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Classification of spherical tilings by congruent quadrangles over pseudo-double wheels (II)—the isohedral case

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Abstract. We classify all edge-to-edge spherical isohedral 4-gonal tilings such that the skeletons are pseudo-double wheels. For this, we characterize these spherical tilings by a quadratic equation for the cosine of an edge-length. By the classification, we see: there are indeed two non-congruent, edge-to-edge spherical isohedral 4-gonal tilings such that the skeletons are the same pseudo-double wheel and the cyclic list of the four inner angles of the tiles are the same. This contrasts with that every edge-to-edge spherical tiling by congruent 3-gons is determined by the skeleton and the inner angles of the skeleton. We show that for a particular spherical isohedral tiling over the pseudo-double wheel of twelve faces, the quadratic equation has a double solution and the copies of the tile also organize a spherical non-isohedral tiling over the same skeleton.

1. Introduction

Throughout this paper, we are concerned with edge-to-edge tilings. A tiling $\mathscr{T}$ is called *isohedral* (or *tile-transitive*), if for any pair of tiles of $\mathscr{T}$, there is a symmetry operation of $\mathscr{T}$ that transforms one tile to the other. In characterizing the *skeletons* of spherical (isohedral) tilings, an important graph is a *pseudo-double wheel* (the dual graph of the skeleton of an antiprism [6, p. 19]. See Figure 1 (above)). It satisfies the following:

- The skeletons of spherical tilings by spherical 4-gons are generated from pseudo-double wheels by means of applications of two local expansions [4].
- The skeletons of spherical isohedral tilings consist of pseudo-double wheels, an infinite series of graphs, and eighteen sporadic graphs [7].

In Section 3, we prove: for every spherical tiling $\mathscr{T}$ by congruent spherical 4-gons with the skeleton being a pseudo-double wheel G, $\mathscr{T}$ is isohedral if and only if every graph automorphism [6, Sect. 1.1] of G respects the edge-lengths and inner angles of $\mathscr{T}$.

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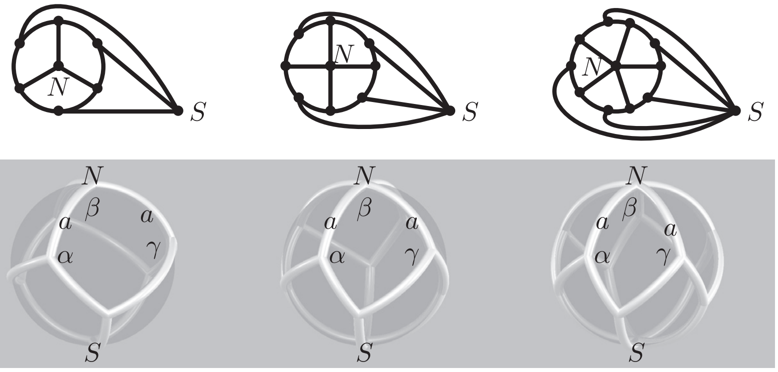

Fig. 1. The above are pseudo-double wheels of $2n$ faces ($n = 3, 4, 5$). The below are spherical isohedral tilings by $2n$ congruent quadrangles such that the skeletons are pseudo-double wheels ($n = 3, 4, 5$). It holds that $(\cos a)^2 - \cot(\pi/n)(\cot\alpha + \cot\gamma)\cos a - \cot\alpha\cot\gamma = 0$.

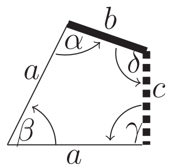

Fig. 2. The notation for angles and edges of the quadrangular tile. Some among α, β, γ, δ are equal, and some among a, b, c are equal.

In any spherical tiling by congruent quadrangles, the tile has a pair of adjacent, equilateral edges [10]. For spherical tilings by congruent quadrangles T with the skeleton being pseudo-double wheels, fix the notation for the angles and the edges of the quadrangular tile T as Figure 2. For the spherical *isohedral* tilings such that the skeletons are pseudo-double wheels, we bijectively parameterize the tiles with the pair of the edge-length a of the tile and a typical angle, in Section 4. Then we characterize the tiles as follows (Section 5). Given a spherical 4-gon T such that α, β, γ are adjacent inner angles and β is an inner angle between two edges of length a. T is a tile of some spherical *isohedral* tiling $\mathscr{T}$ by $2n$ congruent spherical 4-gons with the skeleton of $\mathscr{T}$ being a pseudo-double wheel, if and only if

$$(\cos a)^2 - \cot\frac{\pi}{n}(\cot\alpha + \cot\gamma)\cos a - \cot\alpha\cot\gamma = 0.$$

For notations, see Figure 1 (below). In Section 6, by solving this equation, we classify all the tiles of spherical isohedral tilings such that the skeleton of the tilings are pseudo-double wheels. By the classification, we see: there are indeed two non-congruent, edge-to-edge spherical isohedral 4-gonal tilings such that the skeletons are the same pseudo-double wheel and the cyclic list of the four inner angles of the tiles are the same. This contrasts with that every edge-to-edge spherical tiling by congruent 3-gons is determined by the skeleton and the inner angles of the skeleton [11]. In Section 7, we show that for a particular spherical isohedral tiling over the pseudo-double wheel of twelve faces, the quadratic equation has a double solution. Moreover, the copies of the tile also organize a less symmetric, spherical non-isohedral tiling $\mathscr{T}$ over the same skeleton. Based on this tiling $\mathscr{T}$ and Grünbaum-Shephard's characterization theorem [7] of the skeletons of spherical isohedral tilings, we briefly discuss our classification of spherical isohedral tilings over pseudo-double wheels.

2. Basic definitions

By a *spherical* 4-*gon*, we mean a topological disk T on the two-dimensional unit sphere $\mathbf{S}^2$ such that T is circumscribed by four straight edges, (1) any inner angle between adjacent edges of T is strictly between 0 and 2π but not π, and (2) T is contained in the interior of a hemisphere. By "quadrangle," we mean a "spherical 4-gon." The congruence on the sphere is just the orthogonal transformation, and "sphere" (and "spherical") means the two-dimensional unit sphere $\mathbf{S}^2$. We identify spherical tilings modulo a special orthogonal group $SO(3)$.

Definition 1 (pseudo-double wheel [4]). For an even number $F \geq 6$, a *pseudo-double wheel* of F faces is a map such that

- the graph is obtained from a cycle $(v_0, v_1, v_2, \ldots, v_{F-1})$, by adjoining a new vertex N to each v_{2i} $(0 \leq i < F/2)$ and then by adjoining a new vertex S to each v_{2i+1} $(0 \leq i < F/2)$. We identify the suffix i of the vertex v_i modulo F.
- The cyclic order at the vertex N is defined as follows: the edge Nv_{2i+2} is next to the edge Nv_{2i}. The cyclic order at the vertex v_{2i} $(0 \leq i \leq F/2)$ is: the edge $v_{2i}N$ is next to the edge $v_{2i}v_{2i+1}$, and $v_{2i}v_{2i+1}$ is next to the edge $v_{2i}v_{2i-1}$. The cyclic order at the vertex S is: the edge Sv_{2i-1} is next to the edge Sv_{2i+1}. The cyclic order at the vertex v_{2i+1} $(0 \leq i < F/2)$ is: the edge $v_{2i+1}S$ is next to the edge $v_{2i+1}v_{2i}$, and $v_{2i+1}v_{2i}$ is next to the edge $v_{2i+1}v_{2i+2}$.

The skeleton of the cube is the pseudo-double wheel of six faces.

In the rest of this paper, we fix an orientation of the sphere. By $\angle PQR$, we mean the angle from PQ to RQ in the orientation of the sphere, and assume that (1) $\alpha, \beta, \gamma, \delta \in (0, \pi) \cup (\pi, 2\pi)$, *and* $a, b, c \in (0, \pi)$, *and* (2) *for tiles, edges represented by solid (thick, dotted, resp.) lines have length a (b, c, resp.).* We say a

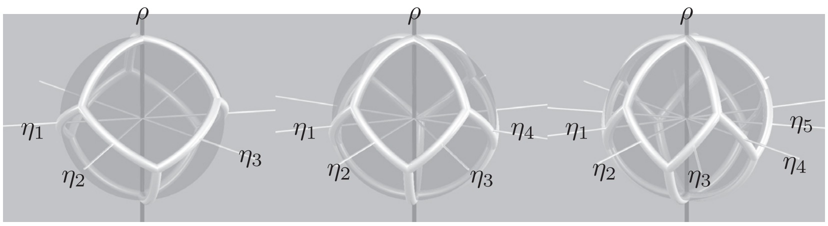

Fig. 3. Spherical tilings by $2n$ congruent quadrangles such that the skeletons are pseudo-double wheels ($n = 3, 4, 5$). Each is isohedral, as any tile is transformed to any tile with the vertical n-fold axis ρ and n horizontal 2-fold axes $\eta_1, \dots, \eta_n$.

quadrangle is *concave*, if it has an inner angle greater than π. We are concerned with all spherical isohedral tilings by congruent possibly concave quadrangles such that the skeletons are pseudo-double wheels.

Proposition 1 ([2, p. 62]). (1) *If* $0 < A, B, C < \pi$, $A + B + C > \pi$, $-A + B + C < \pi$, $A - B + C < \pi$ *and* $A + B - C < \pi$, *then there exists* uniquely *up to congruence a spherical* 3-*gon on the two-dimensional unit sphere* $\mathbf{S}^2$ *such that the inner angles are* A, B *and* C. *The converse is also true.*

(2) *Let* ABC *be a spherical* 3-*gon, and let* a, b, c *be the sides opposite to the inner angles* A, B, C, *respectively. Then*
   (a) *(Dual cosine law for the sphere (Spherical cosine theorem for angles)* [2, p. 65]*)* $\cos A = -\cos B \cos C + \sin B \sin C \cos a$.
   (b) *(Cosine law for the sphere (Spherical cosine theorem)* [2, p. 65]*)* $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

Spherical cosine law is obtained from the spherical cosine theorem for angles, by exchanging the angles A, B, C and the sides a, b, c with $A \leftrightarrow \pi - a$, $B \leftrightarrow \pi - b$, $C \leftrightarrow \pi - c$. By this exchange, the last three inequalities of Proposition 1 (1) become the distance inequalities for spherical 3-gons. For every nonzero real number x, $\operatorname{arccot} x$ is the angle θ such that $0 < |\theta| < \pi/2$ and $\cot \theta = x$. Let $\csc x$ be $1/\sin x$. We say a spherical 4-gon Q is a *copy* of a spherical 4-gon Q', if Q is an orthogonal transformation of Q'.

3. Combinatorial conditions for spherical monohedral quadrangular tilings to be isohedral tilings over pseudo-double wheels

Definition 2. Let PDW_n $(n \geq 3)$ be the set of spherical tilings by $2n$ congruent, possibly concave quadrangles such that (1) the skeleton is a pseudo-double wheel, and (2) the distribution of inner angles and that of the edge-length on the skeleton are as in Figure 4.

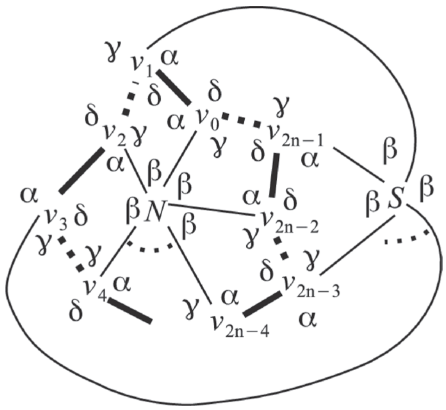

Fig. 4. The solid, the thick, and the dotted thick edges have lengths a, b and c. The vertices N and S are n-valent. Some among α, β, γ, δ are equal, and some among a, b, c are equal.

Note that we do not assume the isohedrality in Definition 2.

For example, all images of Figure 1 and Figure 3 are members of PDW_n $(n = 3, 4, 5)$. The leftmost of Figure 3 is so-called *the central projection* of the cube. They have the vertical n-fold axis ρ of rotation, and n horizontal 2-fold axes η_i $(i = 0, 1, \dots, n-1)$ of rotation such that for each i, η_i is through the midpoint of an edge and $\eta_i \perp \rho$. By these symmetry operations, any tile is transformed to any tile in each tiling. So, they are isohedral.

The two vertices N and S of any tiling $\mathscr{T}$ presented in Figure 4 can be identified with the north pole and the south pole of the unit sphere $\mathbf{S}^2$ respectively, as there are two congruent paths from N to S. For each point V $(\neq N, S)$ on $\mathbf{S}^2$, the *longitude* of V is the angle $\psi \in [-\pi, \pi)$ from the edge Nv_0 to a geodesic segment NV, measured in the direction indicated in Figure 7.

Proposition 2 ([1, Lemma 5]). *Given a spherical tiling by congruent quadrangles such that the quadrangles are as in Figure* 2 *with the edge-length c being the edge-length a. Suppose that* (1) *there is a vertex incident to only three edges of length a, and* (2) *there is a* 3*-valent vertex incident to two edges of length a and to one edge of length b. Then for the inner angles of the tile, we have* $\alpha \neq \delta$ *and* $\beta \neq \gamma$.

Theorem 1. *Let* $\mathscr{T}$ *be a spherical tiling by six congruent quadrangles.*

(1) $\mathscr{T} \in PDW_3$.

(2) $\mathscr{T}$ *has a* 3*-fold axis* ρ *of rotation and three* 2*-fold axes* η_1, η_2, η_3 *of rotation perpendicular to* ρ.

(3) $\mathscr{T}$ *is isohedral.*

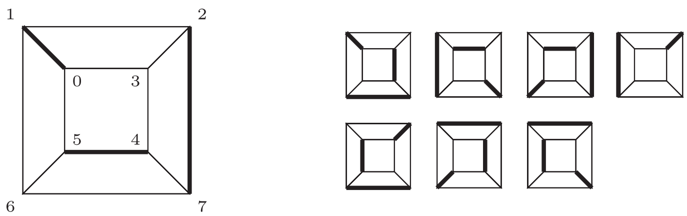

Fig. 5. The eight perfect face-matchings of the skeleton of the cube.

Proof. (1). Sakano proved this assertion by case analysis [8]. We improve the presentation of his proof, by using Proposition 2, [4], and [3, Theorem 8]. By Euler's theorem, every spherical tiling by 4-gons has a 3-valent vertex [10]. From this, we can prove that the skeleton of any spherical tiling by six 4-gons is the skeleton of the cube (By the enumeration of spherical quadrangulations [4], there is only one spherical quadrangulation of eight vertices). Moreover, the cyclic list of edge-lengths of the tile of a spherical tiling by congruent 4-gons is either *aaaa*, *aabb*, *aaab*, or *aabc* where a, b, c are mutually distinct ([10]). When the cyclic list of edge-lengths of the tile of $\mathscr{T}$ is *aaaa* or *aabb*, then $\mathscr{T} \in PDW_3$, by [9].

Let the edge-lengths of the tile be *aaab* ($a \neq b$). Then the spherical tiling by six congruent 4-gons induces a perfect face-matching consisting of three edges of length b. By a *perfect face-matching* of a graph, we mean a perfect matching [6, p. 2] of the dual graph. By [3, Theorem 8], the skeleton of the cube has eight perfect face-matchings.

In Figure 5, the first perfect face-matching is transformed to the other seven perfect face-matchings by seven automorphisms of the skeleton of the cube. Enumerate the vertices v_i ($0 \leq i \leq 7$) of the cube, as in the figure of the first perfect face-matching. The seven automorphisms are represented as seven permutations $(2\ 6)(3\ 5)$, $(0\ 4)(1\ 7)(2\ 6)(3\ 5)$, $(0\ 5)(1\ 6)(2\ 7)(3\ 4)$, $(0\ 3)(1\ 2) \cdot (4\ 5)(6\ 7)$, $(0\ 3\ 4\ 5)(1\ 2\ 7\ 6)$, $(0\ 5\ 4\ 3)(1\ 6\ 7\ 2)$, and $(0\ 4)(1\ 7)$. So we have only to consider the first perfect face-matching. Every inner angle around the vertex v_3 or the vertex v_6 is β or γ, by Figure 2, because v_3 and v_6 are incident to only edges of length a. The number of inner angles β around v_3, say k, is the number of inner angles β around v_6. Otherwise, $\beta = \gamma$, which contradicts Proposition 2.

We will prove $k \neq 2$. Suppose $k = 2$. Without loss of generality, $\angle v_1 v_6 v_7 = \gamma$, $\angle v_7 v_6 v_5 = \angle v_5 v_6 v_1 = \beta$, because the automorphism $(1\ 5\ 7)(0\ 4\ 2)$ of the skeleton of the cube fixes the face-matching edges $v_0 v_1$, $v_4 v_5$ and $v_2 v_7$. Then $\angle v_0 v_5 v_6 = \gamma$, $\angle v_6 v_5 v_4 = \alpha$ and $\angle v_5 v_4 v_7 = \delta$. Here $\angle v_4 v_5 v_0 = \alpha$ or δ. Assume $\angle v_4 v_5 v_0 = \alpha$. Then an opposite inner angle $\angle v_0 v_3 v_4$ is γ. Hence $\angle v_4 v_3 v_2 = \beta$. So the inner angle $\angle v_7 v_4 v_3$ is γ. The three inner angles around

v_4 are γ, δ, X for some $X \in \{\alpha, \delta\}$, while the three inner angles around v_5 are α, α, γ. So $\alpha = \delta$. This contradicts Proposition 2. Hence $\angle v_4 v_5 v_0 = \delta$. Then $\angle v_3 v_4 v_5 = \alpha$. Here $\angle v_7 v_4 v_3 = \beta$ or γ. Assume $\angle v_7 v_4 v_3 = \beta$. Then the three inner angles around v_4 are α, β, δ and those around v_5 are α, γ, δ. So $\beta = \gamma$. This contradicts Proposition 2. Thus $\angle v_7 v_4 v_3 = \gamma$. Hence $\angle v_4 v_3 v_2 = \beta$. $\angle v_2 v_3 v_0 = \gamma$, because the three inner angles around v_3 are β, β, γ. Thus $\angle v_3 v_0 v_1 = \delta$. As $\angle v_5 v_6 v_1 = \beta$, an opposite inner angle $\angle v_1 v_0 v_5$ is δ. Hence the three inner angles around v_0 are γ, δ, δ. On the other hand, those around v_4 are α, γ, δ. So $\alpha = \delta$. This contradicts Proposition 2. Thus, the number k of β around v_3 is not two.

In a similar argument, $k \neq 1$. If $k = 3$, then $\mathcal{T} \in PDW_3$. Otherwise, $k = 0$. But, because the cyclic of the tile is $aaab$ $(a \neq b)$, we have the symmetry $(\alpha, \beta, \gamma, \delta) \leftrightarrow (\delta, \gamma, \beta, \alpha)$. Hence, we have $\mathcal{T} \in PDW_3$, too.

Suppose the cyclic list of the edge-lengths of the tile is $aabc$ with a, b, c are mutually distinct. The distribution of the edges of length b is the first perfect face-matching of Figure 5 without loss of generality. Since every edge of length c should be adjacent to an edge of length b and each face has exactly one edge of length c, the tiling is Figure 4 with $n = 3$. So, $\mathcal{T} \in PDW_3$.

(2). In $\mathcal{T}$, two vertices consisting of three inner angles β are antipodal to each other, because there are three congruent paths between them: "travel straight a, bend in γ angle, travel straight c, bend in $-\gamma$ angle, and travel straight a." Actually there is a 3-fold axis ρ of rotation through the two vertices, by examining the distribution of α, β, γ and the edge-lengths a, b, c. ρ is the black vertical axis in Figure 3 (left). Moreover the midpoint of an edge e of length b is antipodal to the midpoint of the edge e' of length c, where e is not adjacent to e'. It is because there are two congruent paths between them: one is "travel straight $b/2$, bend in δ angle, travel straight a, bend in $-\alpha$ angle, travel straight a, bend in β angle, travel straight $c/2$." The other path is the same with the three angles inverted. Actually an axis through the two midpoints is a 2-fold axis of rotation by examining the distribution of α, β, γ, a, b, c. Similarly we can find three 2-fold axes η_1, η_2, η_3 of rotation. Each η_i is a white horizontal axis in Figure 3 (left).

(3). Let T and T' be tiles of $\mathcal{T}$. Let ρ be the vertical 3-fold axis of rotation and η_i $(i = 1, 2, 3)$ be the horizontal 2-fold axes of rotation, given in (2). If ρ is through a point of $T \cap T'$, then T is transformed to T' by a rotation around the 3-fold axis ρ. Otherwise, if T and T' are adjacent, then T is transformed to T' by some 2-fold axis η_i that is through an edge $T \cap T'$ of length b or c. By repeating these transformations, any tile T is transformed to any other tile T'. So $\mathcal{T}$ is isohedral. This completes the proof of Theorem 1.

□

For spherical tilings by congruent quadrangles, Theorem 2 below provides two necessary and sufficient conditions for spherical tilings such that the skeletons are pseudo-double wheels to be isohedral. The two conditions are somehow combinatorial, and come from those given in Theorem 1. One is being PDW_n, and the other is a condition on the symmetry operations of tilings.

As in [9], a *kite* (*dart*, resp.) is a convex (non-convex, resp.) quadrangle such that the cyclic list of edge-lengths is *aabb* $(a \neq b)$, and a *rhombus* is a quadrangle such that all the edges are equilateral. A kite, a dart and a rhombus enjoy a mirror symmetry.

Lemma 1. *Let $\mathscr{T}$ be a spherical tiling by congruent polygons such that any edge is incident to an odd-valent vertex. If the tile does not have a mirror symmetry, then neither does $\mathscr{T}$.*

Proof. Assume $\mathscr{T}$ has a mirror plane σ. Then σ does not intersect transversely with a tile, since the tile does not have a mirror symmetry. Thus the intersection of σ and $\mathscr{T}$ is the cycle of the edges, because each edge of $\mathscr{T}$ is straight. By σ, each vertex on the cycle has even degree. But all the edges of the tiling $\mathscr{T}$ is incident to an odd-valent vertex. This is a contradiction. This completes the proof of Lemma 1. □

Theorem 2. *For any spherical tiling $\mathscr{T}$ by $2n$ congruent quadrangles $(n \geq 4)$, the following three conditions are equivalent:*
(1) *$\mathscr{T} \in PDW_n$.*
(2) *$\mathscr{T}$ has an n-fold axis ρ of rotation and n 2-fold axes of rotation perpendicular to ρ.*
(3) *$\mathscr{T}$ is isohedral and the skeleton is the pseudo-double wheel of $2n$ faces.*

Proof. ((1) ⇒ (3)) By condition (1), we compute the longitude and the latitude (i.e., the length of the geodesic segment from the north pole) of the vertices v_i's of $\mathscr{T}$. There is an n-fold axis ρ of rotation through the two poles N and S, because there are three congruent paths between them. We see that there is a 2-fold axis ℓ_i of rotation though the midpoint of the edge $v_i v_{i+1}$ and the midpoint of the edge $v_{(i+n \bmod 2n)} v_{(i+1+n \bmod 2n)}$ and that ℓ_i is perpendicular to ρ, for every i. So we have condition (2). By this and Figure 4, we have condition (3).

((2) ⇒ (1)) We verify:

Claim 1. *If $m \geq n \geq 4$, any m-fold axis of rotation of $\mathscr{T}$ is through two vertices.*

Proof. The m-fold axis is not though the midpoint of an edge, by $m \neq 2$. The m-fold axis is not through an inner point of a tile. Otherwise all inner angles of the tile is equal, all edges are equilateral, $m = 4$. By the premise, $m = n = 4$. As the tile is a regular quadrangle, all diagonal segments of the tiles are less than π. Otherwise any pair of incident diagonal segments crosses to each other. By drawing exactly one diagonal, geodesic segment in each quadrangular tile, we have a spherical tiling $\mathcal{T}'$ by $2n \times 2 = 16$ congruent isosceles spherical 3-gons. The inner angles of the isosceles 3-gons are $5\pi/8$, $5\pi/16$, $5\pi/16$. It is because the area of the quadrangular tile of the given tiling $\mathcal{T}$ is $\pi/2$, and the sum of the four equal inner angles is $5\pi/2$. However $\mathcal{T}'$ is impossible, by the classification of all spherical tilings by congruent spherical 3-gons [11, Table]. Thus the axis is though a pair of antipodal vertices. This completes the proof of Claim 1. □

By Claim 1, the n-fold axis ρ of rotation is through two vertices u and v. Both u and v are n-valent. Otherwise, for some positive integers k, n, kn equilateral edges are incident to u, and ℓn equilateral edges are incident to v, because of the n-fold axis of rotation through u and v. The kn pairs of neighboring edges incident to the vertex u cause kn distinct tiles. Since the number of tiles is $2n$, both of k and ℓ are one or two. Let $k = 2$. Then the kn pairs of neighboring edges incident to the vertex u cause already $2n$ tiles. Then v is not a vertex of any of these $2n$ tiles. To see it, assume some tile contains v as a vertex. No vertex is adjacent to both u and v. Otherwise the inner angle is π. u is not adjacent to v, since the length of any edge is less than π. So if two vertices of a tile of $\mathcal{T}$ are incident to u, then some vertex other than v is incident to them, because the tile is a quadrangle. Thus the number of tiles is greater than $2n$. So $k = \ell = 1$. Hence there are exactly n vertices w_i $(0 \le i \le n-1)$ adjacent to u. All edges uw_i's are equilateral by the n-fold axis of rotation through u and v. We assume that $uw_{i+1 \bmod n}$ is next to uw_i, and that the two vertices $w_{i+1 \bmod n}$ and w_i are adjacent to a vertex v_i. Let T_i be a tile $uw_iv_1w_{i+1 \bmod n}$. By the n-fold axis ρ, all v_i's are distinct.

Claim 2. *v_i is adjacent to v $(0 \le i < n)$.*

Proof. Otherwise, there is a non-pole vertex $\tilde{u}_i$ adjacent to v_i such that an edge $v_i\tilde{u}_i$ is a neighbor of $v_iw_{i+1 \bmod n}$ without loss of generality. Then there is a quadrangular tile T_i' having the three vertices $\tilde{u}_i$, v_i, $w_{i+1 \bmod n}$.

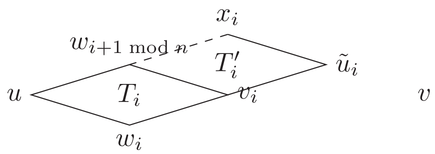

Fig. 6. Proof of Claim 2.

The other vertex, say x_i, of the tile T_i' is not the vertex v. Otherwise, an edge $w_{i+1 \bmod n}x_i$ is transformed to an edge w_iv by the n-fold axis ρ of rotation of $\mathscr{T}$. The edge w_iv cannot be transversal to the edge $v_i\tilde{u}_i$. Hence, the lune (that is, digon) determined by the two edges uw_i and $uw_{i+1 \bmod n}$ contains a tile other than $T_i = uw_iv_iw_{i+1 \bmod n}$ and $T_i' = v_i\tilde{u}_ivw_{i+1 \bmod n}$. Thus $\mathscr{T}$ has more than $2n$ number of tiles, which is absurd. So $x_i \neq v$.

The vertex x_i of the tile T_i' is not the vertex u. Otherwise, an edge $x_i\tilde{u}_i$ has the same length as uw_i and $uw_{i+1 \bmod n}$. Since the edge $uw_{i+1 \bmod n}$ is a neighbor of the edge uw_i, the edge $x_i\tilde{u}_i$ is a neighbor of uw_i or $uw_{i+1 \bmod n}$. Consider the latter case. By the n-fold axis ρ through u and v, the tile $T_i = uw_iv_iw_{i+1 \bmod n}$ is rotated to $x_iw_{i+1 \bmod n}v_i\tilde{u}_i$. But this is impossible because the vertices w_i and $w_{i+1 \bmod n}$ of the former tile $T_i = uw_iv_iw_{i+1 \bmod n}$ and the vertices $w_{i+1 \bmod n}$ and $\tilde{u}_i$ of the latter tile $T_i' = x_iw_{i+1 \bmod n}v_i\tilde{u}_i$ are all incident to the vertex v_i. So $w_{i+1 \bmod n}$ becomes two-valent. When the edge $x_i\tilde{u}_i$ is a neighbor of uw_i, we have similarly a contradiction. Thus $x_i \neq u$.

Any vertex of the tile $T_i' = w_{i+1 \bmod n}v_i\tilde{u}_ix_i$ is neither the n-valent u nor the n-valent v, so the number of the tiles of the tiling $\mathscr{T}$ is greater than $2n$. This is absurd. Thus the vertex v_i is adjacent to v. This establishes Claim 2. □

By Claim 2, the skeleton of $\mathscr{T}$ is the pseudo-double wheel of $2n$ faces. By the n-fold axis ρ through u and v, all n edges incident to the vertex u have the same length a, and all n edges incident to the vertex v have the same length a'.

By the assumption, $\mathscr{T}$ has n horizontal 2-fold rotation axes, each through a pair of midpoints of edges. As they swap the vertices u and v, we have $a = a'$. By computing the longitude and the latitude of each non-pole vertices, the angle-assignment and the length-assignment of $\mathscr{T}$ is exactly as in Figure 4.

((3) $\Rightarrow$ (2)) Suppose that the tile of $\mathscr{T}$ is a rhombus, a kite, or a dart. By the classification of spherical monohedral (kite/dart/rhombus)-faced tilings [9, Table 1], the Schönflies symbol ([5], [6]) of $\mathscr{T}$ is D_{nd}. In the decision tree [5, Fig. 3.10], by going from the leaf "D_{nd}" to the root, we see that D_{nd} must have "n C_2's $\perp$ to C_n" (n 2-fold axes of rotation perpendicular to an n-fold axis of rotation). Thus (2) holds. By the same reasoning, the Schönflies symbol D_n requires (2). So, to complete the proof of ((3) $\Rightarrow$ (2)), we show: if the tile of $\mathscr{T}$ is none of a kite, a dart and a rhombus, then $\mathscr{T}$ has the Schönflies symbol D_n.

The Schönflies symbol of $\mathscr{T}$ is none of T, T_d, T_h, O, O_h, I, and I_h. Otherwise, the tiling $\mathscr{T}$ has more than three 3-fold rotation axes, by [5, Sect. 3.14]. Since the skeleton of $\mathscr{T}$ is the pseudo-double wheel of $2n$ faces ($n \geq 4$), $\mathscr{T}$ has only two vertices N and S of valence more than three. So there is a

3-fold axis ρ of rotation through a three-valent vertex. Thus the rotation in $2\pi/3$ around ρ transforms N (S, resp.) to S (N, resp.), or fixes both of N and S. So the rotation in $4\pi/3$ around ρ fixes both of N and S. This is absurd, since the 3-fold rotation axis is through neither N nor S.

In any pseudo-double wheel, any edge is incident to an odd-valent vertex. Because we assumed that the tile of the tiling $\mathscr{T}$ on a pseudo-double wheel is none of a rhombus, a kite, and a dart, the tile has no mirror symmetry. By Lemma 1, $\mathscr{T}$ has no mirror symmetry.

So the Schönflies symbol of the tiling $\mathscr{T}$ is C_m or D_m for some integer $m \geq 2$. This is due to the systematic procedure to determine the Schönflies symbol [5, Sect. 3.14]. Then the tiling $\mathscr{T}$ has an m-fold axis ρ of rotation. Let G be the symmetry group of the tiling $\mathscr{T}$. Because the tiling $\mathscr{T}$ is isohedral, G acts transitively on the tiles of $\mathscr{T}$. So

(#) the order $\#G$ is a multiple of the number $2n \geq 8$ of tiles of $\mathscr{T}$.

Assume the Schönflies symbol of $\mathscr{T}$ is C_m for some $m \geq 2$. By [5, p. 41], $\#G = m$. By (#), $m \geq 8$. ρ is through a vertex with the valence being a multiple of m. So the m-fold axis ρ of rotation is through the poles N and S, and thus $m = n$. The symmetry operations of $\mathscr{T}$ are exactly m rotations around ρ by C_m [5, p. 41]. No symmetry operation of $\mathscr{T}$ transforms a tile having N as a vertex to a tile having S as a vertex. However $\mathscr{T}$ is isohedral.

Thus the Schönflies symbol of the tiling $\mathscr{T}$ is D_m for some $m \geq 2$. By [5, p. 41], $\#G = 2m$. By (#), m is a multiple of $n \geq 4$. So the m-fold axis ρ of rotation is not through a three-valent vertex of $\mathscr{T}$, but through N and S of the pseudo-double wheel, and $m = n$. Hence the condition (2) holds. This completes the proof of ((3)⇒(2)). □

4. Tiles of spherical isohedral tilings over pseudo-double wheels

Definition 3. For $n \geq 3$, a *PDW_n-quadrangle* is the tile of some $\mathscr{T} \in PDW_n$.

Fact 1. *For given $n \geq 3$, $\alpha, \gamma \in (0, \pi) \cup (\pi, 2\pi)$ and $a \in (0, \pi)$, there is at most one PDW_n-quadrangle, modulo $SO(3)$, such that*

- *the cyclic list of inner angles in the clockwise order is $(\alpha, \beta, \gamma, \delta) = (\alpha, 2\pi/n, \gamma, 2\pi - \alpha - \gamma)$ (cf. Figure 4); and*
- *the edge $\alpha\beta$, the edge $\beta\gamma$, and the geodesic segment $\beta\delta$ have length a, a, $\pi - a$.*

Proof. From a point N on the unit sphere, travel in distance a, bend counterclockwise in $\pi - \alpha$, and travel in 2π. Then, by the last travel, we have a great circle C. The bending angle intends the inner angle α. By abuse of notation, we denote the bending point by α. C is through the point α. Similarly, from N, travel in distance a. Here the angle of this travel from the travel $N\alpha$ of length a is $\beta = 2\pi/n$. By abuse of notation, we often write β for the vertex N. Then bend clockwise in $\pi - \gamma$, and travel in 2π. By the last travel, we have a great circle C'. By abuse of notation, we denote the bending point by γ. C' is through the point γ. Then $C \neq C'$ by $\delta \neq \pi$. So C and C' share exactly two points P, P'. If each of P and P' is a vertex of the PDW_n-quadrangle, the inner angle of a PDW_n-quadrangle $\alpha\beta\gamma P$ which is diagonal to P is β if and only if the inner angle of a PDW_n-quadrangle $\alpha\beta\gamma P'$ diagonal to P' is $2\pi - \beta$. In this case, P' is inappropriate, as the tiles must not overlap. Hence, for n, α, γ, a, there is at most one pair of b, c. Actually, b is determined from α, a by a spherical cosine law (Proposition 1 (2b)) $\cos(\pi - a) = \cos a \cos b + \sin a \sin b \cos \alpha$, and c is determined similarly. □

Definition 4. Let $Q_{n,\alpha,\gamma,a}$ be a PDW_n-quadrangle of Fact 1. We identify $Q_{n,\alpha,\gamma,a}$ modulo $SO(3)$.

In fact, any PDW_n-quadrangle is specified without mentioning a tiling of PDW_n, as in the following Fact. There the vertices A, B, C, D intend the vertices N, v_0, v_1, v_2 of a tiling of PDW_n.

Fact 2. (1) *A PDW_n-quadrangle is exactly a quadrangle $ABCD$ such that $AB = \pi - AC = AD$, the area of $ABCD$ is $2\pi/n$, and the inner angle A is $2\pi/n$.*

(2) *The set of PDW_n-quadrangles bijectively corresponds to PDW_n.*

Proof. (1) As the edges BC and CD have length less than π, we have two spherical 3-gons ABC and CDA. As noted in the caption of Figure 7, $\angle ABC = \pi - \angle BCA$ and $\angle ADC = \pi - \angle DCA$. So the three inner angles of the vertices B, C, D sum up to 2π. Thus the inner angle of the vertex A is $2\pi/n$ since the area of $ABCD$ plus 2π is the sum of all inner angles A, B, C, D. By regarding the inner angles A, B, C, D as β, α, δ, γ and then arranging the $2n$ copies of the quadrangle $ABCD$ as Figure 4, we conclude $ABCD$ is a tile of a tiling of PDW_n. (2) Clear. □

An edge incident to N or S is called a *meridian edge*.

Lemma 2. *Suppose $n \geq 3$, $\alpha, \gamma \in (0,\pi) \cup (\pi, 2\pi)$, and $a \in (0,\pi)$. Every PDW_n-quadrangle $Q_{n,\alpha,\gamma,a}$ satisfies $a \neq \pi/2$, $\alpha \neq \pi/2$, $\gamma \neq \pi/2$,*

$$0 < a < \frac{\pi}{2} \quad \Leftrightarrow \quad 0 < \delta < \pi; \qquad \textit{and} \tag{1}$$

$$\alpha > \pi \textit{ or } \gamma > \pi \quad \Rightarrow \quad \frac{3\pi}{2} > \alpha > \pi > \gamma > \frac{\pi}{2} \textit{ or } \frac{3\pi}{2} > \gamma > \pi > \alpha > \frac{\pi}{2}. \tag{2}$$

Proof. Consider a tiling of PDW_n. If $a = \pi/2$, then any tile has three vertices on the equator and the other vertex is a pole. This contradicts the condition "no inner angle is π" (see Section 2).

Assume $\gamma = \pi/2$. See Figure 7. N and S are the poles, and $\angle N v_2 v_1 = \angle S v_1 v_2 = \pi/2 = \gamma$. Then $\angle N v_1 v_2 = \pi - \angle S v_1 v_2 = \pi/2$. Thus $N v_1 v_2$ is an isosceles triangle. Hence $\pi - a = N v_1 = N v_2 = a$. This contradicts $a \neq \pi/2$ which we have already proved. Hence $\gamma \neq \pi/2$. Similarly, $\alpha \neq \pi/2$.

As for equivalence (1), $a \in (0, \pi/2)$ if and only if v_0 and v_2 are located in the northern hemisphere and v_1 is in the southern. This is equivalent to $\delta \in (0, \pi)$. We will prove the implication (2). First assume the case where γ is too large. Then, the vertex v_1 is in the northern hemisphere and the edge $v_0 v_1$ crosses to the edge $N v_2$. To think of the situation, the leftmost lower tiling in Figure 8 may be useful. In the critical situation, $\alpha + \gamma + \delta = 2\pi$ implies $\alpha = \angle v_1 v_0 N = \pi/2$, $\gamma = 3\pi/2$, and $\delta = 0$. So $\pi/2 < \alpha < \pi < \gamma < 3\pi/2$. The same inequalities with α and γ swapped follows when α is too large. □

For $\mathscr{T} \in PDW_n$, let a be the length of the geodesic segment between N and v_0, and let φ be the longitude of the vertex v_2 minus that of the vertex v_1. See Figure 7.

Definition 5. For $n \geq 3$, define open sets $A_n^{(i)}$ in $\mathbf{R}^2$ $(i = 1, 2, 3, 4)$ as $\left(\frac{2\pi}{n} - \pi, 0\right) \times \left(0, \frac{\pi}{2}\right)$, $\left(0, \frac{2\pi}{n}\right) \times \left(0, \frac{\pi}{2}\right)$, $\left(\frac{2\pi}{n}, \pi\right) \times \left(0, \frac{\pi}{2}\right)$, and $\left(0, \frac{2\pi}{n}\right) \times \left(\frac{\pi}{2}, \pi\right)$. Let A_n be $\bigcup_{i=1}^4 A_n^{(i)}$. See Figure 8.

Theorem 3 (A coordinate system of PDW_n). *For each integer $n \geq 3$, a function $\mathscr{T} \in PDW_n \mapsto \langle \varphi, a \rangle \in A_n$ is a bijection.*

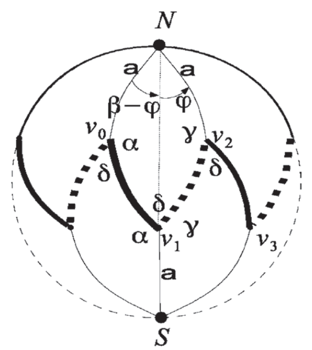

Fig. 7. The coordinate system $\langle \varphi, a \rangle$ of a tiling $\mathscr{T}$ of PDW_n. See the caption of Figure 4. Possibly $\varphi < 0$ and possibly $\varphi > \beta$. Because a straight line from N to the antipodal vertex S is through v_1, and because $v_1 S = a$, we have $N v_1 = \pi - a$, $\angle v_2 v_1 N = \pi - \gamma$ and $\angle N v_1 v_0 = \pi - \alpha$.

Proof. We prove $\langle \varphi, a \rangle \in A_n$ for any $\mathscr{T} \in PDW_n$, as follows: For $\pi/2 < a < \pi$, we have $0 < \varphi < \beta = 2\pi/n$. Otherwise, we can see that an edge crosses to an opposite edge. To think of the situation, see the right upper tiling in Figure 8.

For $0 < a < \pi/2$, φ is strictly between $\beta - \pi = 2\pi/n - \pi$ and π. Otherwise one of the edge v_0v_1 and the edge v_1v_2 contains a pair of antipodal points. Obviously $a \neq 0$. By Lemma 2, $a \neq \pi/2$. $\varphi \neq \beta = 2\pi/n$, by $\alpha \neq \pi$.

We show that the function $\mathscr{T} \mapsto \langle \varphi, a \rangle$ is onto A_n. Take an arbitrary $\langle \varphi, a \rangle$ of A_n. We first construct a quadrangle as follows: Take a point v_2 on the sphere such that the geodesic segment v_2S has length $\pi - a$. Since φ is given and $\beta = 2\pi/n$ is known, the vertex v_1 and v_0 is determined, as in Figure 7.

When $0 < a < \pi$, a pair of antipodal points appears neither in the edge Nv_0 nor in the edge Nv_2. No inner angle is π, as $\varphi \neq 0$, $2\pi/n$ and $a \neq \pi/2$.

We verify no edge contains a pair of antipodal points. Since $\langle \varphi, a \rangle$ is in the union A_n of the four open rectangles of Figure 8, a hemisphere contains all the vertices v_0, v_1, v_2 and the pole N as inner points. Hence, a pair of antipodal points appears in neither the edge v_1v_2 nor the edge v_0v_1, and lengths of the edges Nv_0 and Nv_2 are $a < \pi$.

Moreover, any of the four edges of the tile does not cross to the opposite edge, because when $0 < a < \pi/2$ the vertex v_1 is located in the southern hemisphere and the edges Nv_0 and Nv_2 are in the northern hemisphere. On the other hand, $\pi/2 < a < \pi$ implies $0 < \varphi < 2\pi/n$.

Arranging the $2n$ copies of the quadrangle as Figure 8 results in a tiling of PDW_n. So the function $\mathscr{T} \mapsto \langle \varphi, a \rangle$ is onto A_n. $\pi - a$ is the distance of the vertex v_1 from the pole N while $2\pi/n - \varphi$ is the longitude of v_1, i.e., $\angle v_0Nv_1$. So $\mathscr{T} \mapsto \langle \varphi, a \rangle$ is injective. Hence Theorem 3 is proved. □

5. Quadratic equation of tiles

Theorem 4. *Suppose* $n \geq 3$, $\alpha, \gamma \in (0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2)$, $a \in (0, \pi/2) \cup (\pi/2, \pi)$. *Then a quadrangle is a* PDW_n*-quadrangle* $Q_{n,\alpha,\gamma,a}$, *if and only if* $f_{n,\alpha,\gamma}(\cos a) = 0$ *where*

$$f_{n,\alpha,\gamma}(x) := x^2 - \left(\cot \frac{\pi}{n}\right)(\cot \alpha + \cot \gamma)x - \cot \alpha \cot \gamma.$$

Proof. Assume we are given a quadrangle $Nv_0v_1v_2$. By our definition of quadrangles (see Section 2), the quadrangle is a subset of the interior of an hemisphere. So, Nv_0v_1 and Nv_1v_2 are spherical 3-gons. Let φ be the angle from a geodesic segment Nv_1 to the edge Nv_2 and φ' be the angle from the edge Nv_0 to the geodesic segment Nv_1.

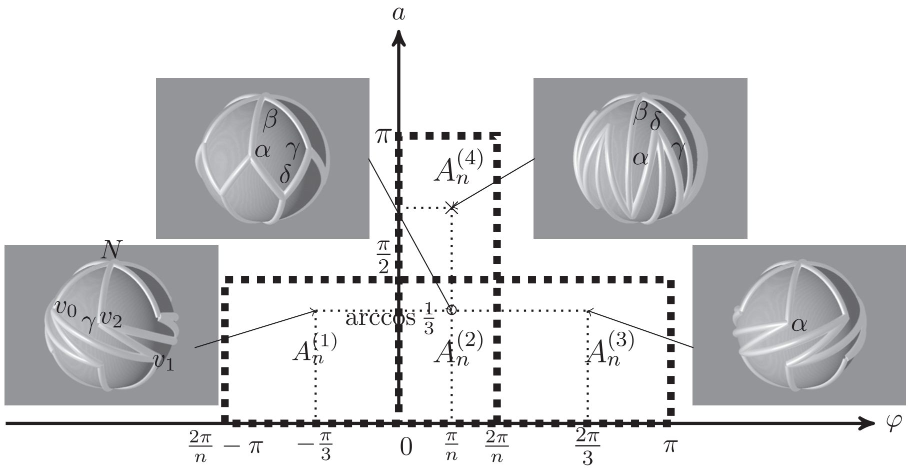

Fig. 8. The open set A_n ($n = 6$) (Definition 5) of $\langle \varphi, a \rangle$ (Figure 7) of $\mathcal{T} \in PDW_n$. A_n bijectively corresponds to PDW_n (Theorem 3). The four images are tilings of PDW_n.

The given quadrangle is $Q_{n,\alpha,\gamma,a}$, if and only if there are φ and φ' such that

$$\varphi + \varphi' = \frac{2\pi}{n}, \qquad \varphi \neq 0, \qquad \varphi' \neq 0 \tag{3}$$

$$\cos\gamma = \cos\varphi\cos\gamma - \sin\varphi\sin\gamma\cos a, \qquad \text{and} \tag{4}$$

$$\cos\alpha = \cos\varphi'\cos\alpha - \sin\varphi'\sin\alpha\cos a. \tag{5}$$

The two equations (4) and (5) are equivalent to spherical cosine theorems for angles (Proposition 1 (2a)) to spherical 3-gons. It is because of applying the last two equations $\angle v_2 v_1 N = \pi - \gamma$ and $\angle N v_1 v_0 = \pi - \alpha$ in the caption of Figure 7.

In the xy-plane, consider two lines

$$\ell\colon\ x - y\tan\gamma\cos a = 1, \qquad m\colon\ x - y\tan\alpha\cos a = 1.$$

They are well-defined, by the premise. Then

$$(*) \quad (4) \quad \Leftrightarrow \quad (\cos\varphi, \sin\varphi) \in \ell, \qquad (5) \quad \Leftrightarrow \quad (\cos\varphi', \sin\varphi') \in m.$$

Let R be the reflection with respect to the x-axis followed by rotation in $2\pi/n$ around the origin O. Then, (3) implies (5) $\Leftrightarrow (\cos\varphi, \sin\varphi) \in R(m)$. To sum up, under the equation (3),

$$(4)\ \&\ (5) \quad \Leftrightarrow \quad (\cos\varphi, \sin\varphi) \in \ell \cap R(m). \tag{6}$$

Let P be a point $(1, 0)$ and C be the unit circle $x^2 + y^2 = 1$.

Claim 3. (1) *For all* α, γ, *there is a unique point* $P' \in C \cap \ell \setminus \{P\}$. *Moreover* $P' = (\cos\varphi, \sin\varphi)$.

(2) *For all* α, γ, *there is a point* $Q' \in C \cap m \setminus \{P\}$. *Moreover* $Q' = (\cos\varphi', \sin\varphi')$ *and* $\angle POQ' = \varphi'$.

Proof. (1). By the premise, $\varphi \neq 0$. So P' is unique. By equivalence $(*)$, $\varphi = \angle POP'$. (2). Similar to (1). □

Let S be a point on the x-axis in the xy-plane. The ray starting from S in the direction of the positive part of x-axis is denoted by xS or Sx. The sum of the three inner angles of the plane triangle OPP' is π. So,

$$u := \angle xPP' = \frac{\varphi + \pi}{2}. \tag{7}$$

The line $R(m)$ is not the x-axis. It is because $R(P) \in C \cap R(m) \setminus (\mathbf{R} \times \{0\})$ by $\angle xOR(P) = 2\pi/n$. Hence, $\#(R(m) \cap (\mathbf{R} \times \{0\})) \leq 1$. Let a point Q be the intersection of the line $R(m)$ and x-axis, if it exists. Define

$$v := \begin{cases} \pi & (R(m) \cap (\mathbf{R} \times \{0\}) = \emptyset); \\ \angle xQR(P) & (\text{otherwise}). \end{cases}$$

See Figure 9.

Claim 4. *If the equation* (3) *holds and* $\pi/n \leq \varphi < \pi$, *then*

$$\tan u = (\tan\gamma \cos a)^{-1}. \qquad \tan v = \frac{\sin\frac{2\pi}{n}\sin\alpha\cos a - \cos\frac{2\pi}{n}\cos\alpha}{\cos\frac{2\pi}{n}\sin\alpha\cos a + \sin\frac{2\pi}{n}\cos\alpha}. \tag{8}$$

$$(4)\ \&\ (5) \quad \Leftrightarrow \quad f_{n,\alpha,\gamma}(\cos a) = 0. \tag{9}$$

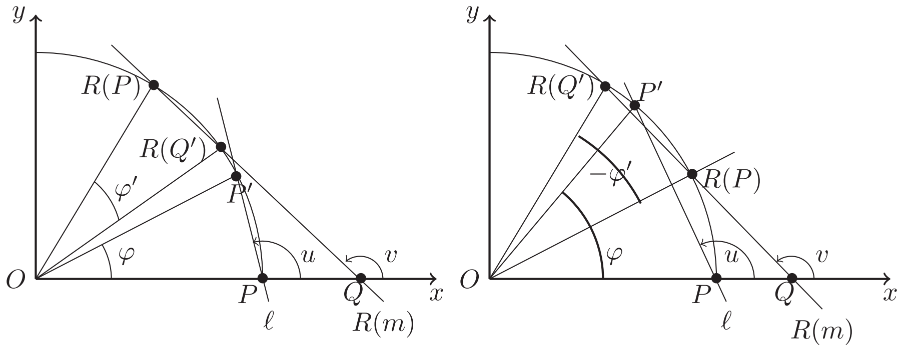

Fig. 9. Proofs of (7) and (13). Case $\varphi' > 0$ (left) and case $\varphi' < 0$ (right).

Proof. (8) The first equation is by the definition of ℓ and Claim 3. The denominator of the left-hand side of the second equation is not zero, by Claim 3 (2).

Next, we prove

$$v = \begin{cases} \frac{2\pi}{n} + \pi - \arctan((\tan\alpha\cos a)^{-1}) & \left(\frac{\pi}{n} \le \varphi < \frac{n-2}{n}\pi\right); \\ \pi & \left(\varphi = \frac{n-2}{n}\pi\right); \\ \frac{2\pi}{n} - \arctan((\tan\alpha\cos a)^{-1}) & \left(\frac{n-2}{n}\pi < \varphi < \pi\right). \end{cases}$$

The proof is as follows: Suppose $\pi/n \le \varphi < (n-2)\pi/n$. Let a point $\overline{Q'}$ be the reflection of the point Q' with respect to the x-axis. $\angle xP\overline{Q'} = \pi - \arctan((\tan\alpha\cos a)^{-1})$. Thus $v = \angle xP\overline{Q'} + 2\pi/n$. Suppose $\varphi = (n-2)\pi/n$. By the equation (3), $\varphi' = (4-n)\pi/n$. By the definition, $\angle xOR(Q') = \angle xOR(P) + \varphi' = \varphi + \varphi'$, which is $2\pi/n$ by (3). Hence, $(\angle xOR(P) + \angle xOR(Q'))/2 = \pi/2$. As $R(P), R(Q') \in C$, $R(m)$ does not intersect with the x-axis. Hence $v = \pi$ by the definition of v. The proof for $(n-2)\pi/n < \varphi \le \pi$ is similar to the proof for $\pi/n \le \varphi < (n-2)\pi/n$. This establishes the desired representation of v.

If $\varphi \ne (n-2)\pi/n$, then by the addition formula of tan, $\tan v$ is as desired. Consider the case $\varphi = (n-2)\pi/n$. Then $\varphi' = (4-n)\pi/n$. By (5), $(\tan\alpha\cos a)^{-1} = \tan(2\pi/n)$. By the addition formula of tan, $\tan v$ is as desired. This completes the proof of the equation (8) of Claim 4.

(9) First we claim

$$\ell \cap R(m) \ni (\cos\varphi, \sin\varphi) \quad \Leftrightarrow \quad R(Q') = P'. \tag{10}$$

The proof is as follows: $(\cos\varphi, \sin\varphi) = P'$ is $R(P)$ or $R(Q')$, because $m \cap C = \{P, Q'\}$ by Claim 3 (2). Here $R(P)$ is $(\cos(2\pi/n), \sin(2\pi/n))$. If $R(P) = P'$, then $2\pi/n = \varphi$, and thus $\varphi' = 2\pi/n - \varphi = 0$, by the equation (3). This is a contradiction. This completes the proof of (10).

Next we claim

$$\tan(v-u) = \tan\frac{\pi}{n} \quad \Leftrightarrow \quad f_{n,\alpha,\gamma}(\cos a) = 0. \tag{11}$$

The left-hand side of equivalence (11) is

$$\frac{\tan u - \tan v}{1 + \tan u \tan v} + \tan\frac{\pi}{n} = 0.$$

Observe that the denominator of the first term of the left-hand side cannot be 0. Assume otherwise. Then $u - v = \pi/2 + i\pi$ for some integer i. Thus $\angle PP'(R(P)) = \pi/2 + i\pi$ for some integer i. Thus $\varphi + \varphi' = \pi$, which contradicts the equation (3). Hence the denominator $1 + \tan u \tan v$ of the first term of the left-hand side is not 0. Also note that the denominator $\cos \pi/n$ of the second term of the left-hand side is not 0. Substitute (8) in the left-hand side. Then we have a quadratic equation of $\cos a$, by calculation. Because $\sin\alpha \sin\gamma \sin(\pi/n) \neq 0$, the quadratic equation is equivalent to the quadratic equation $f_{n,\alpha,\gamma}(x) = 0$ of $x = \cos a$. This completes the proof of (11).

Hence, by equivalences (6), (10), and (11), we have only to prove

$$R(Q') = P' \quad \Leftrightarrow \quad \tan(v - u) = \tan\frac{\pi}{n}, \tag{12}$$

to show (9). If $R(Q') = P'$, then $\angle POR(Q') = \angle POP' = \varphi$. Thus $v - u = (\varphi + \varphi')/2 + k\pi$ for some integer k. The equation (3) implies $\tan(v - u) = \tan(\pi/n)$. To prove the converse of (12), we derive

$$\tan(v - u) = \tan(\pi/n) \quad \Rightarrow \quad \angle POR(Q') = \varphi.$$

Case A. $\pi/n \le \varphi < (n-2)\pi/n$ (See Figure 9).

The mean M of $\angle xOR(P) = 2\pi/n$ and $\angle xOR(Q') = 2\pi/n - \varphi'$ is $2\pi/n - \varphi'/2 = \pi/n + \varphi/2$ by the equation (3). Then $M < \pi/2$, by $\varphi < (n-2)\pi/n$. Therefore, $R(m) \cap ((0,\infty) \times \{0\})$ consists of a unique point Q, where $R(m)$ is a line through the two points $R(P)$ and $R(Q')$. We claim

$$\frac{\pi}{n} \le \varphi < \frac{n-2}{n}\pi \quad \Rightarrow \quad v - u = \frac{\pi}{n} - \varphi + \angle POR(Q') \in \left(\frac{4-n}{2n}\pi, \frac{\pi}{2}\right). \tag{13}$$

The proof is as follows: Observe $v = (\varphi' + \pi)/2 + \angle POR(Q')$. It is clear when $\varphi' > 0$. In case $\varphi' < 0$, the observation follows from $v = (-\varphi' + \pi)/2 + (\angle POR(Q') + \varphi')$. By the equation (8), $v > \pi/2 + 2\pi/n$. Clearly, $v < \pi$. Hence, by $u \in (\pi/2, \pi)$, $v - u$ is in the desired interval. From the equations (7) and (3), the desired equation of (13) follows. This completes the proof of (13).

Assume $\tan(v - u) = \tan(\pi/n)$. By (13), $\varphi = \angle POR(Q')$.

Case B. $(n-2)\pi/n < \varphi < \pi$ (See Figure 10).

We claim:

$$\frac{n-2}{n}\pi < \varphi < \pi \quad \Rightarrow \quad v - u = \frac{\pi}{n} - \varphi + \angle POR(Q') - \pi \in \left(-\pi, \frac{2-n}{2n}\pi\right). \tag{14}$$

The proof is as follows: By Figure 10, $v = (-\varphi' + \pi)/2 + 2\pi/n - \pi > 0$.

So, the desired equation follows from the equations (7) and (3), in a similar argument as Case A. The equation (3) implies $v = \pi/n + \varphi/2 - \pi/2 < \pi/n$. By condition of Case B and the definition (7) of the angle u, we have $(n-1)\pi/n < u < \pi$. So $v - u$ is indeed in the desired interval of (14). This completes the proof of (14).

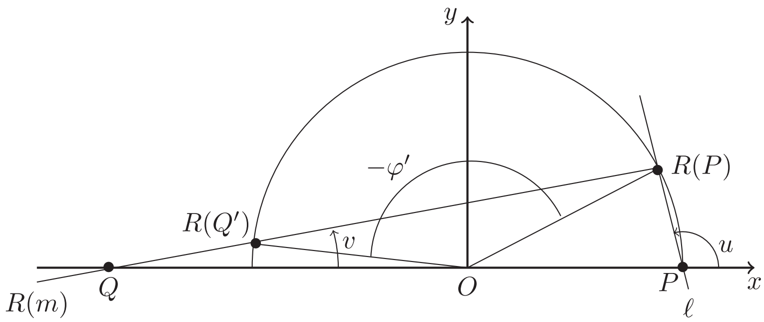

Fig. 10. Proof of (14). $(n-2)\pi/n < \varphi < \pi$.

Assume $\tan(v-u) = \tan(\pi/n)$. The interval $(-\pi, (2-n)\pi/(2n))$ contains $\pi/n + k\pi$ for a unique integer $k = -1$. By (14), $\varphi = \angle POR(Q')$.

Case C. $\varphi = (n-2)\pi/n$. Then the line $R(m)$ does not intersect with the x-axis. As $R(m) \cap C = \{R(P), R(Q')\}$, $\pi/2 = (\angle POR(Q') + \angle POR(P))/2$, $\angle R(Q')OR(P) = -\varphi'$, $\angle POR(P) = \varphi$, and the equation (3), it holds that $\varphi = \angle POR(Q')$. Thus we have proved the converse of (12). This completes the proof of Claim 4. □

By the symmetry $(\varphi, \gamma) \leftrightarrow (\varphi', \alpha)$, (9) of Claim 4 implies: *If the equation* (3) *holds and* $\varphi < \pi/n$, *then* (4) & (5) $\Leftrightarrow f_{n,\gamma,\alpha}(\cos a) = 0$. Here $f_{n,\gamma,\alpha}(\cos a) = f_{n,\alpha,\gamma}(\cos a)$. So, T is a $Q_{n,\alpha,\gamma,a}$ if and only if $f_{n,\alpha,\gamma}(\cos a) = 0$. This establishes Theorem 4. □

6. Range of inner angles of PDW_n-quadrangles

To classify the two opposite inner angles α, γ and the edge-length a of PDW_n-quadrangles $Q_{n,\alpha,\gamma,a}$'s, we solve $f_{n,\alpha,\gamma}(\cos a) = 0$, taking the condition Lemma 2 (Proposition 1 (1), resp.) of quadrangles (spherical 3-gons, resp.) into account. This classifies all tilings of PDW_n, because of Fact 2 (2).

6.1. Discriminant. The equation $f_{n,\alpha,\gamma}(\cos a) = 0$ has at most two solutions $a \in (0, \pi)$, as $\cos a$ is strictly decreasing for $a \in (0, \pi)$ and $f_{n,\alpha,\gamma}(x)$ is quadratic. The smaller solution $a = a^-_{n,\alpha,\gamma}$ is the arccosine of

$$\frac{1}{2} \cot \frac{\pi}{n} (\cot \alpha + \cot \gamma + \sqrt{\Delta_{n,\alpha,\gamma}}),$$

while the larger solution $a = a^+_{n,\alpha,\gamma}$ of $f_{n,\alpha,\gamma}(\cos a) = 0$ is obtained from $a^-_{n,\alpha,\gamma}$ by inverting the sign in front of the square root. Here

$$\Delta_{n,\alpha,\gamma} := \cot^2 \gamma + 2\left(2 \tan^2 \frac{\pi}{n} + 1\right) \cot \alpha \cot \gamma + \cot^2 \alpha.$$

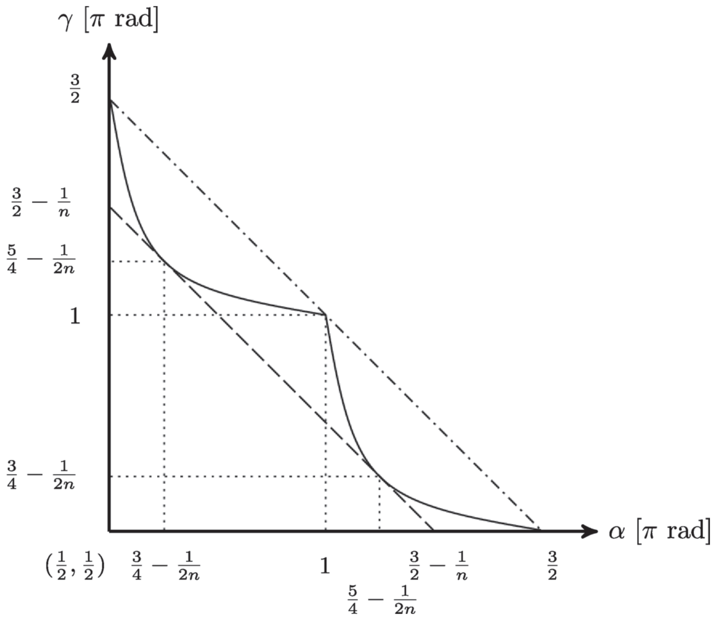

Fig. 11. The curves $\gamma = \mathrm{dgn}_n(\alpha)$ $(\pi/2 < \alpha < \pi)$, $\alpha = \mathrm{dgn}_n(\gamma)$ $(\pi/2 < \gamma < \pi)$, $\alpha + \gamma = 2\pi - \pi/n$ (dash), and $\alpha + \gamma = 2\pi$ (dash-dot), when $n = 4$. See Lemma 3.

Lemma 3. *Let* $\pi/2 < \alpha < \pi < \gamma < 3\pi/2$. *Then*

(1) $\Delta_{n,\alpha,\gamma} \geq 0 \Leftrightarrow \gamma \leq \mathrm{dgn}_n(\alpha)$. *Moreover, the equality of one side implies that of the other side. Here* $\mathrm{dgn}_n : (\pi/2, \pi) \to (\pi, 3\pi/2)$ *is defined as*

$$\mathrm{dgn}_n(\psi) := \pi - \arctan\left(\cos^2 \frac{\pi}{n}\left(\sin \frac{\pi}{n} + 1\right)^{-2} \tan \psi\right).$$

(2) *The curve* $\gamma = \mathrm{dgn}_n(\alpha)$ *is strictly decreasing, convex, and has the tangential line* $\gamma = 2\pi - \pi/n - \alpha$ *at* $\alpha = 3\pi/4 - \pi/(2n)$.

(3) $2\pi - \pi/n - \alpha < \mathrm{dgn}_n(\alpha) < 2\pi - \alpha$ *for all* $\alpha \in (\pi/2, 3\pi/4 - \pi/(2n))$.

Proof. (1). Let $s_n := \sin(\pi/n) - 1 < 0$, $t_n := \sin(\pi/n) + 1 > 0$, and

$$z_1 := -s_n^2 \cot \alpha \sec^2 \frac{\pi}{n}, \qquad z_2 := -t_n^2 \cot \alpha \sec^2 \frac{\pi}{n}.$$

We prove $z_1 < \cot \gamma \Rightarrow$ Lemma 3 (1), as follows: By calculation, z_1 and z_2 are the zeros of the quadratic polynomial

$$p(z) := z^2 + 2 \cot \alpha \left(2 \tan^2 \frac{\pi}{n} + 1\right) z + \cot^2 \alpha.$$

Here $\Delta_{n,\alpha,\gamma} = p(\cot\gamma)$. $z_1 < z_2$ by $\pi/2 < \alpha < \pi$. Clearly $\gamma \le \mathrm{dgn}_n(\alpha)$ if and only if $\gamma - \pi \le -\arctan(\tan\alpha\cos^2(\pi/n)t_n^{-2})$. $\gamma - \pi \in (0, \pi/2)$ by the premise. So, by applying the strictly decreasing function cot, $\gamma \le \mathrm{dgn}_n(\alpha)$ is equivalent to $z_2 \le \cot\gamma$. Hence

Claim 5. *Let $\pi/2 < \alpha < \pi < \gamma < 3\pi/2$. Then*

$$\gamma \le \mathrm{dgn}_n(\alpha) \quad \Leftrightarrow \quad -\cot\alpha\left(1 + \sin\frac{\pi}{n}\right)^2 \sec^2\frac{\pi}{n} \le \cot\gamma.$$

The equality of one side implies that of the other side.

Therefore, Lemma 3 (1) follows from $z_1 < \cot\gamma$, because the polynomial $p(z)$ is quadratic.

We first prove $z_1 < -\cot\alpha$ as follows: The premise $\alpha \in (\pi/2, \pi)$ implies $\tan\alpha < 0$. So, $z_1 < -\cot\alpha$ if and only if $s_n^2\sec^2(\pi/n) < 1$. As $s_n < 0$, the inequality $\sec^2(\pi/n)s_n^2 < 1$ is equivalent to $-\cos(\pi/n) < s_n$, which is equivalent to $1/\sqrt{2} < \sin(\pi/n + \pi/4)$. The last inequality holds for $n = 3$ by calculation. It also holds for $n \ge 4$, by $\pi/n + \pi/4 \in (\pi/4, \pi/2]$. Thus $z_1 < -\cot\alpha$.

Assume $z_1 \ge \cot\gamma$. Then $\cot\gamma < -\cot\alpha$ by $z_1 < -\cot\alpha$. By $\alpha \in (\pi/2, \pi)$ and $\gamma \in (\pi, 3\pi/2)$, we have $\cot\gamma > 0$, $\tan\alpha\tan\gamma < 0$, and thus $-\tan\alpha < \tan\gamma$. Hence $\tan(\alpha+\gamma) = (\tan\alpha + \tan\gamma)/(1 - \tan\alpha\tan\gamma) > 0$, which implies $\alpha + \gamma > 2\pi$ by the premise $\alpha \in (\pi/2, \pi)$, $\gamma \in (\pi, 3\pi/2)$. This contradicts $\alpha + \gamma + \delta = 2\pi$. Hence $z_1 < \cot\gamma$.

To prove Lemma 3 (2), we first verify the curve $\gamma = \mathrm{dgn}_n(\alpha)$ and the line $\gamma = 2\pi - (\pi/n) - \alpha$ intersect at $\alpha = 3\pi/4 - \pi/(2n)$, as follows:

$$\mathrm{dgn}_n\left(\frac{3\pi}{4} - \frac{\pi}{2n}\right) = 2\pi - \frac{\pi}{n} - \left(\frac{3\pi}{4} - \frac{\pi}{2n}\right), \tag{15}$$

if and only if $\arctan(\cos^2(\pi/n)\tan(\pi/4 + \pi/(2n))(\sin(\pi/n) + 1)^{-2})$ is $\pi/4 - \pi/(2n)$. As the right-hand side $\pi/4 - \pi/(2n)$ is strictly between $(0, \pi/2)$, the condition is equivalent to $\cos^2(\pi/n)\tan^2(\pi/4 + \pi/(2n))(\sin(\pi/n) + 1)^{-2} = 1$. Hence the square root of the left-hand side is unity, as $n \ge 3$ implies $0 < (1/4 + 1/(2n))\pi < \pi/2$. By calculation, it is indeed unity from the double-angle formulas. Thus (15) is proved.

By calculation, we have a partial derivative

$$\partial_\alpha\,\mathrm{dgn}_n(\alpha) = -t_n^2\cos^2\frac{\pi}{n}\left(\left(t_n^4 - \cos^4\frac{\pi}{n}\right)\cos^2\alpha + \cos^4\frac{\pi}{n}\right)^{-1}.$$

It is negative because $t_n > 1$. So $\gamma = \mathrm{dgn}_n(\alpha)$ is decreasing. Since

$$\partial_\alpha\,\mathrm{dgn}_n\left(\frac{3\pi}{4} - \frac{\pi}{2n}\right) = -1 \tag{16}$$

by calculation, a line $\alpha + \gamma = 2\pi - \pi/n$ is the tangential line of the curve $\gamma = \mathrm{dgn}_n(\alpha)$. By calculation, the second-order derivative $\partial_\alpha^2 \, \mathrm{dgn}_n(\alpha)$ is

$$-t_n^2 \cos^2 \frac{\pi}{n} \left(t_n^4 - \cos^4 \frac{\pi}{n} \right) \sin 2\alpha \left(\left(t_n^4 - \cos^4 \frac{\pi}{n} \right) \cos^2 \alpha + \cos^4 \frac{\pi}{n} \right)^{-2}.$$

It is positive since $\pi/2 < \alpha < \pi$ by the premise and $t_n > 1$.

To prove Lemma 3 (3), observe that the first inequality $2\pi - \pi/n - \alpha < \mathrm{dgn}_n(\alpha)$ follows from Lemma 3 (2). As for the second inequality $\mathrm{dgn}_n(\alpha) < 2\pi - \alpha$, note that $\mathrm{dgn}_n(\alpha) \to 3\pi/2 - 0$, as $\alpha \to \pi/2 + 0$. By equality (16) and the convexity of the curve $\gamma = \mathrm{dgn}_n(\alpha)$, we have $\partial_\alpha \, \mathrm{dgn}_n(\alpha) \leq -1$ for all $\alpha \in (\pi/2, 3\pi/4 - \pi/(2n))$. Thus $\mathrm{dgn}_n(\alpha) < 2\pi - \alpha$. This establishes Lemma 3. □

6.2. The statement.

Definition 6. For $n \geq 3$, define

$B_n := \{(\alpha, \gamma) \in ((0, \pi) \cup (\pi, 2\pi))^2 \mid \text{A } Q_{n,\alpha,\gamma,a} \text{ exists for some } a \in (0, \pi)\}$.

By simple trigonometric formulas, we describe B_n and the length a of the meridian edge of each PDW_n-quadrangles $Q_{n,\alpha,\gamma,a}$.

Theorem 5 (Inner angles and edge-length of PDW_n-quadrangles). *Assume $n \geq 3$.*

(1) $B_n = \bigcup_{i=1}^{8} B_n^{(i)}$ *where* $B_n^{(i)}$ *is defined in* (4) *below.*

(2) $(\alpha, \gamma) \in B_n^{(4)} \cup B_n^{(8)}$, *if and only if there exist exactly two PDW_n-quadrangles. Here the edge-length a is $a_{n,\alpha,\gamma}^+$ or $a_{n,\alpha,\gamma}^-$.*

(3) $(\alpha, \gamma) \in \bigcup_{1 \leq i \leq 8, i \neq 4, 8} B_n^{(i)}$, *if and only if there exists a unique PDW_n-quadrangle $Q_{n,\alpha,\gamma,a}$. Here the edge-length a is $a_{n,\alpha,\gamma}^-$ for $(\alpha, \gamma) \in B_n^{(1)}$; $a_{n,\alpha,\gamma}^+$ for $(\alpha, \gamma) \in B_n^{(2)} \cup B_n^{(5)} \cup B_n^{(6)}$;*

$$\pi - \arccos\left(\left(\sec \frac{\pi}{n} \right) \left(\sin \frac{\pi}{n} + 1 \right) \cot \alpha \right) = a_{n,\alpha,\gamma}^{\pm} < \frac{\pi}{2} \tag{17}$$

for $(\alpha, \gamma) \in B_n^{(3)}$;

$$\pi - \arccos\left(\left(\sec \frac{\pi}{n} \right) \left(\sin \frac{\pi}{n} + 1 \right) \cot \gamma \right) = a_{n,\alpha,\gamma}^{\pm} < \frac{\pi}{2} \tag{18}$$

for $(\alpha, \gamma) \in B_n^{(7)}$.

(4) *Here*

- $B_n^{(1)}$ *is an open pentagon* $\{(\alpha, \gamma) \mid \pi/2 < \alpha < \pi, \ \pi/2 < \gamma < \pi, \ \alpha + \gamma < 2\pi - \pi/n\}$;
- $B_n^{(2)}$ *is an open rectangular triangle* $\{(\alpha, \gamma) \mid \pi/2 < \alpha, \ \pi < \gamma, \ \alpha + \gamma < 2\pi - \pi/n\}$;

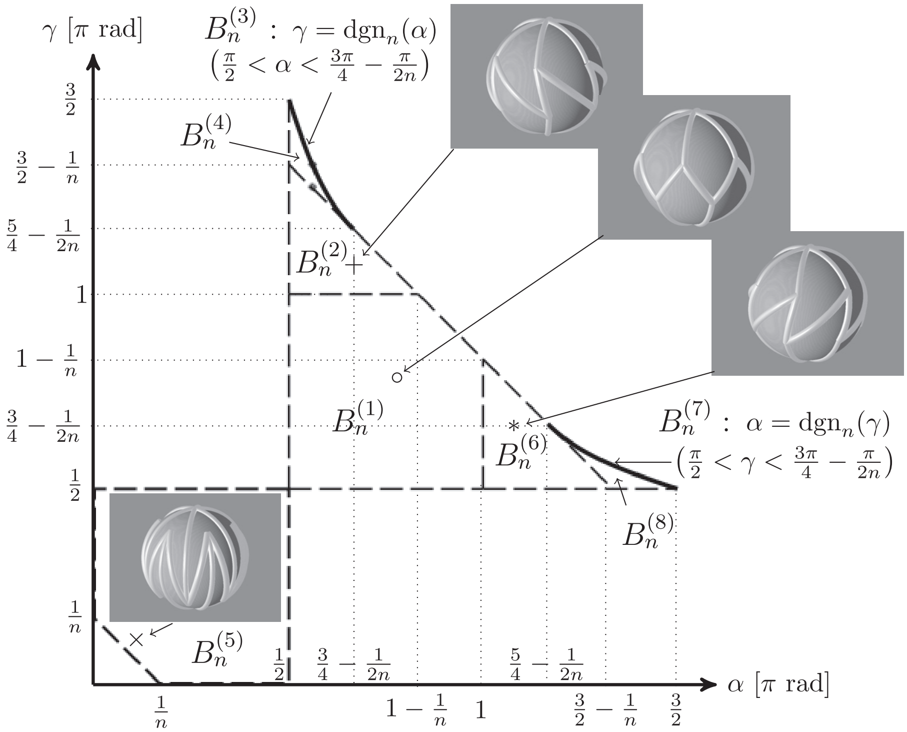

Fig. 12. The set B_n ($n = 6$). See Theorem 5. The edge-length a is $a^+_{n,\alpha,\gamma}$ for $B_n^{(2)} \cup B_n^{(5)} \cup B_n^{(6)}$; $a^-_{n,\alpha,\gamma}$ for $(\alpha, \gamma) \in B_n^{(1)}$; $a^-_{n,\alpha,\gamma}$ or $a^+_{n,\alpha,\gamma}$ for $B_n^{(4)} \cup B_n^{(8)}$; and $a^-_{n,\alpha,\gamma} = a^+_{n,\alpha,\gamma}$ for $B_n^{(3)} \cup B_n^{(7)}$. The point designated by ∘ (×, resp.) corresponds to the upper left (upper right, resp.) image of tiling in Figure 8.

- $B_n^{(3)}$ *is a curve* $\{(\alpha, \mathrm{dgn}_n(\alpha)) \mid \pi/2 < \alpha < 3\pi/4 - \pi/(2n)\}$*;*
- $B_n^{(4)}$ *is a nonempty open set* $\{(\alpha, \gamma) \mid \pi/2 < \alpha,\ 2\pi - \pi/n - \alpha < \gamma < \mathrm{dgn}_n(\alpha)\}$*;*
- $B_n^{(5)}$ *is symmetric to* $B_n^{(1)}$ *around the line* $\alpha + \gamma = \pi$*; and*
- $B_n^{(i+4)}$ $(2 \le i \le 4)$ *is symmetric to* $B_n^{(i)}$ *around the line* $\gamma = \alpha$.

Remark 1. The inner angles (α, γ) of $\mathscr{T} \in PDW_n$ ranges over Figure 12. The coordinate system $\langle \varphi, a \rangle$ for PDW_n is introduced in Definition 5. $\langle \varphi, a \rangle$ ranges over Figure 8. Figure 12 corresponds to Figure 8, as follows:

In Figure 12, the open set above $\gamma = \pi$, the open set right to $\alpha = \pi$, and the open set below $\gamma = \pi/2$ correspond to $A_n^{(1)}$, $A_n^{(3)}$, and $A_n^{(4)}$ of Definition 5 and Figure 8, respectively.

Here is the proof. By Theorem 5 (1) and Figure 7, $\bigcup_{i=2}^4 B_n^{(i)}$ corresponds to $A_n^{(1)}$, and $\bigcup_{i=6}^8 B_n^{(i)}$ to $A_n^{(3)}$. By Theorem 5 (1) and Lemma 2, $B_n^{(5)}$ corresponds to $A_n^{(4)}$. So, by Theorem 5 and Theorem 3, $B_n^{(1)}$ corresponds to $A_n^{(2)}$.

Suppose that the length a of the meridian edges is 0. Then $2n$ non-meridian edges split the sphere where the inner angle of each digon is $\delta = 2\pi - \alpha - \gamma$. So $\delta = \pi/n$. Hence, $a = 0$ implies $\alpha + \gamma = \pi(2 - 1/n)$.

If $a = \pi$, then the tile is the union of a digon of angle α and that of angle γ, so $\beta = \alpha + \gamma = 2\pi/n$.

$a = \pi/2$ corresponds to $\alpha = \pi/2$ or $\gamma = \pi/2$, by the proof of Lemma 2.

Theorem 5 follows from Theorem 6 (Subsection 6.3) and Theorem 7 (Subsection 6.4). Proposition 1 (1) plays an important role in Subsection 6.3. In Figure 11, the two disjoint regions circumscribed by a solid curve, dashed line and dotted line do not correspond to PDW_n-quadrangles. It is because every zero of $f_{n,\alpha,\gamma}$ is greater than 1. See Lemma 5 of Subsection 6.4.

6.3. PDW_n-quadrangles containing the meridian diagonal geodesic segment. In any tiling of PDW_n, the tile $Nv_0v_1v_2$ contains a segment Nv_1, if and only if $\alpha, \gamma \in (0, \pi/2)$ or $\alpha, \gamma \in (\pi/2, \pi)$. Theorem 6 (1) and Theorem 6 (2) correspond to the two open pentagons $B_n^{(5)}$ and $B_n^{(1)}$ of Theorem 5, respectively. For given n, α, γ, a, there is at most one PDW_n-quadrangle $Q_{n,\alpha,\gamma,a}$ (See Fact 1 and Definition 4). We observe

$$f_{n,\alpha,\gamma}(\pm 1) = \mp \csc\alpha \csc\gamma \csc\frac{\pi}{n} \sin\left(\pm\frac{\pi}{n} + \alpha + \gamma\right). \tag{19}$$

The axis of the parabola $y = f_{n,\alpha,\gamma}(x)$ is

$$\operatorname{axis}(n, \alpha, \gamma) := \frac{1}{2} \cot\frac{\pi}{n} (\cot\alpha + \cot\gamma) \tag{20}$$

THEOREM 6. *Let $n \geq 3$.*

(1) *Let $\alpha, \gamma \in (0, \pi/2)$. Then a PDW_n-quadrangle $Q_{n,\alpha,\gamma,a}$ exists for some $a \in (0, \pi)$, if and only if $\alpha + \gamma > \pi/n$. In this case, $a = a^+_{n,\alpha,\gamma}$.*

(2) *Let $\alpha, \gamma \in (\pi/2, \pi)$. Then a PDW_n-quadrangle $Q_{n,\alpha,\gamma,a}$ exists for some $a \in (0, \pi)$, if and only if $\alpha + \gamma < 2\pi - \pi/n$. In this case, $a = a^-_{n,\alpha,\gamma}$.*

PROOF. (Only-if part of Theorem 6 (1)). See Figure 7. By $\alpha, \gamma \in (0, \pi/2)$, a segment Nv_1 is in the quadrangle. To the two spherical 3-gons v_0Nv_1 and v_2Nv_1, apply the last inequality of Proposition 1 (1). Then $-\alpha + \angle v_0Nv_1 + \angle Nv_1v_0 < \pi$ and $-\gamma + \angle v_1Nv_2 + \angle v_2v_1N < \pi$. The sum of the left-hand sides of the two inequalities is $-\alpha - \gamma + \beta + \delta$, and is less than 2π. By $\delta = 2\pi - \alpha - \gamma$ and $\beta = 2\pi/n$, we have $\pi > \alpha + \gamma > \beta/2 = \pi/n$.

(If part of Theorem 6 (1)). For any $a \in (0, \pi)$, there is a spherical isosceles 3-gon v_0Nv_2 such that $Nv_0 = Nv_2 = a$ and $\angle v_0Nv_2 = 2\pi/n$. By $\alpha, \gamma > 0$, there is a vertex v_1 between Nv_0 and Nv_2 such that $\angle v_1v_0N = \alpha$ and $\angle Nv_2v_1 = \gamma$. So, v_0v_1 does not cross to Nv_2. Hence, by Theorem 4, $Q_{n,\alpha,\gamma,a}$ exists if and only if there are a spherical 3-gon $v_0v_1v_2$ and $a \in (0,\pi)$ such that $Nv_0 = Nv_2 = \pi - Nv_1 = a$ and $f_{n,\alpha,\gamma}(\cos a) = 0$.

Put four vertices N, v_0, v_1, v_2 such that N is the north pole, $Nv_0 = Nv_2 = \pi - Nv_1 = a$. Then, v_0 lies in the southern hemisphere if and only if v_1 lies in the northern hemisphere. As $\angle v_1v_0N = \alpha < \pi/2$ by the premise, v_0 lies in the southern hemisphere and $\pi > a > \pi/2$. The geodesic segment between v_0 and v_2 lies inside the southern hemisphere. Hence, the 3-gon $v_0v_1v_2$ is a subset of the 3-gon v_0Nv_2. Let $\theta = \angle v_2v_0N = \angle Nv_2v_0$. The inner angles at v_1 of the 3-gon $v_0v_1v_2$ is $\alpha+\gamma = 2\pi-\delta$, because the assumption $\alpha,\gamma \in (0,\pi/2)$ implies $\alpha+\gamma<\pi$. The other two inner angles of $v_0v_1v_2$ are $\theta-\alpha$, $\theta-\gamma$. Therefore, by Proposition 1 (1), the existence of the 3-gon $v_0v_1v_2$ is equivalent to $2\theta>\pi$, $2\alpha<\pi$, $2\gamma<\pi$, and $2\theta-2\alpha-2\gamma<\pi$. Thus, $Nv_0v_1v_2$ is a quadrangle if and only if $\pi/2<\theta<\pi/2+\alpha+\gamma$. As a spherical 3-gon v_0Sv_2 exists, $2(\pi-\theta)+\beta>\pi$, i.e., $\theta<\pi/2+\beta/2=\pi/2+\pi/n$. The assumption $\pi/n<\alpha+\gamma$ implies $\theta<\pi/2+\alpha+\gamma$. Moreover, $\pi/2<\theta$, as v_0 and v_2 are in the southern hemisphere. Hence, if there is an $a \in (0,\pi)$ such that $f_{n,\alpha,\gamma}(\cos a) = 0$, then the 3-gon $v_0v_1v_2$ exists such that $Nv_0 = Nv_2 = \pi - Nv_1 = a$.

In this case, we show $a = a^+_{n,\alpha,\gamma} \in (0,\pi)$ exists and $f_{n,\alpha,\gamma}(\cos a^+_{n,\alpha,\gamma})$ is 0. As $a \in (\pi/2,\pi)$, $\cos a \in (-1,0)$. By the assumption, $\pi>\alpha+\gamma>-\pi/n+\alpha+\gamma>0$. By (19), $f_{n,\alpha,\gamma}(-1)>0>f_{n,\alpha,\gamma}(0) = -\cot\alpha\cot\gamma$. The axis $\mathrm{axis}(n,\alpha,\gamma)$ of the parabola $y = f_{n,\alpha,\gamma}(x)$ is positive, as $\alpha,\gamma\in(0,\pi/2)$. So the intersection of $(-1,0)\times\{0\}$ and the parabola $y=f_{n,\alpha,\gamma}(x)$ is the smaller intersection point of $\mathbf{R}\times\{0\}$ and the parabola. Hence $a = a^+_{n,\alpha,\gamma}$.

(2). $Q_{n,\alpha,\gamma,a}$ exists if and only if $Q_{n,\pi-\alpha,\pi-\gamma,\pi-a}$ does so. It is because from any $\mathscr{T}\in PDW_n$, we obtain $\mathscr{T}'\in PDW_n$, by joining the vertices N and S to the opposite vertices v_{2i+1} and v_{2i} respectively, and then deleting the $2n$ meridian edges Nv_{2i} and Sv_{2i+1} of $\mathscr{T}$. As $\pi-\alpha,\pi-\gamma\in(0,\pi/2)$, Theorem 6 (1) implies $\pi - a = a^+_{n,\pi-\alpha,\pi-\gamma}$. Hence, $a = a^-_{n,\alpha,\gamma}$, by the explanation at the beginning of Subsection 6.1 and $\arccos(x) = \pi - \arccos(-x)$. This establishes Theorem 6. □

6.4. ***PDW_n*-quadrangles with α or γ greater than π.** Theorem 7 (1) and Theorem 7 (2) correspond to an open set $B_n^{(4)}\cup B_n^{(8)}$ and a set $B_n^{(2)}\cup B_n^{(3)}\cup B_n^{(6)}\cup B_n^{(7)}$ of Theorem 5, respectively.

THEOREM 7. *Suppose $\alpha>\pi$ or $\gamma>\pi$. Then we have the following:*

(1) *There are more than one, actually, exactly two PDW_n-quadrangles $Q_{n,\alpha,\gamma,a}$, if and only if*

$$\frac{\pi}{2} < \alpha < \frac{3\pi}{4} - \frac{\pi}{2n} \qquad \& \qquad 2\pi - \frac{\pi}{n} - \alpha < \gamma < \mathrm{dgn}_n(\alpha); \qquad or \tag{21}$$

$$\frac{\pi}{2} < \gamma < \frac{3\pi}{4} - \frac{\pi}{2n} \qquad \& \qquad 2\pi - \frac{\pi}{n} - \gamma < \alpha < \mathrm{dgn}_n(\gamma). \tag{22}$$

There is indeed a pair (α, γ) *satisfying* (21) *or* (22). *Here the edge-length* a *is* $a^+_{n,\alpha,\gamma}$ *or* $a^-_{n,\alpha,\gamma}$.

(2) *There is a unique* PDW_n*-quadrangle, if and only if*

$$\frac{\pi}{2} < \alpha < 2\pi - \frac{\pi}{n} - \gamma \qquad \& \qquad \pi < \gamma; \tag{23}$$

$$\gamma = \mathrm{dgn}_n(\alpha) \qquad \& \qquad \frac{\pi}{2} < \alpha < \frac{3\pi}{4} - \frac{\pi}{2n}; \tag{24}$$

$$\frac{\pi}{2} < \gamma < 2\pi - \frac{\pi}{n} - \alpha \qquad \& \qquad \pi < \alpha; \qquad or \tag{25}$$

$$\alpha = \mathrm{dgn}_n(\gamma) \qquad \& \qquad \frac{\pi}{2} < \gamma < \frac{3\pi}{4} - \frac{\pi}{2n}.$$

If (23) *or* (25) *hold, then the edge-length* a *is* $a^+_{n,\alpha,\gamma}$. *If* $\gamma = \mathrm{dgn}_n(\alpha)$, *then the edge-length* a *is*

$$\pi - \arccos\left(\left(\sec\frac{\pi}{n}\right)\left(\sin\frac{\pi}{n} + 1\right)\cot\alpha\right) = a^+_{n,\alpha,\gamma} = a^-_{n,\alpha,\gamma}.$$

If $\alpha = \mathrm{dgn}_n(\gamma)$, *then the edge-length* a *is*

$$\pi - \arccos\left(\left(\sec\frac{\pi}{n}\right)\left(\sin\frac{\pi}{n} + 1\right)\cot\gamma\right) = a^+_{n,\alpha,\gamma} = a^-_{n,\alpha,\gamma}.$$

To prove Theorem 7, we prove the following lemma.

Lemma 4. *If some of condition* (21), *condition* (23), *condition* (24) *and the three conditions with* α *and* γ *swapped hold, then for every* $a \in (0, \pi/2)$ *with* $f_{n,\alpha,\gamma}(\cos a) = 0$, *there exists a* $Q_{n,\alpha,\gamma,a}$*-quadrangle.*

Proof. The assumption implies

$$0 < \alpha + \gamma - \frac{3\pi}{2} < \frac{\pi}{2}. \tag{26}$$

Consider a quadrangle $Nv_0v_1v_2$ such that the inner angle between two edges of length a is $\beta = 2\pi/n$, and the two inner angles neighboring to β are α and γ. We prove $Nv_0v_1v_2$ is indeed a *spherical* 4-gon. $\angle v_0Nv_2 = 2\pi/n < \pi$ and $v_0N = v_2N = a < \pi/2$. So, $\theta := \angle v_2v_0N$ is strictly between 0 and $\pi/2$.

As a spherical 3-gon v_0Nv_2 clearly exists, a quadrangle $Nv_0v_1v_2$ exists, if and only if a spherical 3-gon $v_0v_1v_2$ exists. The last condition holds, if and only if $\alpha - \gamma + \delta < \pi$, $-\alpha + \gamma + \delta < \pi$, $(\alpha - \theta) + (\gamma - \theta) - \delta < \pi$, and $(\alpha - \theta) + (\gamma - \theta) + \delta > \pi$, by Proposition 1 (1). $\alpha, \gamma > \pi/2$ holds from the assumption of this lemma. So, $\alpha + \gamma + \delta = 2\pi$ implies the first and the second of the four inequalities. The last inequality follows from $0 < \theta < \pi/2$ and $\alpha + \gamma + \delta = 2\pi$. The third inequality is $\alpha + \gamma - 3\pi/2 < \theta$. By $0 < \theta < \pi/2$ and the range (26) of $\alpha + \gamma$,

$$\text{the quadrangle } Nv_0v_1v_2 \text{ exists} \quad \Leftrightarrow \quad -\tan(\alpha + \gamma) > \cot\theta.$$

Here $\cot\theta = \cos a \tan(\pi/n)$. To see this, represent the length of the base edge of a spherical isosceles 3-gon v_0Nv_2, in terms of a, n, by using spherical cosine law (Proposition 1 (2b)). By applying the spherical cosine law for angles (Proposition 1 (2a)) to v_0Nv_2, we have $\cos(2\pi/n) = -\cos^2\theta + \sin^2\theta(\cos^2 a + \sin^2 a \cos(2\pi/n))$. As $\cos(\pi/n) > 0$ by $n \geq 3$,

$$\sin\theta = \frac{\cos\frac{\pi}{n}}{\sqrt{\sin^2\frac{\pi}{n}\cos^2 a + \cos^2\frac{\pi}{n}}}.$$

So, as $\cos a > 0$ by the premise, $0 < \theta < \pi/2$ implies $\cot\theta = \cos a \tan(\pi/n)$.

Hence, by Theorem 4, Lemma 4 is equivalent to: For every $n \geq 3$, if (21), (23), or (24), then for every $a \in (0, \pi/2)$ with $f_{n,\alpha,\gamma}(\cos a) = 0$, we have

$$\cos a < -\cot\frac{\pi}{n}\tan(\alpha + \gamma). \tag{27}$$

When condition (23) holds, $-\tan(\alpha + \gamma) > \tan(\pi/n)$, and thus inequality (27) holds.

Assume condition (21) or condition (24). By dgn_n,

$$\frac{\pi}{2} < \alpha < \frac{3\pi}{4} - \frac{\pi}{2n}, \qquad \pi < \frac{5\pi}{4} - \frac{\pi}{2n} < \gamma < \frac{3\pi}{2}. \tag{28}$$

Thus $f_{n,\alpha,\gamma}(0) = -\cot\alpha\cot\gamma > 0$. So two solutions of the quadratic equation $f_{n,\alpha,\gamma}(x) = 0$ are of the same sign. Hence inequality (27) follows from

$$n \geq 3, \qquad (21)\ \&\ f_{n,\alpha,\gamma}(x) = 0 \quad \Rightarrow \quad x < -\cot\frac{\pi}{n}\tan(\alpha + \gamma). \tag{29}$$

As $f_{n,\alpha,\gamma}(x)$ is quadratic, condition (29) is equivalent to the conjunction of

$$f_{n,\alpha,\gamma}\left(-\cot\frac{\pi}{n}\tan(\alpha + \gamma)\right) > 0, \tag{30}$$

and the condition $\mathrm{axis}(n, \alpha, \gamma) < -\cot(\pi/n) \tan(\alpha + \gamma)$:

$$\frac{1}{2} \cot \frac{\pi}{n}(\cot \alpha + \cot \gamma) < -\cot \frac{\pi}{n} \tan(\alpha + \gamma). \tag{31}$$

Inequality (30) is proved as follows: By calculation, the left-hand side is

$$\frac{\sin(\alpha + \gamma + \pi/n) \sin(\alpha + \gamma - \pi/n)}{\cos \alpha \sin \alpha \cos \gamma \sin \gamma(\tan \alpha \tan \gamma - 1)^2 \sin^2(\pi/n)}.$$

This is positive, because (28) implies $\cos \alpha \sin \alpha \cos \gamma \sin \gamma < 0$, and Lemma 3 (3) implies $2\pi - \pi/n < \alpha + \gamma < 2\pi$. So (30) is established.

In inequality (31), the right-hand side divided by the left-hand side has absolute value $M = |2 \sin \alpha \sin \gamma / \cos(\alpha + \gamma)|$. The right-hand side of (31) is positive by (21) and $\alpha + \gamma + \delta = 2\pi$. So, we have only to show $M > 1$. The numerator $2 \sin \alpha \sin \gamma$ is negative by (28) and the denominator $\cos(\alpha + \gamma)$ is positive by (21). So $M > 1$ is equivalent to $\cos(\alpha - \gamma) < 0$. By (28), $\pi/2 = (5\pi/4 - \pi/(2n)) - (3\pi/4 - \pi/(2n)) < \gamma - \alpha < 3\pi/2 - \pi/2 = \pi$. This proves inequality (31) and thus the implication (29). So the spherical 3-gon $v_0 v_1 v_2$ exists. This establishes Lemma 4. □

By calculation,

$$(\dagger) \quad \mathrm{axis}(n, \alpha, \mathrm{dgn}_n(\alpha)) = -\sec \frac{\pi}{n} \left(\sin \frac{\pi}{n} + 1 \right) \cot \alpha.$$

Lemma 5. *Let $n = 3, 4, 5, \ldots$. Suppose*

$$\frac{3\pi}{4} - \frac{\pi}{2n} < \alpha < \pi < \gamma < \frac{5\pi}{4} - \frac{\pi}{2n}, \qquad 2\pi - \frac{\pi}{n} < \alpha + \gamma, \qquad \textit{and} \qquad \gamma < \mathrm{dgn}_n(\alpha).$$

Then there is no $a \in (0, \pi)$ such that $f_{n,\alpha,\gamma}(\cos a) = 0$.

Proof. We prove that $f_{n,\alpha,\gamma}(x) = 0 \Rightarrow x \geq 1$. We have only to verify $\mathrm{axis}(n, \alpha, \gamma) > 1$ and $f_{n,\alpha,\gamma}(1) \geq 0$.

We show $\mathrm{axis}(n, \alpha, \gamma) > 1$. By the premise, $\cot(\pi/n) > 0$ and $\pi/2 < \alpha < \pi$. By (1) and (2) of Lemma 3, we have $\pi < \mathrm{dgn}_n(\alpha) < 3\pi/2$. So, by the premise, $\pi < \gamma < \mathrm{dgn}_n(\alpha) < 3\pi/2$. Thus, by (20), $\mathrm{axis}(n, \alpha, \gamma) > \mathrm{axis}(n, \alpha, \mathrm{dgn}_n(\alpha))$. By (†) and $\pi/2 < 3\pi/4 - \pi/(2n) < \alpha < \pi$, $\mathrm{axis}(n, \alpha, \mathrm{dgn}_n(\alpha)) > \mathrm{axis}(n, 3\pi/4 - \pi/(2n), \mathrm{dgn}_n(3\pi/4 - \pi/(2n)))$. The last is 1 by calculation. This concludes $\mathrm{axis}(n, \alpha, \gamma) > 1$.

Next, we verify $f_{n,\alpha,\gamma}(1) \geq 0$. By the first premise $3\pi/4 - \pi/(2n) < \alpha < \pi < \gamma < 5\pi/4 - \pi/(2n)$, we have $\alpha + \gamma + \pi/n < 2\pi + \pi/4 + \pi/(2n)$. So, by $n \geq 3$ and the second premise, $2\pi < \alpha + \gamma + \pi/n < 2\pi + 5\pi/12$. Thus, by the first premise and (19), $f_{n,\alpha,\gamma}(1) \geq 0$. This completes the proof of Lemma 5. □

Proof of Theorem 7. Let $\alpha > \pi$ or $\gamma > \pi$. The edge-length a is smaller than $\pi/2$. Otherwise, equivalence (1) of Lemma 2 implies $\delta > \pi$. So α and γ are both less than π, which is absurd. So $0 < \cos a < 1$.

Theorem 7 (1) is proved as follows: By Theorem 4, the following two assertions are equivalent:

- more than one PDW_n-quadrangles $Q_{n,\alpha,\gamma,a}$ exist.
- the quadratic polynomial $f_{n,\alpha,\gamma}(x)$ has two distinct zeros x_1, x_2 in an open interval $(0,1)$ such that a quadrangle $Q_{n,\alpha,\gamma,\arccos x_i}$ exists for each x_i $(i = 1,2)$.

Here the quadratic polynomial $f_{n,\alpha,\gamma}(x)$ has two distinct zeros x_1, x_2 in an open interval $(0,1)$ if and only if the following three are all true:

(i) $f_{n,\alpha,\gamma}(0) > 0$ and $f_{n,\alpha,\gamma}(1) > 0$;
(ii) $0 < \mathrm{axis}(n,\alpha,\gamma) < 1$; and
(iii) $\Delta_{n,\alpha,\gamma} > 0$.

Hence, more than one PDW_n-quadrangles $Q_{n,\alpha,\gamma,a}$ exist, if and only if condition (21) or condition (22) holds. It is due to (2) of Lemma 2, Lemma 4 and the following:

Claim 6. *For the three conditions mentioned above, the following holds:*
(1) *In case* $\pi/2 < \alpha < \pi < \gamma < 3\pi/2$, *inequality* (21) $\Leftrightarrow$ (i) & (iii).
(2) *In case* $\pi/2 < \gamma < \pi < \alpha < 3\pi/2$, *inequality* (22) $\Leftrightarrow$ (i) & (iii).
(3) *In each of the above-mentioned two cases,* (i) & $\Delta_{n,\alpha,\gamma} \geq 0 \Rightarrow$ (ii).

Proof. Claim 6 (1) is proved as follows: $f_{n,\alpha,\gamma}(0) = -\cot\gamma\cot\alpha > 0$ by the premise. Hence, condition (i) is equivalent to $f_{n,\alpha,\gamma}(1) > 0$. Thus, by the premise and (19),

$$\text{condition (i)} \quad \Leftrightarrow \quad \alpha + \gamma > 2\pi - \frac{\pi}{n}. \tag{32}$$

By the premise and Lemma 3 (1),

$$\text{condition (iii)} \quad \Leftrightarrow \quad \gamma < \mathrm{dgn}_n(\alpha). \tag{33}$$

See Figure 11. By the premise and Lemma 5,

$$(\ddagger) \quad \text{condition (i) \& } \Delta_{n,\alpha,\gamma} \geq 0 \quad \Rightarrow \quad \alpha < \frac{3\pi}{4} - \frac{\pi}{2n} \text{ or } \frac{5\pi}{4} - \frac{\pi}{2n} < \gamma.$$

By (32), condition (i) implies $\alpha < 3\pi/4 - \pi/(2n) \Leftrightarrow 5\pi/4 - \pi/(2n) < \gamma$. Thus, by (32) and (33), we have (21) $\Leftrightarrow$ (i) & (iii). So, Claim 6 (1) holds. The same argument with α and γ swapped proves Claim 6 (2).

We prove Claim 6 (3). $\mathrm{axis}(n,\alpha,\gamma) > 0$ in either case, because $\cot\alpha + \cot\gamma = \sin(\alpha+\gamma)/\sin\alpha\sin\gamma > 0$ follows from $3\pi/2 < \alpha + \gamma = 2\pi - \delta < 2\pi$.

Consider the case $\pi/2 < \alpha < \pi < \gamma < 3\pi/2$. As $\cot\gamma > 0$,

$$\text{condition (ii)} \quad \Leftrightarrow \quad \gamma > \operatorname{arccot}\left(2\tan\frac{\pi}{n} - \cot\alpha\right) + \pi.$$

Hence, by equivalence (32), condition (ii) follows from

$$\pi - \alpha - \frac{\pi}{n} \geq \operatorname{arccot}\left(2\tan\frac{\pi}{n} - \cot\alpha\right). \tag{34}$$

The right-hand side is positive, because $\cot\alpha < 0$ by $\pi/2 < \alpha < \pi$. So, inequality (34) is equivalent to $\cot(\pi - \alpha - \pi/n) \leq 2\tan(\pi/n) - \cot\alpha$. Subtract $\tan(\alpha - \pi/2) + \tan(\pi/n)$ from both hand sides of the last inequality, and then divide them by $\tan(\pi/n)$. By the addition formula of tan, inequality (34) is equivalent to $\tan(\alpha - \pi/2)\tan(\alpha - \pi/2 + \pi/n) \leq 1$. By the assumption $\pi/2 < \alpha < \pi$ and implication (‡), this holds because the two arguments $\alpha - \pi/2$ and $(\alpha - \pi/2 + \pi/n)$ are both in the interval $(0, \pi/2)$ and have mean less than $\pi/4$. Thus inequality (34) holds. The other case $\pi/2 < \gamma < \pi < \alpha < 3\pi/2$ is proved by the same argument with α and γ swapped. This completes the proof of Claim 6. □

Theorem 7 (2) is proved as follows: First observe that there exists exactly one PDW_n-quadrangle, if and only if a quadrangle $Nv_0v_1v_2$ exists and

(a) $f_{n,\alpha,\gamma}(x)$ has a degenerate (i.e., double) zero strictly between 0 and 1; or

(b) $f_{n,\alpha,\gamma}(x)$ has distinct two zeros, but only one in the interval $(0, 1)$.

Here the edge-length a is less than $\pi/2$, from $\alpha > \pi$ or $\gamma > \pi$, by equivalence (1) of Lemma 2.

We prove that (a) $\Leftrightarrow (\gamma - \operatorname{dgn}_n(\alpha))(\alpha - \operatorname{dgn}_n(\gamma)) = 0$, as follows: Note that condition (a) holds if and only if we have all of the conditions (i), (ii) and $\Delta_{n,\alpha,\gamma} = 0$. By Claim 6 (3) and (32), the condition (a) is equivalent to $\alpha + \gamma > 2\pi - \pi/n$ & $\gamma = \operatorname{dgn}_n(\alpha)$ or to $\alpha + \gamma > 2\pi - \pi/n$ & $\alpha = \operatorname{dgn}_n(\gamma)$. Because $2\pi - \pi/n - \alpha < \operatorname{dgn}_n(\alpha)$ by Lemma 3 (3), the equation $\gamma = \operatorname{dgn}_n(\alpha)$ implies $\alpha + \gamma > 2\pi - \pi/n$. So, the condition (a) is equivalent to $\gamma = \operatorname{dgn}_n(\alpha)$ or $\alpha = \operatorname{dgn}_n(\gamma)$. This establishes the desired equivalence.

The quadratic equation $f_{n,\alpha,\gamma}(\cos a) = 0$ of $\cos a$ has the two solutions $a = a^+_{n,\alpha,\gamma}, a^-_{n,\alpha,\gamma}$, presented at the beginning of Subsection 6.1. If the two solutions are a degenerate solution $a = a^+_{n,\alpha,\gamma} = a^-_{n,\alpha,\gamma} = \arccos(\cot(\pi/n)(\cot\alpha + \cot\gamma)/2)$, then $\Delta_{n,\alpha,\gamma} = 0$.

Claim 7. *The arccosine of the degenerate (i.e., double) solution x of $f_{n,\alpha,\gamma}(x) = 0$ is* (17) *for $\gamma = \operatorname{dgn}_n(\alpha)$ and* (18) *for $\alpha = \operatorname{dgn}_n(\gamma)$.*

Proof. By Lemma 3 (1), either $\gamma = \mathrm{dgn}_n(\alpha)$ and $\pi/2 < \alpha < \pi < \gamma < 3\pi/2$, or $\alpha = \mathrm{dgn}_n(\gamma)$ and $\pi/2 < \gamma < \pi < \alpha < 3\pi/2$. Consider the first case. By (†), a is (17) for $\gamma = \mathrm{dgn}_n(\alpha)$. The proof for case $\alpha = \mathrm{dgn}_n(\gamma)$ is similar. This completes the proof of Claim 7. □

It is easy to see that condition (b) $\Leftrightarrow f_{n,\alpha,\gamma}(0) f_{n,\alpha,\gamma}(1) < 0$. As $f_{n,\alpha,\gamma}(0) > 0$ by the implication (2) of Lemma 2,

$$\text{(b)} \quad \Leftrightarrow \quad f_{n,\alpha,\gamma}(1) = -\sin\left(\alpha + \gamma + \frac{\pi}{n}\right) \csc \alpha \csc \gamma \csc \frac{\pi}{n} < 0.$$

By equivalence (32), (b) is equivalent to condition (23) of Theorem 7, for $\pi/2 < \alpha < \pi < \gamma$; and is equivalent to condition (25) of Theorem 7, for $\pi/2 < \gamma < \pi < \alpha$. Because $f_{n,\alpha,\gamma}(0) > 0$ and $f_{n,\alpha,\gamma}(1) < 0$ hold, the unique solution x of the quadratic equation $f_{n,\alpha,\gamma}(x) = 0$ strictly between 0 and 1 is the smaller solution of the equation. Therefore the edge-length a is $a^{+}_{n,\alpha,\gamma}$. Hence Lemma 4 establishes Theorem 7 (2). Thus Theorem 7 is proved. □

From Theorem 6 and Theorem 7, Theorem 5 follows.

7. A quadrangle organizing both non-isohedral tiling and isohedral one over the same skeleton

Recall a PDW_6-quadrangle $Q_{6,\alpha,\gamma,a}$ from Definition 4.

Theorem 8. *Copies of a spherical* 4*-gon* $T := Q_{6,\arccos(-1/2\sqrt{7}),4\pi/3,\arccos(1/3)}$ *organize both an* isohedral *tiling* $\mathscr{T}'$ *(Figure* 13 *(middle, right)) and a* non-isohedral *tiling* $\mathscr{T}$ *(Figure* 13 *(middle, left),* [1]*) such that the skeletons are the same pseudo-double wheel. The quadratic equation associated to* T *of Theorem* 4 *is* $(x - 1/3)^2$.

Proof. The edge-lengths and inner angles of $\mathscr{T}$ are as in Figure 14 (lower). So, a tile (designated $N234$ in the figure) of $\mathscr{T}$ has two edges of length a, both incident to the vertex N. N is antipodal to a vertex S, because there are two congruent paths between the two vertices in Figure 14 (lower). The edge between a vertex δ (designated by 3 in Figure 14 (lower)) and S is a, by the figure. The area of the tile of $\mathscr{T}$ is $4\pi/12$, as $\mathscr{T}$ is a spherical tiling by twelve congruent tiles. So, the tile of $\mathscr{T}$ is a PDW_6-quadrangle, by Fact 2 (2). By the definition of $f_{n,\alpha,\gamma}(x)$ in Theorem 4, we have $f_{6,\arccos(-1/2\sqrt{7}),4\pi/3}(x) = (x - 1/3)^2$. □

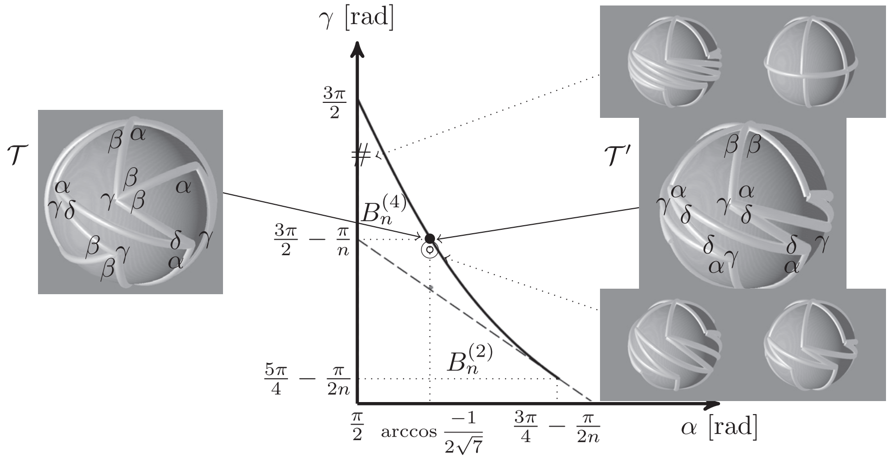

Fig. 13. The copies of the tiles of the rightmost, middle spherical isohedral tiling $\mathscr{T}'$ organize a spherical *non-isohedral* tiling $\mathscr{T}$ over the skeleton of $\mathscr{T}'$. The middle graph is an excerpt of Figure 12. The right four images are *the* spherical isohedral tilings by $Q_{n,\alpha,\gamma,a}$ for $n=6$ and designated (α,γ) on the graph. The distribution of inner angles and that of edge-length on the skeleton of $\mathscr{T}$ is the reflection of Figure 14.

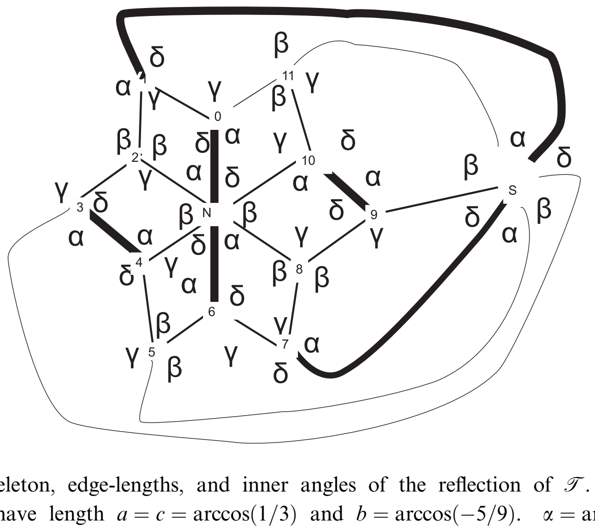

Fig. 14. The skeleton, edge-lengths, and inner angles of the reflection of $\mathscr{T}$. The solid, and the thick edges have length $a=c=\arccos(1/3)$ and $b=\arccos(-5/9)$. $\alpha=\arccos(-1/(2\sqrt{7}))$, $\beta=\pi/3$, $\gamma=4\pi/3$, and $\delta=\arccos(5/(2\sqrt{7}))$. See [1] for detail of $\mathscr{T}$.

We conjecture that $Q_{6,\arccos(-1/2\sqrt{7}),4\pi/3,\arccos(1/3)}$ is the only spherical 4-gon such that copies of it organize both a non-isohedral tiling and an isohedral tiling over a pseudo-double wheel. The conjecture is true by [1, Theorem 2], once the following is proved: from any spherical non-isohedral tiling by congruent $Q_{n,\alpha,\gamma,a}$ over a pseudo-double wheel, we can obtain such a tiling $\mathscr{T}$ satisfying the condition (II) of [1, Theorem 2].

To generalize Theorem 8, we want to enumerate all spherical polygons which organize both *non-isohedral* tilings and *isohedral* tilings over the same skeletons. This is a weak inverse problem of the following theorem:

Proposition 3 (Grünbaum-Shephard [7]). *The skeleton of a spherical isohedral tiling is exactly a pseudo-double wheel, the skeleton of a bipyramid, that of a Platonic solid, or that of an Archimedean dual.*

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References

[1] Y. Akama, Classification of spherical tilings by congruent quadrangles over pseudo-double wheels (I)—a special tiling by congruent concave quadrangles, Hiroshima Math. J., Vol. 43, No. 3, pp. 285–304, 2013.

[2] D. V. Alekseevskij, È. B. Vinberg, and A. S. Solodovnikov, Geometry of spaces of constant curvature, In Geometry, II, Vol. 29 of Encyclopaedia Math. Sci., pp. 1–138. Springer, Berlin, 1993.

[3] A. Behmaram and S. Friedland, Upper bounds for perfect matchings in Pfaffian and planar graphs, Electron. J. Combin., Vol. 20, No. 1, p. 64, 2013.

[4] G. Brinkmann, S. Greenberg, C. Greenhill, B. D. McKay, R. Thomas, and P. Wollan, Generation of simple quadrangulations of the sphere, Discrete Math., Vol. 305, No. 1–3, pp. 33–54, 2005.

[5] F. A. Cotton, Chemical Applications of Group Theory, Wiley India, third edition, 2009.

[6] M. Deza and M. Dutour Sikirić, Geometry of chemical graphs: polycycles and two-faced maps, Cambridge university press, 2008.

[7] B. Grünbaum and G. C. Shephard, Spherical tilings with transitivity properties, In C. Davis, B. Grünbaum, and F. A. Sherk, editors. The geometric vein. The Coxeter Festschrift. Springer-Verlag, New York, 1981.

[8] Y. Sakano, Toward classification of spherical tilings by congruent quadrangles, Master's thesis, Mathematical Institute, Tohoku University, March, 2010. In Japanese.

[9] Y. Sakano and Y. Akama, Anisohedral spherical triangles and classification of spherical tilings by congruent kites, darts, and rhombi, Hiroshima Math. J., Vol. 45, No. 3, pp. 309–339, 2015.

[10] Y. Ueno and Y. Agaoka, Examples of spherical tilings by congruent quadrangles, Mem. Fac. Integrated Arts and Sci., Ser. IV, Vol. 27, pp. 135–144, 2001.

[11] Y. Ueno and Y. Agaoka, Classification of tilings of the 2-dimensional sphere by congruent triangles, Hiroshima Math. J., Vol. 32, No. 3, pp. 463–540, 2002.

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