compactness of $<\vec{K}, A>$, then $T*\psi$ is a limiting element of $\{<\vec{K}, \vec{e}'>*\psi\}$ which converges to zero since $\vec{e}' \to <\vec{K}, \vec{e}'>*\psi$ is continuous. Henec $T*\psi=0$ for any $\psi \in \mathcal{D}_{x,y}$. This implies in turn that $T=0$. Therefore $<\vec{K}, \vec{e}'> \to 0$ as $\vec{e}' \to 0$ in A.

Next we consider the general case. Let $\hat{E}$ be the completion of E. $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y(E))$ is a topological subspace of $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y(\hat{E}))$. By the result established just above, $\vec{K} \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(\hat{E}))$. Let $\{\rho_i\}$ be any sequence of regularizations. $\vec{K}*\rho_i\to\vec{K}$ in $\mathcal{D}'_x((\mathcal{D}'_{L^1})_y(\hat{E}))$ as $i\to\infty$. On the other hand, for every i, $\vec{K}*\rho_i\in\mathcal{D}'_x((\mathcal{D}'_{L^1})_y(E))$, the quasi-complete space. Henec we have $\vec{K}\in\mathcal{D}'_x((\mathcal{D}'_{L^1})(E))$, completing the proof.

If E is a scalar field, Prop. 6 tells us that a subset A of $\mathcal{D}'_{L^1}$ is relatively compact if and only if $A*\varphi$ is relatively compact in L^1 (resp. $\mathcal{D}'_{L^1}$, $\mathcal{D}_{L^1}$) for any $\varphi\in\mathcal{D}$. In particular, it follows that a sequence $\{T_k\}_{k=1,2,\dots}$ of $\mathcal{D}'_{L^1}$ converges to zero when $T_k*\varphi\to 0$ in L^1 for any $\varphi\in\mathcal{D}$ as $k\to\infty$.

§ 2. Convolution of a kernel distribution

We shall define the convolution of a distribution $K(\hat{x},\hat{y})$ on $R^n\times R^n$ under the following condition:

(1) $$\varphi(\hat{x}+\hat{y})K(\hat{x},\hat{y})\in(\mathcal{D}'_{L^1})_{x,y} \text{ for every } \varphi\in\mathcal{D}_x.$$

If this is the case, the convolution which we denote be $\overset{*}{K}$ is defined by the relation:

$$<\overset{*}{K},\varphi>=\iint_{R^n\times R^n}\varphi(x+y)K(x+y)dxdy.$$

$\overset{*}{K}$ is a distribution on R^n since the application $\varphi\to\varphi(\hat{x}+\hat{y})K(\hat{x},\hat{y})$ of $\mathcal{D}_x$ into $(\mathcal{D}'_{L^1})_{x,y}$ is continuous ([14], Theorm 2). According to the definition of the space $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$, (1) is equivalent to

(2) $$\delta(\hat{z}-\hat{x}-\hat{y})K(\hat{x},\hat{y})\in\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y}),$$

where $\delta(\hat{z}-\hat{x}-\hat{y})K(\hat{x},\hat{y})$ is a distribution on $R^n\times R^n\times R^n$ issuing from $\delta(\hat{z})\otimes K(\hat{x},\hat{y})$ by change of variables.

Owing to Prop. 2 and by making use of change of variables we can deduce from (1) or (2) the following equivalent conditions:

(3) $$\varphi(\hat{x})K(\hat{x}-\hat{y},\hat{y})\in(\mathcal{D}'_{L^1})_{x,y} \text{ for every } \varphi\in\mathcal{D}_x,$$

(4) $$\int\varphi(x)K(x-\hat{y},\hat{y})dx\in(\mathcal{D}'_{L^1})_y \text{ for every } \varphi\in\mathcal{D}_x,$$

(5) $K(\hat{x}-\hat{y},\hat{y})$ is partially summable with respect to y,

(6) $$\varphi(\hat{x})K(\hat{y},\hat{x}-\hat{y})\in(\mathcal{D}'_{L^1})_{x,y} \text{ for every } \varphi\in\mathcal{D}_x,$$

(7) $$\int\varphi(x)K(\hat{y},x-\hat{y})dx\in(\mathcal{D}'_{L^1})_y \text{ for every } \varphi\in\mathcal{D}_x,$$

(8) $K(\hat{y},\hat{x}-\hat{y})$ is partially summable with respect to y.

The convolution $\overset{*}{K}$ is also given by the integrals:

$$\overset{*}{K}(\hat{x})=\int K(\hat{x}-y,y)dy=\int K(y,\hat{x}-y)dy$$

$$\overset{*}{K}(\hat{z})=\iint\delta(\hat{z}-x-y)K(x,y)dxdy$$

In fact, $<\int K(\hat{x}-y,y)dy,\ \varphi(\hat{x})>=\iint\varphi(x)K(x-y,y)\,dxdy$

$$=\iint\varphi(x+y)K(x,y)\,dxdy=<\overset{*}{K},\varphi>,\ \text{ for every } \varphi\in\mathcal{D}.$$

Hence $\overset{*}{K}(\hat{x})=\int K(\hat{x}-y,y)dy$. In like manner we have $\overset{*}{K}(\hat{x})=\int K(y,\hat{x}-y)dy$. As regards the last equation, its validity results from

$$<\iint\delta(\hat{z}-x-y)K(x,y)\,dxdy,\ \varphi(\hat{z})>=\iint\varphi(x+y)K(x,y)\,dxdy=<\overset{*}{K},\varphi>.$$

When K is decomposable, i. e., $K(\hat{x},\hat{y})=S(\hat{x})\otimes T(\hat{y})$, then the above-mentioned conditions give rise to the equivalent conditions for two distributions S, T to be defined the usual convolution $S*T$, which coincides with $\overset{*}{K}$ [13].

For example, $\psi\in\mathcal{D}_{x,y}$ satisfies the equivalent conditions and $\overset{*}{\psi}(\hat{x})=\int\psi(\hat{x}-y,y)dy$. For any $K(\hat{x},\hat{y})\in\mathcal{E}'_{x,y}$, it is clear that $\varphi(\hat{x}+\hat{y})K(\hat{x},\hat{y})\in(\mathcal{D}'_{L^1})_{x,y}$ since its support is bounded. This is also the case for $\varphi\in\mathcal{E}_x$ and the application $\varphi\to\varphi(\hat{x}+\hat{y})K(\hat{x},\hat{y})$ of $\mathcal{E}_x$ into $(\mathcal{D}'_{L^1})_{x,y}$ is continuous. Then $\overset{*}{K}\in\mathcal{E}'$ and $<\overset{*}{K},\varphi>=\int\varphi(x+y)K(x,y)\,dxdy$.

We define K^s by $K^s(\hat{x},\hat{y})=K(\hat{y},\hat{x})$, that is, $<K^s,\psi(\hat{x},\hat{y})>=<K,\psi(\hat{y},\hat{x})>$ for every $\psi\in\mathcal{D}_{x,y}$. The condition (1) yields that $\overset{*}{K^s}$ is defined if and only if $\overset{*}{K}$ is also, and $\overset{*}{K^s}=\overset{*}{K}$.

Now consider the condition:

(9) the convolution of $K*\psi$ is defined for every $\psi\in\mathcal{D}_{x,y}$.

If we put $K_1(\hat{x},\hat{y})=K(\hat{x}-\hat{y},\hat{y})$, $\psi_1(\hat{x},\hat{y})=\psi(\hat{x}-\hat{y},\hat{y})$, then it is easy to verify that $(K*\psi)(\hat{x}-\hat{y},\hat{y})=(K_1*\psi_1)(\hat{x},\hat{y})$. Then it follows from Corollary of Prop. 5 that $K(\hat{x}-\hat{y},\hat{y})$ is partially summable. The converse is true also. Therefore $\overset{*}{K}$ is defined if and only if $(K*\psi)^*$ is defined for every $\psi\in\mathcal{D}_{x,y}$. We show that

$$(K*\psi)^*=\overset{*}{K}*\overset{*}{\psi}.$$

By a similar reasoning as in the proof of the implication (1)→(2′) of Prop. 2, we can show that the condition (9) is equivalent to

(10) $$\varphi(\hat{x}+\hat{y})K(\hat{x}-\hat{\xi},\hat{y}-\hat{\eta})\check{\psi}(\hat{\xi},\hat{\eta})\in(\mathcal{D}'_{L^1})_{x,y,\xi,\eta}$$

for every $\varphi\in\mathcal{D}_x$ and $\psi\in\mathcal{D}_{x,y}$.

Then

$$\begin{aligned}<(K*\psi)^*, \varphi> &= \iint_{R^n\times R^n}\varphi(x+y)\left\{\iint_{R^n\times R^n}K(x-\xi, y-\eta)\psi(\xi, \eta)\, d\xi\, d\eta\right\} dxdy \\ &= \iint_{R^n\times R^n}\psi(\xi, \eta)\left\{\iint_{R^n\times R^n}\varphi(x+y)K(x-\xi, y-\eta)\, dxdy\right\} d\xi\, d\eta \\ &= \iint_{R^n\times R^n}\psi(\xi, \eta)\left\{\iint_{R^n\times R^n}\varphi(x+y+\xi+\eta)K(x, y)\, dxdy\right\} d\xi\, d\eta \\ &= \iint_{R^n\times R^n}\psi(\xi, \eta)\,(\check{\overset{*}{K}}*\varphi)\,(\xi+\eta)\, d\xi\, d\eta \\ &= <\psi, \check{\overset{*}{K}}*\varphi> = <\overset{*}{K}*\psi, \varphi>.\end{aligned}$$

Hence we have that $(K*\psi)^*=\overset{*}{K}*\psi$.

We have proved in the preceding discussions

PROPOSITION 7. *For any distribution $K(\hat{x}, \hat{y})$ on $R^n\times R^n$, the conditions (1)$\sim$(8) are equivalent. The convolutions $\overset{*}{K}$ and $\overset{*}{K}^s$ are defined and coincide if $\overset{*}{K}$ is defined. $\overset{*}{K}$ is defined if and only if $(K*\varphi)^*$ is defined for every $\varphi\in\mathcal{D}_{x,y}$.*

Suppose $\overset{*}{K}$ is defined. Let T be any distribution on $R^n\times R^n$ with compact support. We show that $(K*T)^*$ is defined and

$$(K*T)^*=\overset{*}{K}*\overset{*}{T}. \tag{11}$$

Let ψ be any element of $\mathcal{D}_{x,y}$. $T*\psi$ is also an element of $\mathcal{D}_{x,y}$. Then $(K*T)*\psi=K*(T*\psi)$, Hence $K*T$ astisfies the condition (9). Therefore

$$\begin{aligned}(K*T)^**\psi &= ((K*T)*\psi)^*=(K*(T*\psi))^* \\ &= \overset{*}{K}*(T*\psi)^*=\overset{*}{K}*(\overset{*}{T}*\psi)=(\overset{*}{K}*\overset{*}{T})*\psi.\end{aligned}$$

Hence $(K*T)^*=\overset{*}{K}*\overset{*}{T}$.

Let $\mathcal{O}'_{x,y}$ be the set of distributions $K(\hat{x}, \hat{y})$ on $R^n\times R^n$ for which the convolutions are defined: K satisfies the equivalent conditions (1)$\sim$(8) discussed above. By change of variables $x=\xi-\eta$, $y=\eta$, $\mathcal{O}'_{x,y}$ is transformed into $\mathcal{D}'_\xi((\mathcal{D}'_{L^1})_\eta)$. On $\mathcal{O}'_{x,y}$ we put the topology so that the application $\mathcal{O}'_{x,y}\to\mathcal{D}'_\xi((\mathcal{D}'_{L^1})_\eta)$ is isomorphic. Then the application $K(\hat{x}, \hat{y})\to\delta(\hat{z}-\hat{x}-\hat{y})K(\hat{x}, \hat{y})$ of $\mathcal{O}'_{x,y}$ into $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$ is monomorphic, because the application is decomposed into $K(\hat{x}, \hat{y})\to K(\hat{\xi}-\hat{\eta}, \hat{\eta})\to\delta(\zeta-\hat{\xi})K(\hat{\xi}-\hat{\eta}, \hat{\eta})\to\delta(\hat{z}-\hat{x}-\hat{y})K(\hat{x}, \hat{y})$, where $K(\hat{\xi}-\hat{\eta}, \hat{\eta})\to\delta(\zeta-\hat{\xi})K(\hat{\xi}-\hat{\eta}, \hat{\eta})$ is the isomorphism of $\mathcal{D}'_\xi((\mathcal{D}'_{L^1})_\eta)$ into $\mathcal{D}'_\zeta((\mathcal{D}'_{L^1})_{\xi,\eta})$, and $\delta(\zeta-\hat{\xi})K(\hat{\xi}-\hat{\eta}, \hat{\eta})\to\delta(\hat{z}-\hat{x}-\hat{y})K(\hat{x}, \hat{y})$ is caused by change of variables $\xi=x+y$, $\eta=y$, $\zeta=z$ defining the isomorphism of $\mathcal{D}'_\zeta((\mathcal{D}'_{L^1})_{\xi,\eta})$ onto $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$. It follows from Prop. 3 that $\mathcal{O}'_{x,y}$ is a permitted, ultra-bornological, complete space of distribution with the approximation properties by truncature and regularization. Let $\dot{\mathcal{O}}_{x,y}$ be the strict inductive limit of an increasing sequence of spaces $(\dot{\mathcal{O}}_{x,y})_p$ of type $(\boldsymbol{F})$, where $(\dot{\mathcal{O}}_{x,y})_p$ is a subspace of $\dot{\mathcal{B}}_{x,y}$ consisting of functions with supports in the cylinder $|x+y|\leq p$, p being a positive integer. By change of variables $x=\xi-\eta$, $y=\eta$, we see that $\dot{\mathcal{O}}_{x,y}$ is isomorphic with $\bar{\mathcal{D}}_x(\dot{\mathcal{B}}_y)$. Therefore it follows from Prop. 3 that $\mathcal{O}'_{x,y}$ is the strong dual of $\dot{\mathcal{O}}_{x,y}$, and the duality between $\mathcal{O}'_{x,y}$ and $\dot{\mathcal{O}}_{x,y}$ is given by

(12) $$<K, \varphi>=\iint K(x, y)\varphi(x, y)\, dxdy,$$

where $K \in \mathcal{O}'_{x,y}$ and $\varphi \in \dot{\mathcal{O}}_{x,y}$.

It is clear that the strong dual $\mathcal{O}_{x,y}$ of $\mathcal{O}'_{x,y}$ results from $\bar{\mathcal{D}}_{\xi}(\mathcal{B}_{\eta})$ by change of variables $x=\xi-\eta$, $y=\eta$, that is, $\mathcal{O}_{x,y}$ is the strict inductive limit of an increasing sequence $(\mathcal{O}_{x,y})_p$ of type $(\boldsymbol{F})$, where $(\mathcal{O}_{x,y})_p$ is the subspace of $\mathcal{B}_{x,y}$ consisting of functions with supports in the cylinder $|x+y|\leq p$. The duality between $\mathcal{O}'_{x,y}$ and $\mathcal{O}_{x,y}$ is given by (12) with $K \in \mathcal{O}'_{x,y}$ and $\varphi \in \mathcal{O}_{x,y}$. Any bounded subset of $\mathcal{O}_{x,y}$ is equicontinuous. The topology of compact convergence on compact subsets of $\mathcal{O}_{x,y}$ coincides on bounded subsets of $\mathcal{O}_{x,y}$ with the topology induced by $\mathcal{E}_{x,y}$.

It is to be noticed that the application $(\varphi, \psi, K)\to\varphi(\hat{x}+\hat{y})(K*\psi)(\hat{x}, \hat{y})$ of $\mathcal{D}_x\times\mathcal{D}_{x,y}\times\mathcal{O}'_{x,y}$ into $(\mathcal{D}'_{L^1})_{x,y}$ is hypocontinuous. In fact, since $\mathcal{D}_x$, $\mathcal{D}_{x,y}$ and $\mathcal{O}'_{x,y}$ are barrelled, it suffices to see that the application is separately continuous. This is almost obvious from Theorem 2 of [14].

Now we give an $\mathcal{S}'$-convolution of a distribution $K(\hat{x}, \hat{y})$ on $R^n\times R^n$ under the condition:

(1′) $$\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y}) \in (\mathcal{D}'_{L^1})_{x,y} \text{ for every } \varphi \in \mathcal{S}_x.$$

The $\mathcal{S}'$-convloution which we denote also by $\overset{*}{K}$ is defined by

$$<\overset{*}{K}, \varphi>=\iint_{R^n\times R^n}\varphi(x+y)K(x, y)dxdy.$$

Since the application $\varphi\to\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$ of $\mathcal{S}_x$ into $(\mathcal{D}'_{L^1})_{x,y}$ is continuous ([14], Theorem 2), $\overset{*}{K}$ is tempered, that is, $\overset{*}{K} \in \mathcal{S}'$. It is clear from (1′) that the $\mathcal{S}'$-convolution $\overset{*}{K}$, if defined, coincides with the previously defined convolution. We note that (1′) implies $K \in \mathcal{S}'_{x,y}$. In fact, let φ be the Fourier transform $\hat{\psi}$, $\psi \in \mathcal{D}$, then the application $\psi\to\hat{\psi}(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$ of $\mathcal{D}$ into $(\mathcal{D}'_{L^1})_{x,y}$ is continuous. It is almost obvious that we can make use of a parametrix for an iterated Laplacian as in the proof of Prop. 1 to conclude that $K \in \mathcal{S}'_{x,y}$. $\mathcal{S}'_x(\mathcal{D}'_{L^1})_y$ is the strong dual of $\mathcal{S}_x(\dot{\mathcal{B}}_y)$ ([12], p. 103), and we can apply Lemma 1 of §1 to show that $\mathcal{S}'_x(\mathcal{D}'_{L^1})_y$ is a permitted, ultra-bornological complete space of type $(\boldsymbol{DF})$, with the approximation properties by truncature and regularization. The strong dual of $\mathcal{S}'_x(\mathcal{D}'_{L^1})_y$ is $\mathcal{S}_x(\dot{\mathcal{B}}_y)$ ([12], p. 103).

By similar reasoning as in the preceding discussions, we can find the equivalent alternatives of (1′). Suppose K is tempered. Taking into consideration the definition of $\mathcal{S}'_z((\mathcal{D}'_{L^1})_{x,y})$, we see immediately that (1′) is equivalent to

(2′) $$\delta(\hat{z}-\hat{x}-\hat{y})K(\hat{x}, \hat{y}) \in \mathcal{S}'_z((\mathcal{D}'_{L^1})_{x,y}).$$

By change of variables we obtain from (2′), the equivalent conditions:

(5′) $$K(\hat{x}-\hat{y}, \hat{y}) \in \mathcal{S}'_x((\mathcal{D}'_{L^1})_y),$$

(8′) $$K(\hat{y}, \hat{x}-\hat{y}) \in \mathcal{S}'_x((\mathcal{D}'_{L^1})_y).$$

The $\mathcal{S}'$-convolution $\overset{*}{K}$ is also expressed by the following integrals:

$$\int_{R^n} K(\hat{x}-y, y)\,dy, \quad \int_{R^n} K(y, \hat{x}-y)\,dy, \quad \iint_{R^n \times R^n} \delta(\hat{z}-x-y)K(x, y)\,dxdy.$$

With necessary modifications of the proof of (11) we can show that if the $\mathcal{S}'$-convolution $\overset{*}{K}$ is defined, then for any $T \in (\mathcal{O}'_c)_{x,y}$ the $\mathcal{S}'$-convolution $(K*T)^*$ is defined and

$$(K*T)^* = \overset{*}{K} * \overset{*}{T}.$$

Finally we turn to the simultaneous convolution of a distribution on $R^n \times R^n \times \cdots \times R^n$. For the sake of simplicity we consider a distribution $K(\hat{x}, \hat{y}, \hat{z})$ on $R^n \times R^n \times R^n$. We say that the convolution of K is defined if the following condition:

(1″) $$\varphi(\hat{x}+\hat{y}+\hat{z})K(\hat{x}, \hat{y}, \hat{z}) \in (\mathcal{D}'_{L^1})_{x,y,z} \text{ for every } \varphi \in \mathcal{D}_x$$

is satisfied.

The convolution which we denote also by $\overset{*}{K}$ is given by the relation:

$$<\overset{*}{K}, \varphi> = \iiint_{R^n \times R^n \times R^n} \varphi(x+y+z)K(x, y, z)\,dxdydz.$$

$\overset{*}{K}$ is a distribution on R^n since the application $\varphi \to \varphi(\hat{x}+\hat{y}+\hat{z})K(\hat{x}, \hat{y}, \hat{z})$ of $\mathcal{D}_x$ into $(\mathcal{D}'_{L^1})_{x,y,z}$ is continuous ([14], Theorem 2). It is easy to see that (1″) is equivalent to the following conditions:

(2″) $$\delta(\hat{u}-\hat{x}-\hat{y}-\hat{z})K(\hat{x}, \hat{y}, \hat{z}) \in \mathcal{D}'_u((\mathcal{D}'_{L^1})_{x,y,z}),$$

($3''_1$) $$K(\hat{z}-\hat{x}-\hat{y}, \hat{y}, \hat{x}) \in \mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y}),$$

($3''_2$) $$K(\hat{x}, \hat{z}-\hat{x}-\hat{y}, \hat{y}) \in \mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y}),$$

($3''_3$) $$K(\hat{x}, \hat{y}, \hat{z}-\hat{x}-\hat{y}) \in \mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y}).$$

The convolution $\overset{*}{K}$ is also expressed by the integrals

$$\iiint_{R^n \times R^n \times R^n} \delta(\hat{u}-x-y-z)K(x, y, z)\,dxdydz, \quad \iint_{R^n \times R^n} K(\hat{z}-x-y, x, y)\,dxdy,$$

$$\iint_{R^n \times R^n} K(x, \hat{z}-x-y, y)\,dxdy, \quad \iint_{R^n \times R^n} K(x, y, \hat{z}-x-y)\,dxdy.$$

The spaces $\dot{\mathcal{O}}_{x,y,z}$, $\mathcal{O}'_{x,y,z}$, $\mathcal{O}_{x,y,z}$ can be defined in the same way as $\dot{\mathcal{O}}_{x,y}$, $\mathcal{O}'_{x,y}$, $\mathcal{O}_{x,y}$.

We say that the partial convolution of $K(\hat{x}, \hat{y}, \hat{z})$ with respect to x, y is defined if $K(\hat{x}, \hat{y}, \hat{z}) \in \mathcal{O}'_{x,y}(\mathcal{D}'_z)$. This will be a special case of the convolution of a vector valued kernel distribution treated in the next section. The condition that the partial convolution of $K(\hat{x}, \hat{y}, \hat{z})$ with respect to x, y is defined is the following

(13) $\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y}, \hat{z}) \in (\mathcal{D}'_{L^1})_{x,y}(\mathcal{D}'_z)$ for every $\varphi \in \mathcal{D}_x$.

This is also equivalent to any of the following conditions:

(14) $$\delta(\hat{u}-\hat{x}-\hat{y})K(\hat{x}, \hat{y}, \hat{z}) \in \mathcal{D}'_{u,z}((\mathcal{D}'_{L^1})_{x,y}),$$

(15) $$K(\hat{x}-\hat{y}, \hat{y}, \hat{z}) \in \mathcal{D}'_{x,z}((\mathcal{D}'_{L^1})_y).$$

The partial convolution $H(\hat{x}, \hat{z})$ of $K(\hat{x}, \hat{y}, \hat{z})$ is given by the relation:

$$\varphi\cdot H(\hat{z})=\iint_{R^n\times R^n}\varphi(x+y)K(x, y, \hat{z})\, dxdy.$$

H is also expressed by the integrals:

$$\iint_{R^n\times R^n}\delta(\hat{u}-x-y)K(x, y, \hat{z})\, dxdy, \quad \int_{R^n}K(\hat{x}-y, y, \hat{z})dy.$$

In fact, let these integrals be denoted by H_1 and H_2 respectively. For any φ, $\psi \in \mathcal{D}$, we have

$$<\varphi\cdot H_1, \psi>=<H_1, \varphi(\hat{u})\psi(\hat{z})>$$
$$=\iint\varphi(x+y)(K\cdot\psi)(x, y)dxdy=<\iint\varphi(x+y)K(x, y, \hat{z})\, dxdy, \psi(\hat{z})>$$
$$<\varphi\cdot H_2, \psi>=<H_2, \varphi(\hat{x})\psi(\hat{y})>$$
$$=\int\left\{\int\varphi(x)(K\cdot\psi)(x-y, y)dx\right\}dy=\iint\varphi(x+y)(K\cdot\psi)(x, y)\, dxdy$$
$$=<\iint\varphi(x+y)K(x, y, \hat{z})\, dxdy, \psi(\hat{z})>.$$

PROPOSITION 8. *If the convolution of a distribution $K(\hat{x},\hat{y}, \hat{z})$ on $R^n\times R^n\times R^n$ is defined, then the partial convolution H of K with respect to x, y is defined and the convolution of H is also defined and we have*

$$\overset{*}{K}=\overset{*}{H}.$$

PROOF. Since the convolution of K is defined, we have

$$\delta(\hat{u}-\hat{x}-\hat{y}-\hat{z})K(\hat{x}, \hat{y}, \hat{z}) \in \mathcal{D}'_{u,z}((\mathcal{D}'_{L^1})_{x,y}).$$

By change of variables $u\to u+z$, $x\to x$, $y\to y$, $z\to z$, we have

$$\delta(\hat{u}-\hat{x}-\hat{y})K(\hat{x}, \hat{y}, \hat{z}) \in \mathcal{D}'_{u,z}((\mathcal{D}'_{L^1})_{x,y}).$$

Hence the partial convolution H of K is defined, and $H(\hat{x}, \hat{z})=\int K(\hat{x}-y-\hat{z}, y, \hat{z})$ dy. Now it is easy to see that

$$H(\hat{x}-\hat{z},\ \hat{z})=\int K(\hat{x}-y-\hat{z},\ y,\ \hat{z})\,dy.$$

Since $K(\hat{x}-\hat{y}-\hat{z},\ \hat{y},\ \hat{z}) \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_{y,z})$ and $\overset{*}{K}=\iint K(\hat{x}-y-z, y, z)dydz$. Therefore $\overset{*}{H}$ is defined and $\overset{*}{K}=\overset{*}{H}$.

We can apply Prop. 8 to deduce the equation (11). If K is decomposable i. e., $K=S\otimes T\otimes U$, $U\neq 0$, then Prop. 8 is equivalent to saying that if the simultaneous convolution $S*T*U$ is defined, then $(S*T)*U$ is defined and

$$S * T * U=(S * T) * U.$$

The result was already established in Shiraishi [13].

We can define the simultaneous $\mathcal{S}'$-convolution of $K(\hat{x}, \hat{y}, \hat{z})$ and show the analogue of Prop. 8.

§ 3. Convolution of a vector valued kernel distribution

Let E and F be two locally convex Hausdorff vector spaces, not necessarily quasi-complete. $E\hat{\otimes}_\iota F$ denotes the quasi-completion of inductive tensor product $E\otimes_\iota F$, the locally convex topology $\mathcal{T}$ of $E\otimes_\iota F$ is a unique locally convex Hausdorff topology such that under the usual correspondence between the bilinear application of $E\times F$ into an arbitrary locally convex Hansdorff space G and the linear applications of $E\otimes F$ into G, the separately continuous bilinear applications of $E\times F$ into G precisely correspond to the continuous linear applications of $E\otimes F$ provided with the topology $\mathcal{T}$.

Let $\vec{S}$ and $\vec{T}$ be vector valued distributions on R^n with values in E and F respectively, that is, $\vec{S} \in \mathcal{D}'_x(E)$, $\vec{T} \in \mathcal{D}'_y(F)$. The tensor product $\vec{S}\otimes\otimes_\iota\vec{T}$ is, by definition, an $E\hat{\otimes}_\iota F$-valuded distribution on $R^n\times R^n$ defined by the following relation ([12], p. 145):

$$\vec{S}_x\otimes\otimes_\iota\vec{T}_y \cdot u(x)v(y)=(\vec{S}\cdot u)\otimes(\vec{T}\cdot v) \text{ for every } u,\ v \in \mathcal{D}.$$

From this definition of tensor product of vector valued distributions we can easily prove

LEMMA 3. (i) *$(\vec{S}_x*\varphi)\otimes\otimes_\iota(\vec{T}_y*\psi)=(\vec{S}_x\otimes\otimes_\iota\vec{T}_y)*(\varphi\otimes\psi)$ for every φ, $\psi \in \mathcal{D}$. And $(\vec{S}_x\otimes\otimes_\iota\vec{T})*(\varphi\otimes\psi)$ is an indefinitely differentiable function $(x, y)\to(\vec{S}*\varphi)(x)\otimes(\vec{T}*\psi)(y)$ with values in $E\hat{\otimes}_\iota F$.*

(ii) *If U, $V \in \mathcal{E}'$, then $(\vec{S}_x * U)\otimes\otimes_\iota(\vec{T}_y*V)=(\vec{S}_x\otimes\otimes_\iota\vec{T}_y) * (U\otimes V)$,*

(iii) *For any $\varphi \in \mathcal{D}$, $\int\varphi(x+\hat{y})(\vec{S}_x\otimes\otimes_\iota\vec{T}_y)dx=[(\check{\vec{S}} * \varphi)\vec{T}]_\iota$*

and

$$\int\varphi(\hat{x}+y)(S_x\otimes\otimes_\iota\vec{T}_y)dy=[\vec{S}(\check{\vec{T}}*\varphi)]_\iota.$$

(iv) *For any* $\varphi, \psi \in \mathcal{D}$, $\int \varphi(x+\hat{y})((\vec{S}_x \otimes\otimes_\iota \vec{T}_y) *_y \psi)dx$ *is a distribution of* $\mathcal{E}_y(E \hat{\otimes}_\iota F)$ *and is written as* $(\vec{\check{S}} * \varphi)(y) \otimes_\iota (\vec{T} * \psi)(y)$, *where* $[\quad]_\iota$ *denotes the multiplicative product of two vector valued distributions defined by L. Schwartz* ([12], p. 57).

Proof. First we prove (ii). Let u, v be any elements of $\mathcal{D}$. Then

$$
\begin{aligned}
&(\vec{S} * U) \otimes\otimes_\iota (\vec{T} * V) \cdot (u \otimes v) = (\vec{S} * U) \cdot u \otimes (\vec{T} * V) \cdot v \\
&= \vec{S} \cdot (\check{U} * u) \otimes \vec{T} \cdot (\check{V} * v) = (\vec{S} \otimes\otimes_\iota \vec{T}) \cdot (\check{U} * u) \otimes (\check{V} * v) \\
&= (\vec{S} \otimes\otimes_\iota \vec{T}) \cdot (U \otimes V)\check{} * (u \otimes v) = (\vec{S} \otimes\otimes_\iota \vec{T}) * (U \otimes V) \cdot u \otimes v.
\end{aligned}
$$

Hence we have the equation (ii).

Specially, $(\vec{S} * \varphi) \otimes\otimes_\iota (\vec{T} * \psi) = (\vec{S} \otimes\otimes_\iota \vec{T}) * (\varphi \otimes \psi)$ for every $\varphi, \psi \in \mathcal{D}$. Let $\vec{K}$ denote the second member of the equation. Then $\vec{K} \in \mathcal{E}_{x,y}(E \hat{\otimes}_\iota F)$. Let δ_x, δ_y denote Dirac measures located at points x, y respectively. Since $\delta_x \otimes \delta_y \in \mathcal{E}'_{x,y}$, so that $\vec{K}(x, y) = \vec{K} \cdot \delta_x \otimes \delta_y$.

$$
\begin{aligned}
\vec{K} \cdot \delta_x \otimes \delta_y &= (\vec{S} \otimes\otimes_\iota \vec{T}) * (\varphi \otimes \psi) \cdot \delta_x \otimes \delta_y \\
&= (\vec{S} \otimes\otimes_\iota \vec{T}) \cdot (\check{\varphi} * \delta_x) \otimes (\check{\psi} * \delta_y) = \vec{S} \cdot \tau_x \check{\varphi} \otimes \vec{T} \cdot \tau_y \check{\psi} \\
&= (\vec{S} * \varphi)(x) \otimes (\vec{T} * \psi)(y).
\end{aligned}
$$

The equation (iii) is proved in ([12], p. 181).

Before proceeding to the proof of the final part (iv), some preliminary discussions on the multiplicative product $\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$, where $\varphi \in \mathcal{D}_x$, $K \in \mathcal{D}'_x(\mathcal{E}_y)$, are given.

Let E be a quasi-complete locally convex space. $\bar{\mathcal{E}}_x(E)$ is, by definition, the space of indefinitely differentiable E-valued functions $\vec{f}(x)$ on R^n with usual topology ([9], *p.* 94). $\bar{\mathcal{E}}_x(E)$ is isomorphic to $\mathcal{E}_x(E)$ by the application $\vec{f}(x) = \delta_x \cdot \vec{T}$, $\vec{T}$ being an element of $\mathcal{E}_x(E)$. Now let $\alpha \in \mathcal{E}_{x,y}$, $K \in \mathcal{D}'_x(\mathcal{E}_y)$, and we denote by $\vec{\alpha}(y)$ and $\vec{K}(y)$ the corresponding functions in $\bar{\mathcal{E}}_y(\mathcal{E}_x)$ and $\bar{\mathcal{E}}_y(\mathcal{D}'_x)$ respectively. Since the multiplicative product between $\mathcal{E}_x$ and $\mathcal{D}'_x$ is hypocontinuous, the function $\vec{T}(y) = \vec{\alpha}(y)\vec{K}(y)$ is an indefinitely differentiable function with values in $\mathcal{D}'_x$, that is, $\vec{T}(y) \in \bar{\mathcal{E}}_y(\mathcal{D}'_x)$. Evidentely the application $(\vec{\alpha}, \vec{K}) \to \vec{T}$ is separately continuous. We show that the distribution T on $R^n \times R^n$ corresponding to $\vec{T}$ is αK, the multiplicative product of α and K. The application $(\alpha, K) \to \alpha K$ is also separately continuous from $\mathcal{E}_{x,y} \times \mathcal{D}'_x(\mathcal{E}_y)$ into $\mathcal{D}'_{x,y}$. For the end of the proof it suffices to show that it is the case where α and K are decomposable. Let $\alpha(\hat{x}, \hat{y}) = \xi(\hat{x}) \otimes \eta(\hat{y})$, $K = S_x \otimes \zeta(\hat{y})$, where $\xi, \eta, \zeta \in \mathcal{E}$ and $S \in \mathcal{D}'$. Then $\vec{\alpha}(y) = \xi(\hat{x})\eta(y)$, $\vec{K}(y) = S_x \zeta(y)$, $\vec{T}(y) = \xi(\hat{x}) S_x \eta(y) \zeta(y)$. For any $u \in \mathcal{D}_x$, $v \in \mathcal{D}_y$ we have

$$
\langle \alpha K, u \otimes v \rangle = \langle \xi S_x \otimes \eta\zeta, u \otimes v \rangle = \langle \xi S_x, u \rangle \langle \eta\zeta, v \rangle,
$$

$$
\langle T, u \otimes v \rangle = \langle T \cdot v, u \rangle = \left\langle \int \xi S_x \eta(y) \zeta(y) v(y)\, dy, u \right\rangle
$$

$$=\int <\xi S_x, u> \eta(y)\zeta(y)v(y)dy = <\xi S_x, u><\eta\zeta, v>.$$

From these equations we obtain that $T=\alpha K$, as desired. Moreover, since $\vec{\alpha}\vec{K} \in \bar{\mathcal{E}}_y(\mathcal{D}'_x)$, we see that $\alpha K \in \mathcal{D}'_x(\mathcal{E}_y)$. Now we let $\alpha(\hat{x}, \hat{y})=\varphi(\hat{x}+\hat{y})$, where $\varphi \in \mathcal{D}_x$. Then we see from the above facts that $K_1(\hat{x}, \hat{y})=\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y}) \in \mathcal{D}'_x(\mathcal{E}_y)$. We can show $K_1 \in \mathcal{D}'_{L^1}(\mathcal{E}_y)$. In fact, let G be any compact subset of R^n, and H the support of φ. Then it is easy to see that the support of $K_1 \cdot v$, $v \in \mathcal{E}'_G$, is contained in $H-G$, a compact subset of R^n. It follows that $K_1 \in \mathcal{E}'_x(\mathcal{E}_y)$ and therefore $K_1 \in (\mathcal{D}'_{L^1})_x(\mathcal{E}_y)$. It is not difficult to see that the application $(\varphi, K)\to\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$ of $\mathcal{D}_x \times \mathcal{D}'_x(\mathcal{E}_y)$ into $(\mathcal{D}'_{L^1})_x(\mathcal{E}_y)$ is separately continuous. Let $T(\hat{y})=\int\varphi(x+\hat{y})K(x, \hat{y})dx$. Then T is an element of $\mathcal{E}_y$, and the value of T at y_o is given by $T(y_o)=\int \vec{\varphi}(y_o)\vec{K}(y_o)\, dx$, where $\vec{\varphi}(y)=\varphi(\hat{x}+y) \in \bar{\mathcal{E}}_y(\mathcal{E}_x)$. This is because $T(y_o)=\delta_{y_o} \cdot T=\int\vec{\varphi}(y_o)\vec{K}(y_o)dx$.

In a similar way we can show that $\varphi(\hat{x}+\hat{z})K(\hat{x}, \hat{y}) \in (\mathcal{D}'_{L^1})_x(\mathcal{E}_{y,z})$, and $S(\hat{y}, \hat{z})=\int\varphi(x+\hat{z})K(x, \hat{y})dx \in \mathcal{E}_{y,z}$. The value taken by S at (y_o, z_o) is $S(y_o, z_o) = \int\vec{\varphi}(z_o)\vec{K}(y_o)dx$. From the expressions of $T(y_o)$ and $S(y_o, z_o)$ it is clear that $T(y_o)= S(y_o, y_o)$.

We come to the proof of (iv).

Since $\vec{S}_x\otimes\otimes_\iota\vec{T}_y \in \mathcal{D}'_{x,y}(E\otimes_\iota F)$, $(\vec{S}_x\otimes\otimes_\iota\vec{T}_y) *_y\psi \in \mathcal{D}'_x(\mathcal{E}_y(E\otimes_\iota F))$ for any $\psi \in \mathcal{D}_y$. On account of the fact that the application $(\varphi, K)\to\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$ of $\mathcal{D}_x \times \mathcal{D}'_x(\mathcal{E}_y)$ into $(\mathcal{D}'_{L^1})_x(\mathcal{E}_y)$ is separately continuous, we see that $\varphi(\hat{x}+\hat{y})((\vec{S}_x\otimes\otimes_\iota\vec{T}_y)*_y\psi) \in (\mathcal{D}'_{L^1})_x\,(\mathcal{E}_y(E\otimes_\iota F))$, whence $\int\varphi(x+\hat{y})((\vec{S}_x\otimes\otimes_\iota\vec{T}_y)*_y\psi)dx$ is an element $\vec{U}(\hat{y})$ of $\mathcal{E}_y(E\otimes_\iota F)$. Let $\bar{g}'$ be any element of the dual of $E\otimes_\iota F$. Then

$$<\vec{U}, \bar{g}'>=\int\varphi(x+y)(<\vec{S}_x\otimes\otimes_\iota\vec{T}_y, \bar{g}'>*_y\psi)dx \in \mathcal{E}_y.$$

Next we note that the symmetry of $(\vec{S}_x\otimes\otimes_\iota\vec{T}_y) *_y\psi$ with respect to the variable x is $(\check{\vec{S}}_x\otimes\otimes_\iota\vec{T}_y) *_y\psi$. For this follows from the following equations:

$$\begin{aligned}(\check{\vec{S}}\otimes\otimes_\iota\vec{T}) *_y\psi \cdot \check{u}\otimes v &= \check{\vec{S}}\otimes\otimes_\iota\vec{T} \cdot \check{u}\otimes v * \check{\psi}\\ &=(\check{\vec{S}} \cdot \check{u})\otimes(\vec{T} \cdot v * \check{\psi})=(\vec{S} \cdot u)\otimes(\vec{T} * v * \check{\psi})\\ &=(\vec{S}\otimes\otimes_\iota\vec{T}) *_y\psi \cdot u\otimes v,\ u \in \mathcal{D}_x,\ v \in \mathcal{D}_y.\end{aligned}$$

We can also show that $\varphi(\hat{x}+\hat{z})((\vec{S}_x\otimes\otimes_\iota\vec{T}_y) *_y\psi) \in (\mathcal{D}'_{L^1})_x(\mathcal{E}_{y,z}(E\otimes_\iota F))$. $\int \varphi(x+\hat{z})((\vec{S}\otimes\otimes_\iota\vec{T})*_y\psi)\, dx$ is an element $\vec{V}(\hat{y}, \hat{z})$ of $\mathcal{E}_{y,z}(E\otimes_\iota F)$. Now using the remark stated above, we have

$$\vec{V}(\hat{y}, \hat{z})=\int\varphi(-x+\hat{z})((\check{\vec{S}}_x\otimes\otimes_\iota\vec{T}_y) *_y\psi)dx$$

$$=(\check{\vec{S}}_x\otimes\otimes_\iota\vec{T}_y)*(\varphi\otimes\psi)=(\check{\vec{S}}*\varphi)\otimes(\vec{T}*\psi).$$

For any $\vec{g}' \in (E\otimes F)'_\iota$, we have

$$<\vec{V}, \vec{g}'>=\int\varphi(x+\hat{z})(<(\vec{S}_x\otimes\otimes_\iota\vec{T}_y), \vec{g}'>*_y\psi)dx.$$

Consider the values of $<\vec{U}, \vec{g}'>$ and $<\vec{V}, \vec{g}'>$ taken at y_o and at (y_o, z_o) respectively.

$$<\vec{U}, \vec{g}'>(y_o)=<\vec{U}(y_o), \vec{g}'>$$
$$<\vec{V}, \vec{g}'>(y_o, z_o)=<\vec{V}(y_o, z_o), \vec{g}'>$$
$$=<(\check{\vec{S}}*\varphi)(z_o)\otimes_\iota(\vec{T}*\psi)(y_o), \vec{g}'>.$$

Then it follows from our preceding discussions for prelimaries that

$$\vec{U}(y_o)=(\check{\vec{S}}*\varphi)(y_o)\otimes_\iota(\vec{T}*\psi)(y_o),$$

which completes the proof.

REMARK. We can show that

$$(1) \qquad D^p\{(\vec{S}*\varphi)(y)\otimes_\iota(\vec{T}*\psi)(y)\}=\sum_{q\leqslant p}\binom{p}{q}\{(\vec{S}*D^q\varphi)(y)\otimes_\iota(\vec{T}*D^{p-q}\psi)(y)\}.$$

For our later use we show the following

LEMMA 4. *Let E be a locally convex Hausdorff vector space and $\mathscr{H}$ a space of distributions. Suppose $\mathscr{H}$ is a permitted barrelled space or has the approximation properties by truncature and regularization. Let j be the continuous injection: $\mathscr{H}(E)\to\mathscr{D}'(E)$. Any linear application $\mathscr{L}$ of a barrelled space F into $\mathscr{H}(E)$ is continuous if $j\circ\mathscr{L}$ is continuous.*

PROOF. In Shiraishi ([13], p. 21) it was shown that any linear application $\mathscr{L}$ of a barrelled space F into a locally convex space G is continuous if G is a subspace of a locally convex space H with the continuous injection j and if $I\in\mathscr{L}_s(G; G)$ is strictly adherent to a subset A of $\mathscr{L}_s(G; G)$ such that each $u\in A$ is a restriction of a continuous linear application of H into G and such that $j\circ\mathscr{L}$ is continuous. Put $G=\mathscr{H}(E)$ and $H=\mathscr{D}'(E)$. In case $\mathscr{H}$ is a permitted space, we take A to be the set of application u_k: $\vec{T}\to(\alpha_k\vec{T})*\rho_k$ of $\mathscr{H}(E)$ into itself, where $\{\alpha_k\}$ and $\{\rho_k\}$ are any sequence of multiplicators and regularizations respectively. An application u_k, as an application of $\mathscr{D}'(E)$ into $\mathscr{H}(E)$, is continuous and the identical application I of $\mathscr{H}(E)$ into itself is strictly adherent to a subset A. In fact $(\alpha_k\vec{T})*\rho_k\to\vec{T}$ in $\mathscr{H}(E)$ since $<(\alpha_k\vec{T})*\rho_k, \vec{e}'>=(\alpha_k<\vec{T}, \vec{e}'>)*\rho_k\to<\vec{T}, \vec{e}'>$ uniformly in $\mathscr{H}$ when $\vec{e}'$ runs through any equicontinuous subset of E'. In case $\mathscr{H}$ has the properties by truncature and regularization we take A to be $\vec{T}\to(\alpha_k\vec{T})*\rho_k$. Then the proof will be carried out in the same way as above, completing the proof.

PROPOSITION 9. *Let G be any quasi-complete locally convex Hausdorff vector space. For any G-valued distribution* $\vec{K}(\hat{x}, \hat{y})$ *on* $R^n \times R^n$, *the following conditions are equivalent to each other:*

(1) $\vec{K}(\hat{x}, \hat{y}) \in \mathcal{O}'_{x,y}(G)$,

(2) $\varphi(\hat{x}+\hat{y})\vec{K}(\hat{x}, \hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}(G)$ *for every* $\varphi \in \mathcal{D}_x$,

(3) $\delta(\hat{z}-\hat{x}-\hat{y})\vec{K}(\hat{x}, \hat{y}) \in \mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y}(G))$,

(4) $\int \varphi(x+\hat{y})\vec{K}(x, \hat{y})dx \in (\mathcal{D}'_{L^1})_y(G)$ *for every* $\varphi \in \mathcal{D}_x$,

(5) $\vec{K}(\hat{x}-\hat{y}, \hat{y}) \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(G))$, *that is,* $\vec{K}(\hat{x}-\hat{y}, \hat{y})$ *is partially summable with respect to y,*

(6) $\int \varphi(\hat{x}+y)\vec{K}(\hat{x}, y)dy \in (\mathcal{D}'_{L^1})_x(G)$ *for every* $\varphi \in \mathcal{D}_y$,

(7) $\vec{K}(\hat{y}, \hat{x}-\hat{y}) \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(G))$, *that is,* $\vec{K}(\hat{y}, \hat{x}-\hat{y})$ *is partially summable with respect to y,*

(8) $\vec{K} *_x \xi \in \mathcal{O}'_{x,y}(G)$ *for every* $\xi \in \mathcal{D}_x$,

(9) $\vec{K} *_y \eta \in \mathcal{O}'_{x,y}(G)$ *for every* $\eta \in \mathcal{D}_y$,

(10) $\vec{K} * \zeta \in \mathcal{O}'_{x,y}(G)$ *for every* $\zeta \in \mathcal{D}_{x,y}$,

(11) $\vec{K} * (\xi \otimes \eta) \in \mathcal{O}'_{x,y}(G)$ *for every* $\xi \in \mathcal{D}_x$ *and* $\eta \in \mathcal{D}_y$,

(12) $\varphi(x+y)(\vec{K} * \zeta) \in L^1_{x,y}\varepsilon G$ *for every* $\varphi \in \mathcal{D}_x$ *and* $\zeta \in \mathcal{D}_{x,y}$.

Let one of the equivalent conditions (1)~(12) *be satisfied. We define* $\overset{*}{\vec{K}}$ *by the relation:*

(13) $<\overset{*}{\vec{K}}, \varphi> = \iint_{R^n \times R^n} \varphi(x+y)\vec{K}(x, y)dxdy$ *for every* $\varphi \in \mathcal{D}_x$. $\overset{*}{\vec{K}}$ *is a G-valued distribution and is also written as the following integrals:*

$\int_{R^n} \vec{K}(\hat{x}-y, y)dy$, $\int_{R^n} \vec{K}(y, \hat{x}-y)dy$, $\iint_{R^n \times R^n} \delta(\hat{z}-x-y)\vec{K}(x, y)dxdy$. *And we have*

(14) *Let* $\vec{K}^s(\hat{x}, \hat{y}) = \vec{K}(\hat{y}, \hat{x})$. *Then* $\vec{K}^s$ *satisfies the above equivalent conditions, and* $\overset{*}{\vec{K}^s} = \overset{*}{\vec{K}}$.

(15) $(\vec{K} * \psi)^* = \overset{*}{\vec{K}} * \overset{*}{\psi}$ *for every* $\psi \in \mathcal{D}_{x,y}$, *more generally this equation holds also for* $T \in \mathcal{E}'_{x,y}$.

PROOF. Ad (1)$\rightleftarrows$(3). By Prop. 4 the application $<\vec{K}(\hat{x}, \hat{y}), \vec{g}'> \to <\delta(\hat{z}-\hat{x}-\hat{y})\vec{K}(\hat{x},\hat{y}), \vec{g}'>$ of $\mathcal{O}'_{x,y}$ into $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$ is monomorphic. Let $\vec{g}' \to 0$ in G'_c. $<\vec{K}(\hat{x},\hat{y}), \vec{g}'> \to 0$ in $\mathcal{O}'_{x,y}$, therefore $<\delta(\hat{z}-\hat{x}-\hat{y})\vec{K}(\hat{x}, \hat{y}), \vec{g}'> \to 0$ in $\mathcal{D}'_z((\mathcal{D}'_{L^1})_{x,y})$, and vice versa.

Ad (3)$\leftrightarrows$(2). Clearly (3) implies (2). If (2) holds, then it follows from Lemma 4 that the application $\varphi(\hat{x}) \to \varphi(\hat{x}+\hat{y})\vec{K}(\hat{x}, \hat{y})$ of $\mathcal{D}_x$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ is continuous. This implies (3), since $\varphi(\hat{z}) \cdot \delta(\hat{z}-\hat{x}-\hat{y})\vec{K}(\hat{x}, \hat{y}) = \varphi(\hat{x}+\hat{y})\vec{K}(\hat{x}, \hat{y})$.

Ad (2)$\to$(4). If (2) holds, the integration of $\varphi(\hat{x}+\hat{y})K(\hat{x}, \hat{y})$ with respect to x yields (4).

Ad (4)→(5). By change of variables we have $\int \varphi(x+\dot{y})\vec{K}(x, \dot{y})\, dx = \int \varphi(x)\vec{K}(x-\dot{y}, \dot{y})dx \in (\mathcal{D}'_{L^1})(G)$. The application $\varphi \to \int \varphi(x)\vec{K}(x-\dot{y}, \dot{y})dx$ of $\mathcal{D}$ into $\mathcal{D}'_{L^1}(G)$ is continuous (Lemma 4). Hence $\vec{K}(\dot{x}-\dot{y}, \dot{y}) \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(G))$.

Ad (5)→(1). This implication is clear from the fact that $\mathcal{O}'_{x,y}$ is isomorphic to $\mathcal{D}'_x(\mathcal{D}'_{L^1})_y$ by change of variables.

Ad (2)→(6)→(7)→(1). We can show these implications in a similar way as in the proof of (2)→(4)→(5)→(1).

Ad (1)→(8). The application $\xi \to \vec{K} *_x \xi$ of $\mathcal{D}$ into $\mathcal{O}'_{x,y}(G)$ is continuous (Lemma 4). Therefore (1) implies (8).

Ad (8)→(11). The application $\vec{K} *_x \xi \to \vec{K} * (\xi \otimes \eta)$ of $\mathcal{O}'_{x,y}(G)$ into $\mathcal{O}'_{x,y}(G)$ is continuous, so that this implication holds.

Ad (1)→(9)→(11). The proof is very similar to the preceding case: (1)→(8)→(11).

Ad (11)→(10). The bilinear application $(\xi, \eta) \to \vec{K} * (\xi \otimes \eta)$ of $\mathcal{D}_x \times \mathcal{D}_y$ into $\mathcal{O}_{x,y}(G)$ being separately continuous, it is also continuous in the topology of $\mathcal{D}_{x,y}$ [11]. Hence (11) yields (10).

Ad (10)→(5). If we put $\vec{K}_1(\dot{x}, \dot{y}) = \vec{K}(\dot{x}-\dot{y}, \dot{y})$ and $\zeta_1(\dot{x}, \dot{y}) = \zeta(\dot{x}-\dot{y}, \dot{y})$ we have $(\vec{K}_1 * \zeta_1)(\dot{x}, \dot{y}) = (\vec{K} * \zeta)(\dot{x}-\dot{y}, \dot{y})$. Therefore (10) implies that $\vec{K}_1 * \zeta_1 \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(G))$. Then it follows from Prop. 6 that $\vec{K}_1 \in \mathcal{D}'_x((\mathcal{D}'_{L^1})_y(G))$.

Ad (10)→(12). We note that $\vec{K}_1(\dot{x}, \dot{y}) = \varphi(\dot{x}+\dot{y})(\vec{K} * \zeta)$ is distinguished in $(\mathcal{D}'_{L^1})_{x,y}(G)$ (see Miyazaki [6] for the definition of "distinguished element"). In fact there exists a sequence of positive number $\{\lambda_{\bar{p}}\}$ such that $\{\lambda_{\bar{p}} D^{\bar{p}} \vec{K}_1\}$ is bounded in $(\mathcal{D}'_{L^1})_{x,y}(G)$. Any derivative $D^{\bar{p}}\vec{K}_1$ is a linear combination of the element of $(\mathcal{D}'_{L^1})_{x,y}(G)$ written in the form $(D^{\bar{q}}\varphi)(\dot{x}+\dot{y})(\vec{K} * D^{\bar{p}-\bar{q}}\zeta)$. There exists a sequence of positive numbers $\{\mu_{\bar{p}}\}$ (resp. $\{\nu_{\bar{p}}\}$) such that $\{\mu_{\bar{p}}(D^{\bar{p}}\varphi)(\dot{x}+\dot{y})\}$ (resp. $\{\nu_{\bar{p}} D^{\bar{p}-\bar{q}}(\zeta)\}$) is bounded in $\mathcal{D}_x$ (resp. $\mathcal{D}_{x,y}$). On account of the fact that the application $(\varphi, \zeta) \to \varphi(x+y)(K * \zeta)$ of $\mathcal{D}_x \times \mathcal{D}_{x,y}$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ is hypocontinuous, we can easily see that $\vec{K}_1(\dot{x}, \dot{y})$ is distinguished. Applying a result due to Miyazaki [6], we see that $\vec{K}_1(x, y)$ is in $\mathcal{D}_{L^1}(G)$, and *a fortiori* in $(L^1)_{x,y} \varepsilon\, G$, as desired.

Ad (12)→(10). Since the injection $(L^1)_{x,y}(G)$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ is continuous, (12) implies (10).

Suppose $\vec{K}$ satisfies the equivalent conditions proved above. The application $\varphi \to \varphi(\dot{x}+\dot{y})\vec{K}(\dot{x}, \dot{y})$ of $\mathcal{D}$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ will be continuous by Lemma 4. Hence the application $\varphi \to \int \varphi(x+y)K(x, y)\, dxdy$ of $\mathcal{D}$ into G is continuous. This shows that $\overset{*}{\vec{K}}$ defined by (13) is a G-valued distribution. Let $\vec{g}'$ be any element of G', then $\langle \vec{K}, \vec{g}' \rangle \in \mathcal{O}'_{x,y}$. It follows then from Prop. 7 that $\overset{*}{\vec{K}}$ is written by the integrals: $\int \vec{K}(\dot{x}-y, y)dy$, $\int \vec{K}(y, \dot{x}-y),dy$, $\iint \delta(\dot{x}-x-y)\, K(x, y)\, dxdy$.

Ad (14). It is clear from Prop. 7.

Ad (15). Let T be any element of $\mathcal{E}'_{x,y}$. Then $T * \zeta \in \mathcal{D}_{x,y}$ for every $\zeta \in$

$\mathcal{D}_{x,y}$. $(\vec{K} * T) * \zeta = \vec{K} * (T * \zeta) \in \mathcal{O}'_{x,y}(G)$. Hence $\vec{K} * T \in \mathcal{O}'_{x,y}(G)$. Then it follows from Prop. 7 that $(\vec{K} * T)^* = \overset{*}{\vec{K}} * \overset{*}{T}$.

If $\vec{K}(\hat{x}, \hat{y}) \in \mathcal{D}'_{x,y}(G)$ satisfies the equivalent conditions (1)~(12), then we shall call $\overset{*}{\vec{K}}$ defined by (13) the convolution of $\vec{K}$.

Now we apply Prop. 9 to a tensor product of vector valued distribution on R^n.

THEOREM. *Let E and F be two locally convex vector spaces. For any distributions $\vec{S} \in \mathcal{D}'_x(E)$ and $\vec{T} \in \mathcal{D}'_y(F)$ the following conditions (1)~(8) are equivalent:*

(1) $(\vec{S}_x \otimes \otimes_\iota \vec{T}_y) \in \mathcal{O}'_{x,y}(E \otimes_\iota F)$,

(2) $\varphi(\hat{x}+\hat{y})(\vec{S}_x \otimes \otimes_\iota \vec{T}_y) \in (\mathcal{D}'_{L^1})_{x,y}(E \otimes_\iota F)$ *for every* $\varphi \in \mathcal{D}_x$,

(3) $\delta(\hat{z}-\hat{x}-\hat{y})(\vec{S}_x \otimes \otimes_\iota \vec{T}_y) \in (\mathcal{D}'_z(\mathcal{D}'_{L^1})_{x,y})(E \otimes_\iota F)$,

(4) $[(\check{\vec{S}} * \varphi)\vec{T}]_\iota \in \mathcal{D}'_{L^1}(E \otimes_\iota F)$ *for every* $\varphi \in \mathcal{D}_x$,

(5) $[\vec{S}(\check{\vec{T}} * \varphi)]_\iota \in \mathcal{D}'_{L^1}(E \otimes_\iota F)$ *for every* $\varphi \in \mathcal{D}_y$,

(6) $\vec{S}(\hat{x}-\hat{y}) \otimes \otimes_\iota \vec{T}(\hat{y})$ *is partially summable with respect to y*,

(7) $\vec{S}(\hat{y}) \otimes \otimes_\iota \vec{T}(\hat{x}-\hat{y})$ *is partially summable with respect to y*,

(8) $(\check{\vec{S}} * \varphi)(x) \otimes_\iota (\check{\vec{T}} * \psi)(x) \in (L^1)_x \mathcal{E}(E \otimes_\iota F)$ *for every* $\varphi, \psi \in \mathcal{D}$.

$\vec{S}$, $\vec{T}$ satisfy the above equivalent conditions if and only if $\vec{S} * \varphi$, $\vec{T}$ (resp. $\vec{S}$, $\vec{T} * \varphi$) satisfy also, for every $\varphi \in \mathcal{D}_x$.

Let $\vec{S}$, $\vec{T}$ satisfy the equivalent conditions (1)~(8). We define $\vec{S} *_\iota \vec{T}$ by the relation:

(9) $$<\varphi, (\vec{S} *_\iota \vec{T})> = \iint_{R^n \times R^n} \varphi(x+y)(\vec{S}_x \otimes \otimes_\iota \vec{T}_y)\, dxdy \text{ for every } \varphi \in \mathcal{D}_x.$$

$\vec{S} *_\iota \vec{T}$ is an $(E \otimes_\iota F)$-valued distribution (which we shall call the ι-convolution of $\vec{S}$ and $\vec{T}$). $\vec{S} *_\iota \vec{T}$ is written as the integrals: $\int_{R^n} \vec{S}(\hat{x}-y) \otimes \otimes_\iota \vec{T}(y) dy$, $\int_{R^n} \vec{S}(y) \otimes \otimes_\iota \vec{T}(\hat{x}-y)\, dy$, $\iint_{R^n \times R^n} \delta(\hat{z}-x-y)(\vec{S}_x \otimes \otimes_\iota \vec{T}_y)\, dxdy$, and $\vec{S} *_\iota \vec{T} \cdot \varphi\psi = \int (\check{\vec{S}} * \varphi)(x) \otimes_\iota (\check{\vec{T}} * \psi)(x)\, dx$ for every $\varphi, \psi \in \mathcal{D}$. And we have

(10) *$\vec{T} *_\iota \vec{S}$ can be defined and is equal to the image of $\vec{S} *_\iota \vec{T}$ under the canonical application: $E \otimes_\iota F \to F \otimes_\iota E$.*

(11) *If U is any element of $\mathcal{E}'_x$, then $(\vec{S} *_\iota \vec{T}) * U = (\vec{S} * U) *_\iota \vec{T} = \vec{S} *_\iota (\vec{T} * U)$.*

PROOF. Put $\vec{K} = \vec{S} \otimes \otimes_\iota \vec{T}$ and $G = E \otimes_\iota F$. With the aid of Lemma 3 we have

$$\int \varphi(x+\hat{y})\vec{K}(x, \hat{y})dx = [(\check{\vec{S}} * \varphi)\vec{T}]_\iota,$$

$$\int \varphi(\hat{x}+y)\vec{K}(\hat{x}, y)dy = [\vec{S}(\check{\vec{T}} * \varphi)]_\iota,$$

$$\int\varphi(x+y)(\vec{\breve{K}} * \psi)dx=(\check{\vec{S}} * \varphi)(y)\otimes_\iota(\vec{T} * \psi)(y),$$
$$\vec{K} *_x\varphi=\vec{S}*_x\varphi\otimes\otimes_\iota \vec{T}, \quad \vec{K} *_y\varphi=\vec{S}\otimes\otimes_\iota(\vec{T} * \varphi).$$

Then it follows from Prop. 9 that the conditions (1)~(7) are equivalent. As regards (8), the conditions are equivalent to

$$(8') \qquad (\check{\vec{S}} * \varphi)(y)\otimes_\iota(\vec{T} * \psi)(y) \in \mathcal{D}'_{L^1}(E\otimes_\iota F).$$

We can show that the left member of (8′) is distinguished in $\mathcal{D}'_{L^1}(E\otimes_\iota F)$. In fact, owing to the relation (1) (Remark after Lemma 3), the proof can be carried out along the similar line as in the proof of the implication (10)→(12) of Prop. 9. Therefore $(\check{\vec{S}} * \varphi)(y)\otimes_\iota(\vec{T} * \psi)(y) \in \mathcal{D}_{L^1}(E\otimes_\iota F)$. It follows since the injection $\mathcal{D}_{L^1}(E\otimes_\iota F)\to L^1\mathcal{E}(E\otimes_\iota F)$ is continuous that the implication (8′)→(8) holds. The converse is trivial.

That $\vec{S}$, $\vec{T}$ satisfy the equivalent conditions (1)~(8) if and only if $\vec{S} * \varphi$, $\vec{T}$ (resp. $\vec{S}$, $\vec{T} * \varphi$) satisfy also for any $\varphi \in \mathcal{D}$ is a consequence of the equivalence of (1), (8), (9) of Prop. 9.

Suppose $\vec{S}$, $\vec{T}$ satisfy the equivalent relations (1)~(8). $\vec{S} *_\iota \vec{T}$ is nothing but $\overset{*}{\vec{K}}$, so that the representations of $\vec{S} * \vec{T}$ by the integrals in our Theorem are obvious. (10) is almost evident. (11) follows from (15) of Prop. 9 if we consider $U\otimes\delta$ or $\delta\otimes U$ instead of T in (15) of Prop. 9. Thus the proof is completed.

This theorem is an extension of the result of Shiraishi [13] concerning the convolution of two scalar valued distributions.

Let $\vec{K}(\hat{x}, \hat{y})$ be any G-valued distribution on $R^n\times R^n$ satisfying the condition:

$$(1)' \qquad \varphi(\hat{x}+\hat{y})\vec{K}(\hat{x}, \hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}(G) \text{ for every } \varphi \in \mathcal{S}_x.$$

By Lemma 4 the application $\varphi\to\varphi(\hat{x}+\hat{y})\vec{K}(\hat{x}, \hat{y})$ of $\mathcal{S}_x$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ is continuous, so that the application $\overset{*}{\vec{K}}$ defined by the relation:

$$<\overset{*}{\vec{K}}, \varphi>=\iint_{R^n\times R^n}\varphi(x+y)\vec{K}(x, y)\, dxdy \quad \text{for every } \varphi \in \mathcal{S}_x,$$

is continuous from $\mathcal{S}_x$ into G, that is, $\overset{*}{\vec{K}}$ (called $\mathcal{S}'$-convolution) is an element of $\mathcal{S}'(G)$. We note that $\vec{K}(\hat{x}, \hat{y}) \in \mathcal{S}'_{x,y}(G)$. In fact, (1)′ implies that $\varphi(\hat{x}+\hat{y}) <\vec{K}(\hat{x}, \hat{y}), \vec{g}'> \in (\mathcal{D}'_{L^1})_{x,y}$ for every $\varphi \in \mathcal{S}_x$ and $\vec{g}' \in G'$, and in turn $<\vec{K}(\hat{x}, \hat{y}), \vec{g}'> \in \mathcal{S}'_{x,y}$ as remarked in § 2. Since $\mathcal{S}'$ has an (ε)-property ([11], p. 54), we have $\vec{K} \in \mathcal{S}'_{x,y}(G)$. In particular, if $\vec{K}(\hat{x}, \hat{y})$ is a tensor product $\vec{S}_x\otimes\otimes_\iota\vec{T}_y$, where $\vec{S} \in \mathcal{D}'(E)$, $\vec{T} \in \mathcal{D}'(F)$ and $\vec{S}\neq0$, $\vec{T}\neq0$, we can conclude that $\vec{S} \in \mathcal{S}'_x(\widehat{E})$ and $\vec{T} \in \mathcal{S}'_y(\widehat{F})$. For in this case $\vec{S}_x\otimes\otimes_\iota\vec{T}_y \in \mathcal{S}'_{x,y}(E\otimes_\iota F)$ implies that $<\vec{S}_x, \vec{e}'>\otimes<\vec{T}_y, \vec{f}'> \in \mathcal{S}'_{x,y}$ for every $\vec{e}' \in E'$ and $\vec{f}' \in F'$. In fact, $<\vec{S}_x\otimes\otimes_\iota\vec{T}_y \cdot u(x)\otimes v(y), \vec{e}'\otimes\vec{f}'>=<\vec{S}_x \cdot u, \vec{e}'><\vec{T}_y\cdot v, \vec{f}'>=(<\vec{S}_x, \vec{e}'> \cdot u)(<\vec{T}_y, \vec{f}'> \cdot v)=(<\vec{S}_x, \vec{e}'>\otimes<\vec{T}_y, \vec{f}'>)\cdot u\otimes v$ for every $u, v \in \mathcal{D}$, $\vec{e}' \in E'$, $\vec{f}' \in F'$, hence $<\vec{S}_x\otimes\otimes_\iota \vec{T}_y, \vec{e}'\otimes\vec{f}'>=<\vec{S}_x, \vec{e}>\otimes<\vec{T}_y, \vec{f}'>$. Since $\vec{T}\neq0$ by our assumption, there is an $\vec{f}'$ such that $<\vec{T}, \vec{f}'>\neq0$. Then it follows from a result of Shiraishi

([13], p. 27) that $<\vec{S}_x, \check{e}'> \in \mathscr{S}'$ for every $\check{e}' \in E'$. This implies $\vec{S} \in \mathscr{S}'_x(\hat{E})$, since $\mathscr{S}'$ has an (ε)-property. Similarly we have $\vec{T} \in \mathscr{S}'_y(\hat{F})$. Conversely if $\vec{S} \in \mathscr{S}'_x(E)$ and $\vec{T} \in \mathscr{S}'_y(F)$, then $\vec{S}\otimes\otimes_\iota\vec{T} \in \mathscr{S}'_{x,y}(E\otimes_\iota F)$. This follows from the equation:

$$(\vec{S}_x\otimes\otimes_\iota\vec{T}_y)\cdot u\otimes v=\vec{S}_x\cdot u\otimes\vec{T}_y\cdot v \text{ for every } u\in\mathscr{D}_x,\ v\in\mathscr{D}_y.$$

Suppose $\vec{K} \in \mathscr{S}'_{x,y}(G)$. This implies $\varphi(\check{x}+\check{y})\vec{K}(\check{x}, \check{y}) \in \mathscr{S}'_x((\mathscr{D}'_{L^1})_y(G))$ for every $\varphi \in \mathscr{S}_x$. $\varphi(\check{x}+\check{y})\alpha(\check{x}) \in \mathscr{S}_{x,y}$ for every $\alpha \in \mathscr{S}_y$. Then $\alpha(\check{x})\varphi(\check{x}+\check{y})T(\check{x}, \check{y}) \in (\mathscr{O}'_c)_{x,y}=(\mathscr{O}'_c)_x((\mathscr{O}'_c)_y)$ for every $T \in \mathscr{S}_{x,y}$ and in turn $\varphi(\check{x}+\check{y})T(\check{x}, \check{y}) \in \mathscr{S}'_x((\mathscr{O}'_c)_y)$ and *a fortiori* $\mathscr{S}'_x((\mathscr{D}'_{L^1})_y)$. It follows from this that the application $(\varphi, T)\to\varphi(\check{x}+\check{y})T(\check{x}, \check{y})$ of $\mathscr{S}_x\times\mathscr{S}'_{x,y}$ into $\mathscr{S}'_x((\mathscr{D}'_{L^1})_y)$ is hypocontinuous. From this remark we can conclude $\varphi(\check{x}+\check{y})\vec{K}(\check{x}, \check{y}) \in \mathscr{S}'_x((\mathscr{D}'_{L^1})_y(G))$.

With the aid of the remarks just mentioned, we can state and prove an analogue of Prop. 9. Here we mention some properties of $\mathscr{S}'$-convolutions without proof. For any $\vec{K} \in \mathscr{D}'_{x,y}(G)$, $\vec{K}(\check{x}-\check{y}, \check{y})$ (resp. $\vec{K}(\check{y}, \check{x}-\check{y})$) $\in (\mathscr{D}'_{L^1})_y(\mathscr{S}'_x(G))$ if and only if $\varphi(\check{x}+\check{y})\vec{K}(\check{x}, \check{y}) \in (\mathscr{D}'_{L^1})_{x,y}(G)$ for every $\varphi \in \mathscr{S}_x$. And if the $\mathscr{S}'$-convolution of $\vec{K}$ is defined, then this is defined also for $\vec{K} * U$ for every $U \in (\mathscr{O}'_c)_{x,y}$ and $(\vec{K} * U)^* = \overset{*}{\vec{K}} * \overset{*}{U}$. As regards the tensor product of vector valued distributions, if we let $\vec{S} \in \mathscr{S}'_x(E)$ and $\vec{T} \in \mathscr{S}'_y(F)$, then the analogous statements of Theorem will hold. The proof is not so difficult that we can omit it.

Next we turn to the simultaneous convolution of a $\vec{K}(\check{x}, \check{y}, \check{z}) \in \mathscr{D}'_{x,y,z}(G)$. Consider the condition:

$$(1)'' \qquad \varphi(\check{x}+\check{y}+\check{z})\vec{K}(\check{x}, \check{y}, \check{z}) \in (\mathscr{D}'_{L^1})_{x,y,z}(G) \text{ for every } \varphi \in \mathscr{D}_x.$$

If $\vec{K}$ satisfies (1)″, we define $\overset{*}{\vec{K}}$ (called the convolution of $\vec{K}$) by the relation:

$$<\overset{*}{\vec{K}}, \varphi>=\iiint_{R^n\times R^n\times R^n}\varphi(\check{x}+\check{y}+\check{z})\vec{K}(\check{x}, \check{y}, \check{z})\, dxdydz$$

$\overset{*}{\vec{K}}$ will be a G-valued distribution on R^n. We can state the conditions equivalent to (1)″ as in § 2.

PROPOSITION 10. *Let $\vec{K}(\check{x}, \check{y}, \check{z})$ be any G-valued distribution on $R^n\times R^n\times R^n$, for which the convolution $\overset{*}{\vec{K}}$ is defined. Then $\vec{K}(\check{x}, \check{y}, \check{z}) \in \mathscr{O}'_{x,y}(\mathscr{D}'_z(G))$ and if we denote by $\vec{H}(\check{x}, \check{z})=\int\vec{K}(\check{x}-y, y, \check{z})dy$, then $\vec{H} \in \mathscr{O}'_{x,z}(G)$ and moreover $\overset{*}{\vec{H}}=\overset{*}{\vec{K}}$.*

PROOF. By our hypothesis, $\vec{K}(\check{x}, \check{y}, \check{z}) \in \mathscr{O}'_{x,y,z}(G)$. It follows since the injection $\mathscr{O}'_{x,y,z}\to\mathscr{O}'_{x,y}(\mathscr{D}'_z)$ is continuous that $\vec{K}(\check{x}, \check{y}, \check{z}) \in \mathscr{O}'_{x,y}(\mathscr{D}'_z(G))$. Put $\vec{K}_1(\check{x}, \check{y})=\vec{K}(\check{x}, \check{y}, \check{z})$. Then $\vec{K}_1 \in \mathscr{O}'_{x,y}(\mathscr{D}'_z(G))$ so that $\overset{*}{\vec{K}}_1(\check{x}, \check{z})$ exists and equals $H(\check{x}, \check{z})$ defined in the Proposition. It is easy to see that $\vec{H}(\check{x}-\check{z},\check{z})=\int\vec{K}(\check{x}-y-\check{z}, y, \check{z})dy$. Since $\vec{K}(\check{x}, \check{y}, \check{z}) \in \mathscr{O}'_{x,y,z}(G)$ implies $\vec{K}(\check{x}-\check{y}-\check{z}, \check{y}, \check{z}) \in \mathscr{D}'_x((\mathscr{D}'_{L^1})_{y,z}(G))$, it follows from a theorem of Fubini ([11], p, 106) that $\overset{*}{\vec{H}}(\check{x}-\check{z}, \check{z}) \in \mathscr{D}'_x$

$((\mathcal{D}'_{L^1})_z(G))$ and $\overset{*}{\vec{K}}=\int\vec{H}(\hat{x}-z, z)dz$, that is $\overset{*}{\vec{H}}=\overset{*}{\vec{K}}$, as desired.

Let E, F and H be three locally convex Hausdorff spaces. $\vec{S}$ (resp. $\vec{T}$, $\vec{U}$) be any E-valued (resp. F-valued, H-valued) distribution on R^n. When $((\vec{S}\otimes\otimes\vec{T}\otimes\otimes\vec{U})_\iota^*$ is defined, we shall denote this convolution by $(\vec{S}*\vec{T}*\vec{U})_\iota$. The properties of the simultaneous convolution of scalar valued distributions were studied by Shiraishi [13]. The analogous statements for the case of vector valued distributions can be proved, as an example we shall show the associative property which is a generalization of a result of L. Schwartz ([12], p. 169).

Proposition 11. *Let the convolution $(\vec{S}*\vec{T}*\vec{U})_\iota$ be defined. If $\vec{U}\neq 0$, then $\vec{S}*_\iota\vec{T}$ and $(\vec{S}*_\iota\vec{T})*_\iota\vec{U}$ are defined and the latter is the image of $(\vec{S}*\vec{T}*\vec{U})_\iota$ under the canonical appilcation $j: (E\otimes F\otimes H)\widehat{}_\iota\to(E\hat{\otimes}_\iota F)\hat{\otimes}_\iota H$.*

Proof. First of all we show that $\vec{S}*_\iota\vec{T}$ is defined. For any $\overleftarrow{h}'\in H'$, we denote by $\theta_{\overleftarrow{h}'}$ the multilinear application $E\times F\times H\to E\hat{\otimes}_\iota F$ defined $\theta_{\overleftarrow{h}'}(\vec{e}, \vec{f}, \vec{h})=(\vec{e}\otimes\vec{f})<\vec{h}, \overleftarrow{h}'>$, $\vec{e}\in E$, $\vec{f}\in F$, $\vec{h}\in H$. Clearly $\theta_{\overleftarrow{h}'}$ is separately continuous. Hence $\theta_{\overleftarrow{h}'}$ is uniquely extended to a linear continuous applicaion $\tilde{\theta}_{\overleftarrow{h}'}$: $(E\otimes F\otimes H)\widehat{}_\iota\to E\hat{\otimes}_\iota F$. It is easy to verify the equation:

$$(1)\qquad (I\otimes\tilde{\theta}_{\overleftarrow{h}'})(\vec{S}\otimes\otimes\vec{T}\otimes\otimes\vec{U})_\iota=(\vec{S}\otimes\otimes_\iota\vec{T})\otimes<\vec{U}, \overleftarrow{h}'>\in\mathcal{O}'_{x,y,z}(E\hat{\otimes}_\iota F),$$

where I is the identical application of $\mathcal{D}'_{x,y,z}$. It follows from (1) that $(\vec{S}\otimes\otimes_\iota\vec{T})<\vec{U}\cdot u, \overleftarrow{h}'>\in\mathcal{O}'_{x,y}(E\hat{\otimes}_\iota F)$ for any $u\in\mathcal{D}$, since the injection $\mathcal{O}'_{x,y,z}\to\mathcal{O}'_{x,y}(\mathcal{D}'_z)$ is continuous. We can choose u and $\overleftarrow{h}'$ such that $<\vec{U}\cdot u, \overleftarrow{h}'>=1$. Hence $\vec{S}*_\iota\vec{T}$ is defined.

We put $\vec{K}(\hat{x}, \hat{y}, \hat{z})=(\vec{S}\otimes\otimes\vec{T}\otimes\otimes\vec{U})_\iota$ and $\vec{H}(\hat{x}, \hat{z})=\int\vec{K}(\hat{x}-y, y, \hat{z})dy$. If we can show that

$$(2)\qquad (I\otimes j)\vec{H}(\hat{x}, \hat{z})=(\vec{S}*_\iota\vec{T})\otimes\otimes_\iota\vec{U}\in\mathcal{O}'_{x,z}((E\hat{\otimes}_\iota F)\hat{\otimes}_\iota H),$$

we shall have $(I\otimes j)\overset{*}{\vec{K}}=(I\otimes j)\overset{*}{\vec{H}}=(\vec{S}*_\iota\vec{T})*_\iota\vec{U}$ by Prop. 10. Hence the proof will be through.

Now, for any $\varphi\in\mathcal{D}_x$, $v\in\mathcal{D}_y$, $u\in\mathcal{D}_z$

$$(3)\qquad (I\otimes j)\vec{K}(\hat{x}-\hat{y}, \hat{y}, \hat{z})\cdot\varphi(\hat{x})v(\hat{y})u(\hat{z})=\int[(\check{\vec{S}}*\varphi)\vec{T}]_\iota(y)v(y)dy\otimes\vec{U}\cdot u.$$

Indeed, the left membər of (3) equals

$$\begin{aligned}&(I\otimes j)\vec{K}(\hat{x}, \hat{y}, \hat{z})\cdot\varphi(\hat{x}+\hat{y})v(\hat{y})u(\hat{z})=(\vec{S}\otimes\otimes_\iota\vec{T})\otimes\otimes_\iota\vec{U}\cdot\varphi(\hat{x}+\hat{y})v(\hat{y})u(\hat{z})\\&=(\vec{S}\otimes\otimes_\iota\vec{T})\cdot\varphi(x+y)v(y)\otimes\vec{U}\cdot u=\int[(\check{\vec{S}}*\varphi)\vec{T}]_\iota(y)v(y)dy\otimes\vec{U}\cdot u.\end{aligned}$$

Let v tend to 1 in $\mathcal{B}_c$. The right member of (3) tends to $\int[(\check{\vec{S}}*\varphi)\vec{T}]_\iota(y)dy\otimes\vec{U}\cdot u$, that is, $(\vec{S} *_\iota \vec{T}) \otimes \otimes_\iota \vec{U} \cdot \varphi u$. On the other hand

$$(I \otimes j)\vec{K}(\hat{x}-\hat{y},\ \hat{y},\ \hat{z}) \cdot \varphi(\hat{x})v(\hat{y})u(\hat{z}) = \varphi(\hat{x})u(\hat{z}) \cdot \int v(y)(I \otimes j)K(\hat{x}-y,\ y,\ \hat{z})dy.$$

Since $(I \otimes j)\vec{K}(\hat{x}-\hat{y},\ \hat{y},\ \hat{z}) \in (\mathcal{D}'_{L^1})_y(\mathcal{D}'_{x,z}((E \otimes_\iota F) \otimes_\iota H))$, it follows that $\int v(y)(I \otimes j)$ $\vec{K}(\hat{x}-y,\ y,\ \hat{z})dy$ tends to $(I \otimes j)\vec{H}(\hat{x},\ \hat{z})$ in $\mathcal{D}'_{x,z}((E \otimes_\iota F) \otimes_\iota H)$ when v tends to 1 in $\mathcal{B}_c$. Thus the left member of (3) tends to $(I \otimes j)\vec{H}(\hat{x},\ \hat{z}) \cdot \varphi u$. Since $\vec{H}(\hat{x},\ \hat{z}) \in \mathcal{O}'_{x,z}((E \otimes_\iota F) \otimes_\iota H)$. Therefore we have (2), completing the proof.

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