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Besov and Triebel–Lizorkin space estimates for fractional diffusion

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Abstract. We study Besov and Triebel–Lizorkin space estimates for fractional diffusion. We measure the smoothing effect of the fractional heat flow in terms of the Besov and Triebel–Lizorkin scale. These estimates have many applications to various partial differential equations.

1. Introduction

In this paper we shall consider the Cauchy problem for the following fractional power dissipative equation

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u = f, & (0,T) \times \mathbb{R}^d \\ u = g, & \{0\} \times \mathbb{R}^d. \end{cases} \tag{1}$$

where f and g are given data. Recently, there have been many interests on the fractional diffusion from the theory of probability related to heavy tail probability distribution and Lévy processes. Fractional Laplacians also have many applications to various non-linear PDEs related to non-local phenomena in science and engineering problems. See for example [2], [11], [3], [10], [6], [5], and the references therein.

By the Duhamel principle, the solution u to the above linear equation (1) can be formally written as

$$u = e^{-t(-\Delta)^{\alpha/2}} g + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} f(s)\,ds.$$

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The action of these operators can be understood naturally in the Fourier space. This motivates us to define the following two integral operators S^α and T^α.

Definition 1. Let $0 < \alpha < \infty$. The kernel $P^\alpha(t,x)$ is defined by

$$P^\alpha(t,x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-t|\xi|^\alpha}\, d\xi.$$

The operator S^α is defined by

$$S^\alpha f(t,x) = \int_{\mathbb{R}^d} P^\alpha(t, x-y) f(y) dy. \tag{2}$$

The operator T^α is defined by

$$T^\alpha f(t,x) = \int_0^t \int_{\mathbb{R}^d} P^\alpha(t-s, x-y) f(s,y) dy ds. \tag{3}$$

The present paper is devoted to understand the mapping properties of these operators among the refined function spaces. In particular, we measure the smoothing effect of the fractional heat flow in terms of the Besov and Triebel–Lizorkin scaling. Several important function spaces in analysis can be thought of as elements of the Besov–Lipschitz and Triebel–Lizorkin spaces. So, we consider their natural parabolic extensions. Our main objective here is to understand mapping properties of the above operators in the corresponding parabolic spaces.

Definition 2. Let $Q = [0,T] \times \mathbb{R}^d$ with $0 < T < \infty$ and $\sigma \in \mathbb{R}$. The space $L^r B_\sigma^{p,q}(Q)$ is the set of measurable functions $u : Q \to \mathbb{C}$ such that

$$\|u\|_{L^r B_\sigma^{p,q}(Q)} := \left(\int_0^T \|u(t,\cdot)\|_{B_\sigma^{p,q}(\mathbb{R}^d)}^r dt \right)^{1/r} < \infty,$$

where $B_\sigma^{p,q}(\mathbb{R}^d)$ is the standard Besov space. Similarly, we define the space $L^r F_\sigma^{p,q}(Q)$ for standard Triebel–Lizorkin spaces $F_\sigma^{p,q}(\mathbb{R}^d)$.

We shall recall the exact definitions of Besov and Triebel–Lizorkin spaces in the next section. We mainly focus on the parabolic Triebel–Lizorkin space estimates and deduce the parabolic Besov space estimates by a slight modification of the former analysis. We note that for the development of the parabolic Besov space estimates of the Gaussian heat flow and their applications to nonlinear PDEs one may consult the following books [4], [9] and [1].

Throughout the paper, we shall use the notation $A \lesssim B$, which means that there is an absolute positive constant C such that $|A| \le CB$.

Now we state our main theorems. Our first theorem is the simplest case of parabolic Triebel–Lizorkin space estimates.

Theorem 1. *For* $1 \le p \le \infty$

$$\|T^{\alpha}f\|_{L^pF^{p,p}_{\sigma+\alpha}(Q)} \lesssim (1+T)\|f\|_{L^pF^{p,p}_{\sigma}(Q)}.$$

We note it is necessary that the operator norm of T^{α} depends on T. See Remark 3. The next theorem is a general version of parabolic Triebel–Lizorkin space estimates.

Theorem 2. *If one of the following conditions holds;*
- $r = 1$ *and* $1 \le q \le p < \infty$,
- $1 < r < \infty$ *and* $1 < p, q < \infty$,
- $r = \infty$ *and* $1 < p \le q \le \infty$,

then for $\beta < \alpha$

$$\|T^{\alpha}f\|_{L^rF^{p,q}_{\sigma+\beta}(Q)} \lesssim (1+T)\|f\|_{L^rF^{p,q}_{\sigma}(Q)}.$$

Remark 1. *This is an extension of the main theorem for the range of exponents* p, q, r, *and* α *in the paper* [13].

Remark 2. *When* $p = q = r = 2$, *Theorem* 1 *is better. For the general exponents we don't know the end point case* $\beta = \alpha$ *continues to hold.*

We also deduce the corresponding parabolic Besov space estimates.

Theorem 3. *If* $1 < p < \infty$, $1 < q = r < \infty$, *or* $1 \le p < \infty$, $q = r = 1$, *or* $1 < p \le \infty$, $q = r = \infty$, *then*

$$\|T^{\alpha}f\|_{L^rB^{p,q}_{\sigma+\alpha}(Q)} \lesssim (1+T)\|f\|_{L^rB^{p,q}_{\sigma}(Q)}.$$

Theorem 4. *If* $(1/p, 1/q, 1/r)$ *belongs to the interior of the octahedron* $P_1P_2P_3$–$P_5P_7P_8$, *or the triangle* $P_1P_2P_3 \backslash$ *the segment* P_1P_2, *or the triangle* $P_5P_7P_8 \backslash$ *the segment* P_5P_8, *then for* $\beta < \alpha$

$$\|T^{\alpha}f\|_{L^rB^{p,q}_{\sigma+\beta}(Q)} \lesssim (1+T)\|f\|_{L^rB^{p,q}_{\sigma}(Q)},$$

where $P_1 = (0,0,0)$, $P_2 = (1,0,0)$, $P_3 = (1,1,0)$, $P_5 = (0,0,1)$, $P_7 = (1,1,1)$ *and* $P_8 = (0,1,1)$.

Finally, related to the initial value problem, we have the following two theorems.

Theorem 5. (i) *If* $1 \le r \le \infty$ *and* $1 \le q \le p < \infty$, *then for* $\beta < \alpha/r$

$$\|S^{\alpha}g\|_{L^rF^{p,q}_{\sigma+\beta}(Q)} \lesssim (1+T^{1/r})\|g\|_{F^{p,q}_{\sigma}(\mathbb{R}^d)}.$$

(ii) *For* $1 \le p \le r < \infty$

$$\|S^{\alpha}g\|_{L^rF^{p,p}_{\sigma+\alpha/r}(Q)} \lesssim (1+T^{1/r})\|g\|_{F^{p,p}_{\sigma}(\mathbb{R}^d)}.$$

The implied constants do not depend on T.

Theorem 6. (i) *Let* $1 \le p < \infty$, $1 \le q \le r < \infty$, $0 < \alpha < \infty$ *and* $\beta \le \alpha/r$. *Then*

$$\|S^{\alpha} g\|_{L^r B^{p,q}_{\sigma+\beta}(Q)} \lesssim (1 + T^{1/r}) \|g\|_{B^{p,q}_{\sigma}(\mathbb{R}^d)}.$$

(ii) *Let* $1 \le p < \infty$, $1 \le r \le q < \infty$, $0 < \alpha < \infty$ *and* $\beta < \alpha/r$. *Then*

$$\|S^{\alpha} g\|_{L^r B^{p,q}_{\sigma+\beta}(Q)} \lesssim (1 + T^{1/r}) \|g\|_{B^{p,q}_{\sigma}(\mathbb{R}^d)}.$$

(iii) *Let* $1 \le p \le r < \infty$ *and* $0 < \alpha < \infty$. *Then*

$$\|S^{\alpha} g\|_{L^r B^{p,p}_{\sigma+\alpha/r}(Q)} \lesssim (1 + T^{1/r}) \|g\|_{B^{p,p}_{\sigma}(\mathbb{R}^d)}.$$

2. Preliminaries

The Littlewood–Paley theory becomes apparent in numerous applications. In particular, it is remarkable that the Littlewood–Paley theory characterizes various function spaces. A classical treatment of the theory is contained in the book of Stein [12]. For a modern and comprehensive treatment we refer to the books of Grafakos [7, 8].

Now, we recall some standard definitions. We fix a radial Schwartz function Φ on $\mathbb{R}^d$ whose Fourier transform is nonnegative, supported in the ball $|\xi| \le 2$, equal to 1 in the ball $|\xi| \le 1$ and define $\hat{\Psi}(\xi) = \hat{\Phi}(\xi) - \hat{\Phi}(2\xi)$. We define the partition of unity

$$\hat{\Phi}(\xi) + \sum_{j=1}^{\infty} \hat{\Psi}(2^{-j}\xi) = 1.$$

Using this, the Littlewood–Paley operators S_0 and Δ_j for $j \in \mathbb{N}$ are defined as

$$S_0 f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{\Phi}(\xi) \hat{f}(\xi) d\xi \tag{4}$$

and

$$\Delta_j f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{\Psi}(2^{-j}\xi) \hat{f}(\xi) d\xi. \tag{5}$$

Besov–Lipschitz and Triebel–Lizorkin spaces are defined as follows. For $\sigma \in \mathbb{R}$, $0 < p, q \le \infty$, the Besov space $B^{p,q}_{\sigma}$ is the space of all tempered distributions f with

$$\|f\|_{B^{p,q}_{\sigma}} := \|S_0 f\|_{L^p} + \left(\sum_{j=1}^{\infty} \|2^{j\sigma} |\Delta_j f| \|_{L^p}^q \right)^{1/q} < \infty. \tag{6}$$

For $\sigma \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$, the Triebel–Lizorkin space $F_\sigma^{p,q}$ is the space of all tempered distributions f with

$$\|f\|_{F_\sigma^{p,q}} := \|S_0 f\|_{L^p} + \left\| \left(\sum_{j=1}^{\infty} (2^{j\sigma} |\Delta_j f|)^q \right)^{1/q} \right\|_{L^p} < \infty. \tag{7}$$

The following lemma is a point-wise estimate of the localized kernel, which is obtained by the standard integration by parts argument.

Lemma 1. *Let $\lambda > 0$ and*

$$K_\lambda(t,x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \hat{\Psi}(\xi/\lambda) e^{-t|\xi|^\alpha}\, d\xi.$$

Then there exists a positive constant c such that for all $\lambda > 0$

$$K_\lambda(t,x) \lesssim \frac{\lambda^d \exp(-ct\lambda^\alpha)}{(1+|\lambda x|^2)^{d+1}}.$$

The implied constant does not depend on λ.

Proof. By a change of variables we have

$$K_\lambda(t,x) = \lambda^d \int_{\mathbb{R}^d} e^{2\pi i \lambda x\cdot\xi} \hat{\Psi}(\xi) e^{-t\lambda^\alpha|\xi|^\alpha}\, d\xi.$$

Using the identity

$$(I - \Delta_\xi) e^{2\pi i \lambda x\cdot\xi} = (1 + 4\pi^2 |\lambda x|^2) e^{2\pi i \lambda x\cdot\xi}$$

we can carry out repeated integration by parts to get the result. □

3. Proof of Theorem 1

The adjoint operator of T^α is useful. It is given by

$$\widetilde{T^\alpha} f(t,x) = \int_t^T \int_{\mathbb{R}^d} P^\alpha(s-t, y-x) f(s,y)\, dy ds. \tag{8}$$

We may prove the theorem with assuming $\sigma = 0$. We divide its proof into four steps.

- First we show that for $1 \leq p, r \leq \infty$

$$\|S_0 T^\alpha f\|_{L^r L^p(Q)} \lesssim T \|S_0 f\|_{L^r L^p(Q)}. \tag{9}$$

It is easy to see that the kernel

$$P^\alpha(t,x) = t^{-d/\alpha} P^\alpha(1, t^{-1/\alpha} x)$$

is integrable uniformly in t. Hence we have for all $1 \le p \le \infty$

$$\|S_0(T^\alpha f)(t)\|_p = \|T^\alpha(S_0 f)(t)\|_p \le \int_0^t \|P^\alpha(t-s) * (S_0 f)(s)\|_p ds$$
$$\lesssim \int_0^t \|(S_0 f)(s)\|_p ds.$$

Integrating in time gives (9).

- We shall prove the case $p = 1$. By (9) it is enough to show

$$\int_0^T \int_{\mathbb{R}^d} \sum_{j=1}^\infty |T_j(t,x)| dx dt \lesssim \int_0^T \int_{\mathbb{R}^d} \sum_{j=1}^\infty |\Delta_j f(s,y)| dy ds, \qquad (10)$$

where

$$T_j(t,x) = 2^{j\alpha} \Delta_j T^\alpha \Delta_j f(t,x)$$
$$= \int_0^t \int_{\mathbb{R}^d} P_j(t-s, x-y) \Delta_j f(s,y) dy ds$$

and

$$P_j(t,x) = 2^{j\alpha} \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \hat{\Psi}(2^{-j}\xi) e^{-t|\xi|^\alpha} \, d\xi.$$

By Tonelli's theorem

$$\int_0^T \int_{\mathbb{R}^d} \sum_{j=1}^\infty |T_j(t,x)| dx dt$$
$$\le \sum_{j=1}^\infty \int_0^T \int_{\mathbb{R}^d} \left(\int_s^T \int_{\mathbb{R}^d} |P_j(t-s, x-y)| dx dt \right) |\Delta_j f(s,y)| dy ds.$$

Using the pointwise estimate for the kernel, Lemma 1, we obtain

$$\sup_{T,j,s,y} \int_s^T \int_{\mathbb{R}^d} |P_j(t-s, x-y)| dx dt \lesssim \sup_j \int_0^\infty 2^{j\alpha} e^{-ct2^{j\alpha}} \, dt < \infty.$$

Hence we get the estimate (10).

- We shall prove the case $p = 2$. By (9) it is enough to show

$$\sum_{j=1}^\infty 2^{j2\alpha} \int_0^T \int_{\mathbb{R}^d} |\Delta_j T^\alpha f(t,x)|^2 dx dt \lesssim \int_0^T \int_{\mathbb{R}^d} |f(t,x)|^2 dx dt. \qquad (11)$$

By Plancherel's theorem

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |\Delta_j T^\alpha f(t,x)|^2 dx dt \\
&\quad = \int_0^T \int_{\mathbb{R}^d} \left| \int_0^t \hat{\Psi}(2^{-j}\xi) e^{-(t-s)|\xi|^\alpha} \hat{f}(s,\xi) ds \right|^2 d\xi dt \\
&\quad = \int_{\mathbb{R}^d} |\hat{\Psi}(2^{-j}\xi)|^2 \int_0^T \left| \int_0^t e^{-(t-s)|\xi|^\alpha} \hat{f}(s,\xi) ds \right|^2 dt d\xi.
\end{aligned}
$$

Using Young's inequality we estimate the time convolution as

$$
\begin{aligned}
&\int_0^T \left| \int_0^t e^{-(t-s)|\xi|^\alpha} \hat{f}(s,\xi) ds \right|^2 dt \\
&\quad \leq \left(\int_0^T e^{-t|\xi|^\alpha} dt \right)^2 \left(\int_0^T |\hat{f}(t,\xi)|^2 dt \right) \\
&\quad \lesssim |\xi|^{-2\alpha} \int_0^T |\hat{f}(t,\xi)|^2 dt.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\sum_{j=1}^\infty 2^{j2\alpha} \int_0^T \int_{\mathbb{R}^d} |\Delta_j T^\alpha f(t,x)|^2 dx dt \\
&\quad \lesssim \sum_{j=1}^\infty 2^{j2\alpha} \int_{\mathbb{R}^d} |\hat{\Psi}(2^{-j}\xi)|^2 \left(|\xi|^{-2\alpha} \int_0^T |\hat{f}(t,\xi)|^2 dt \right) d\xi \\
&\quad = \int_0^T \int_{\mathbb{R}^d} \sum_{j=1}^\infty (|\hat{\Psi}(2^{-j}\xi)|^2 |2^{-j}\xi|^{-2\alpha}) |\hat{f}(t,\xi)|^2 d\xi dt \\
&\quad \lesssim \int_0^T \int_{\mathbb{R}^d} |\hat{f}(t,\xi)|^2 d\xi dt.
\end{aligned}
$$

By Plancherel's theorem we get the estimate (11).

- By an interpolation we have for $1 \leq p \leq 2$

$$\|T^\alpha f\|_{L^p F^{p,p}_{\sigma+\alpha}(Q)} \lesssim (1+T) \|f\|_{L^p F^{p,p}_{\sigma}(Q)}.$$

By the similar way, the adjoint operator also satisfies for $1 \leq p \leq 2$

$$\|\widetilde{T^\alpha} f\|_{L^p F^{p,p}_{\sigma+\alpha}(Q)} \lesssim (1+T) \|f\|_{L^p F^{p,p}_{\sigma}(Q)}.$$

By duality, we get the result for $2 \leq p \leq \infty$. This completes the proof of Theorem 1. □

Remark 3. *We note the necessity of the final time T in the operator norm of T^α. Even when $p = 2$, we see that the operator norm of $S_0 T^\alpha$ is greater than or equal to $T/\sqrt{12}$. Indeed, we fix an arbitrary final time $T > 1$ and suppose g depends only on x with* $\operatorname{supp} \hat{g} \subset \{\xi \in \mathbb{R}^d : |\xi| < T^{-1/\alpha}\}$. *Using Plancherel's theorem and the support of $\hat{g}$, we obtain that*

$$\begin{aligned}\|S_0 T^\alpha g\|_{L^2L^2(Q)}^2 &= \int_0^T \int_{\mathbb{R}^d} \left| \int_0^t e^{-(t-s)|\xi|^\alpha} \hat{\Phi}(\xi)\hat{g}(\xi) ds \right|^2 d\xi dt \\ &= \int_{\mathbb{R}^d} \int_0^T \left| \int_0^t e^{-(t-s)|\xi|^\alpha} ds \right|^2 dt |\hat{g}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} \int_0^T \left(\frac{1 - e^{-t|\xi|^\alpha}}{t|\xi|^\alpha} \right)^2 t^2 \, dt |\hat{g}(\xi)|^2 d\xi.\end{aligned}$$

Since the function $(1 - e^{-u})/u$ is monotonically decreasing for $0 < u \le 1$, we have for $0 < t < T$ and $|\xi| < T^{-1/\alpha}$,

$$\frac{1 - e^{-t|\xi|^\alpha}}{t|\xi|^\alpha} \ge 1 - e^{-1} \ge \frac{1}{2}.$$

Thus,

$$\|S_0 T^\alpha g\|_{L^2L^2(Q)}^2 \ge \frac{T^3}{12} \int_{\mathbb{R}^d} |\hat{g}(\xi)|^2 d\xi = \frac{T^3}{12} \|g\|_2^2.$$

Since $\|g\|_{L^2L^2(Q)} = \sqrt{T}\|g\|_2$, we conclude that

$$\|S_0 T^\alpha\|_{L^2L^2(Q) \to L^2L^2(Q)} \ge T/\sqrt{12}.$$

Actually, we can make $\operatorname{supp} \hat{g}$ *concentrated at the origin so that the operator norm of $S_0 T^\alpha$ is greater than or equal to $T/\sqrt{3}$.*

4. Proof of Theorem 2

We may prove the theorem with assuming $\sigma = 0$. We divide its proof into three steps.

- We shall show that if $1 \le q \le p < \infty$ and $1 \le r < \infty$, then for $\beta < \alpha/r$

$$\|T^\alpha f\|_{L^r F_\beta^{p,q}(Q)} \lesssim (1 + T)\|f\|_{L^1 F_0^{p,q}(Q)}. \tag{12}$$

Consider

$$B(t) := \left\| \left(\sum_{j=1}^\infty |2^{j\beta} \Delta_j T^\alpha \Delta_j f(t,x)|^q \right)^{1/q} \right\|_p.$$

For notational convenience, we denote

$$T_j(t,x) := 2^{j\beta} \Delta_j T^{\alpha} \Delta_j f(t,x) = \int_0^t \int_{\mathbb{R}^d} P_j(t-s, x-y) \Delta_j f(s,y) dyds$$

where

$$P_j(t,x) = 2^{j\beta} \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \hat{\Psi}(2^{-j}\xi) e^{-t|\xi|^{\alpha}} \, d\xi.$$

By the triangle inequality and Lemma 1

$$B(t) = \left\| \left(\sum_{j=1}^{\infty} |T_j(t)|^q \right)^{1/q} \right\|_p \le \left\| \sum_{j=1}^{\infty} |T_j(t)| \right\|_p \le \sum_{j=1}^{\infty} \int_0^t 2^{j\beta} e^{-c(t-s)2^{j\alpha}} \|\Delta_j f(s)\|_p ds.$$

By the triangle inequality and Young's inequality for the time variable

$$\begin{aligned} \|B\|_{L^r([0,T])} &\lesssim \sum_{j=1}^{\infty} \left\| \int_0^t 2^{j\beta} e^{-c(t-s)2^{j\alpha}} \|\Delta_j f(s)\|_p ds \right\|_{L^r([0,T])} \\ &\lesssim \sum_{j=1}^{\infty} \int_0^T 2^{j\beta} \left(\int_s^T e^{-cr(t-s)2^{j\alpha}} \, dt \right)^{1/r} \|\Delta_j f(s)\|_p ds \\ &\lesssim \int_0^T \sum_{j=1}^{\infty} 2^{j(\beta-\alpha/r)} \|\Delta_j f(s)\|_p ds. \end{aligned}$$

If $\beta < \alpha/r$ and $q \le p$, then we use Hölder's inequality to get

$$\begin{aligned} &\sum_{j=1}^{\infty} 2^{j(\beta-\alpha/r)} \|\Delta_j f(s)\|_p \\ &\lesssim \left(\sum_{j=1}^{\infty} 2^{jp'(\beta-\alpha/r)} \right)^{1/p'} \left(\sum_{j=1}^{\infty} \|\Delta_j f(s)\|_p^p \right)^{1/p} \\ &\lesssim \left\| \left(\sum_{j=1}^{\infty} |\Delta_j f(s)|^p \right)^{1/p} \right\|_p \le \left\| \left(\sum_{j=1}^{\infty} |\Delta_j f(s)|^q \right)^{1/q} \right\|_p \\ &\le \|f(s)\|_{F_0^{p,q}(\mathbb{R}^d)}. \end{aligned} \tag{13}$$

Hence

$$\|B\|_{L^r([0,T])} \lesssim \|f\|_{L^1 F_0^{p,q}(Q)}.$$

This together with (9) yields the result.

- We shall show that if $1 \le q \le p < \infty$, $\beta < \alpha$, and $1/p \ge 1/r > \beta/\alpha - 1 + 1/p$, then

$$\|T^\alpha f\|_{L^r F_\beta^{p,q}(Q)} \lesssim (1+T)\|f\|_{L^p F_0^{p,q}(Q)}. \tag{14}$$

By the triangle inequality and Young's inequality for the time variable

$$\begin{aligned}\|B\|_{L^r([0,T])} &\lesssim \sum_{j=1}^{\infty} \left\| \int_0^t 2^{j\beta} e^{-c(t-s)2^{j\alpha}} \|\Delta_j f(s)\|_p ds \right\|_{L^r([0,T])} \\ &\lesssim \sum_{j=1}^{\infty} 2^{j\beta} \left(\int_0^T e^{-cp^\star t 2^{j\alpha}}\, dt \right)^{1/p^\star} \left(\int_0^T \|\Delta_j f(t)\|_p^p dt \right)^{1/p} \\ &\lesssim \sum_{j=1}^{\infty} 2^{j(\beta - \alpha/p^\star)} \left(\int_0^T \|\Delta_j f(t)\|_p^p dt \right)^{1/p}\end{aligned}$$

where $p^\star$ satisfies $1 + 1/r = 1/p^\star + 1/p$. If $\beta < \alpha/p^\star$, then by Hölder's inequality

$$\|B\|_{L^r([0,T])} \lesssim \left(\sum_{j=1}^{\infty} \int_0^T \|\Delta_j f(t)\|_p^p dt \right)^{1/p} \le \|f\|_{L^p F_0^{p,p}(Q)}.$$

This together with (9) yields the result.

- Interpolating (12) with $r = 1$ and (14) with $r = p$, we obtain that if $1 \le q \le p < \infty$, $\beta < \alpha$, and $1 \le r \le p$, then

$$\|T^\alpha f\|_{L^r F_{\sigma+\beta}^{p,q}(Q)} \lesssim (1+T)\|f\|_{L^r F_\sigma^{p,q}(Q)}. \tag{15}$$

If $r = 1$ and $1 \le q \le p < \infty$, then we get the desired estimate for T^α directly from (15).

Next, let $1 < r \le \infty$, $\sigma \in \mathbb{R}$ and $0 < \alpha < \infty$. It is not difficult to see that the adjoint operator $\widetilde{T^\alpha}$ satisfies the same estimate. Indeed, for $1 \le r' \le p' < \infty$ and $1 \le q' \le p' < \infty$, we have

$$\|\widetilde{T^\alpha} f\|_{L^{r'} F_{-\sigma}^{p',q'}(Q)} \lesssim (1+T)\|f\|_{L^r F_{-\sigma-\beta}^{p,q}(Q)}.$$

Hence if $p \le q \le \infty$, and $1 < p \le r \le \infty$, then by duality we have

$$\|T^\alpha f\|_{L^r F_{\sigma+\beta}^{p,q}(Q)} \lesssim (1+T)\|f\|_{L^r F_\sigma^{p,q}(Q)}. \tag{16}$$

Interpolating this with (15), we obtain the desired estimate for T^α for $1 < p, q, r < \infty$. If $r = \infty$ and $1 < p \le q \le \infty$, we get the desired estimate for T^α from (16). This completes the proof of Theorem 2. $\square$

Remark 4. *It is not difficult to see that the adjoint operator $\widetilde{T^\alpha}$ satisfies the same kinds of estimates. Indeed,*

$$\|\widetilde{T^\alpha}f\|_{L^1F^{1,1}_{\sigma+\alpha}(Q)} \lesssim (1+T)\|f\|_{L^1F^{1,1}_\sigma(Q)}.$$

If $1 \le q \le p < \infty$ and $1 \le r < \infty$, then for $\beta < \alpha/r$

$$\|\widetilde{T^\alpha}f\|_{L^rF^{p,q}_{\sigma+\beta}(Q)} \lesssim (1+T)\|f\|_{L^1F^{p,q}_\sigma(Q)}.$$

If $1 \le q \le p < \infty$, $\beta < \alpha$, and $1/p \ge 1/r > \beta/\alpha - 1 + 1/p$, then

$$\|\widetilde{T^\alpha}f\|_{L^rF^{p,q}_{\sigma+\beta}(Q)} \lesssim (1+T)\|f\|_{L^pF^{p,q}_\sigma(Q)}.$$

If $1 \le q \le p < \infty$, $\beta < \alpha$, and $1 \le r \le p$, then

$$\|\widetilde{T^\alpha}f\|_{L^rF^{p,q}_{\sigma+\beta}(Q)} \lesssim (1+T)\|f\|_{L^rF^{p,q}_\sigma(Q)}.$$

5. Proofs of Theorem 3 and 4

These follow from Proposition 1, a fact on Triebel–Lizorkin spaces and duality.

Proposition 1. *For $1 \le p < \infty$*

$$\|T^\alpha f\|_{L^1B^{p,1}_{\sigma+\alpha}(Q)} \lesssim (1+T)\|f\|_{L^1B^{p,1}_\sigma(Q)}. \tag{17}$$

Proof. It suffices to show that for $1 \le p < \infty$

$$A := \int_0^T \sum_{j=1}^\infty \|2^{(\sigma+\alpha)j}\Delta_j T^\alpha \Delta_j f(t,\cdot)\|_p dt \lesssim \|f\|_{L^1B^{p,1}_\sigma(Q)}.$$

We use the notation in the proof of (14). However, in this case we set

$$B(t) = \|2^{(\sigma+\alpha)j}\Delta_j T^\alpha \Delta_j f(t,\cdot)\|_p.$$

Then we have by Lemma 1

$$B(t) \lesssim \int_0^t 2^{\alpha j} e^{-c(t-s)2^{\alpha j}} \|F_j(s,\cdot)\|_p ds$$

where $F_j = 2^{\sigma j}\Delta_j f$. Hence

$$A \lesssim \sum_{j=1}^{\infty} \int_0^T \int_0^t 2^{\alpha j} e^{-c(t-s)2^{\alpha j}} \|F_j(s,\cdot)\|_p ds dt$$
$$= \sum_{j=1}^{\infty} \int_0^T \left(\int_s^T 2^{\alpha j} e^{-c(t-s)2^{\alpha j}}\, dt \right) \|F_j(s,\cdot)\|_p ds$$
$$\lesssim \sum_{j=1}^{\infty} \int_0^T \|F_j(s,\cdot)\|_p ds = \int_0^T \sum_{j=1}^{\infty} \|F_j(s,\cdot)\|_p ds \le \|f\|_{L^1 B_\sigma^{p,1}(Q)}.$$

Similarly we can get the same estimate for $\widetilde{T^\alpha}$. □

Remark 5. *Modifying the proof of Proposition* 1 *somewhat, we have the following estimates. If* $\beta < \alpha/r$, $q \le r$, *and* $1 \le p < \infty$, *then*

$$\|T^\alpha f\|_{L^r B_{\sigma+\alpha}^{p,q}(Q)} \lesssim (T + T^{1-1/r}) \|f\|_{L^r B_\sigma^{p,q}(Q)},$$
$$\|\widetilde{T^\alpha} f\|_{L^1 B_{\sigma+\alpha}^{p,1}(Q)} \lesssim (T + T^{1-1/r}) \|f\|_{L^1 B_\sigma^{p,1}(Q)}.$$

Proof (Proof of Theorem 3.). Proposition 1 also holds for the adjoint operator $\widetilde{T^\alpha}$. Hence by duality we have for $1 < p \le \infty$

$$\|T^\alpha f\|_{L^\infty B_{\sigma+\alpha}^{p,\infty}(Q)} \lesssim (1+T) \|f\|_{L^\infty B_\sigma^{p,\infty}(Q)}.$$

Hence, the desired conclusion follows by interpolating this case with (17). □

Proof (Proof of Theorem 4.). (a) Since $\beta < \alpha$, we see that

$$B_{\sigma+\alpha}^{p,1}(\mathbb{R}^d) \subset B_{\sigma+\beta}^{p,1}(\mathbb{R}^d).$$

By Theorem 3, if (p,q,r) belongs to the set

$$\{(p,1,1) : 1 \le p < \infty\} \cup \{(p,\infty,\infty); 1 < p \le \infty\},$$

then

$$\|T^\alpha f\|_{L^r B_{\sigma+\beta}^{p,q}(Q)} \lesssim (1+T) \|f\|_{L^r B_\sigma^{p,q}(Q)}.$$

(b) Since $B_\sigma^{p,q}(\mathbb{R}^d) = F_\sigma^{p,q}(\mathbb{R}^d)$ for $p = q$, by Theorem 2 in the case of Triebel–Lizorkin spaces, we have that if $(1/p, 1/q, 1/r)$ belongs to the set $\{(1/p,1/q,1/r); 1 < p = q < \infty, 1 < r < \infty)\} \cup \{(1/p,1/q,1/r); 1 \le p = q < \infty, r = 1\} \cup \{(1/p,1/q,1/r); 1 < p = q \le \infty, r = \infty\}$, then

$$\|T^\alpha f\|_{L^r B_{\sigma+\beta}^{p,q}(Q)} \lesssim (1+T) \|f\|_{L^r B_\sigma^{p,q}(Q)}.$$

Interpolating between the cases (a) and (b), we obtain the desired conclusion. □

6. Proofs of Theorems 5 and 6

First we shall prove Theorem 5.

Proof (Proof of Theorem 5.). We recall the operator

$$S^{\alpha}g(t,x) = \int_{\mathbb{R}^d} P^{\alpha}(t,x-y)g(y)dy.$$

We may assume to prove the theorem with assuming $\sigma = 0$. We divide its proof into three steps.

- First we show that for $1 \le p, r \le \infty$

$$\left(\int_0^T \|S_0 S^{\alpha} g(t)\|_p^r dt\right)^{1/r} \lesssim T^{1/r} \|S_0 g\|_p. \tag{18}$$

As in the proof of (9), we have for $1 \le p \le \infty$

$$\|S_0 S^{\alpha} g(t)\|_p = \|S^{\alpha} S_0 g(t)\|_p \lesssim \|S_0 g\|_p.$$

Integrating in time gives the result.

- Next, we shall show (ii). Consider

$$A(t) := \left\|\left(\sum_{j=1}^{\infty} |2^{j\alpha/r} \Delta_j S^{\alpha} \Delta_j g(t,x)|^p\right)^{1/p}\right\|_p.$$

For notational convenience, we denote

$$S_j(t,x) := 2^{j\alpha/r} \Delta_j S^{\alpha} \Delta_j g(t,x) = \int_{\mathbb{R}^d} P_j(t,x-y) \Delta_j g(y) dy$$

where

$$P_j(t,x) = 2^{j\alpha/r} \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \hat{\Psi}(2^{-j}\xi) e^{-t|\xi|^{\alpha}}\, d\xi.$$

Applying Young's inequality and using Lemma 1, we have

$$\|S_j(t)\|_p \le \left(\int_{\mathbb{R}^d} |P_j(t,x)| dx\right) \|\Delta_j g\|_p \lesssim 2^{j\alpha/r} e^{-ct2^{j\alpha}} \|\Delta_j g\|_p.$$

Hence

$$A(t) \le \left(\sum_{j=1}^{\infty} \|S_j(t)\|_p^p\right)^{1/p} \lesssim \left(\sum_{j=1}^{\infty} (2^{j\alpha/r} e^{-ct2^{j\alpha}} \|\Delta_j g\|_p)^p\right)^{1/p}.$$

Since $p \le r$, we use the triangle inequality to get

$$\|A\|_{L^r([0,T])} \lesssim \left(\sum_{j=1}^{\infty} \|2^{j\alpha/r} e^{-ct2^{j\alpha}} \|\Delta_j g\|_p\|_{L^r([0,T])}^p\right)^{1/p}$$

$$\lesssim \left(\sum_{j=1}^{\infty} \|\Delta_j g\|_p^p\right)^{1/p} = \|g\|_{\dot{F}_0^{p,p}(\mathbb{R}^d)}.$$

So, (ii) follows from (18).

- Finally we shall show (i). Consider

$$A(t) := \left\|\left(\sum_{j=1}^{\infty} |2^{j\beta}\Delta_j S^\alpha \Delta_j g(t,x)|^q\right)^{1/q}\right\|_p.$$

For notational convenience, we denote

$$S_j(t,x) := 2^{j\beta}\Delta_j S^\alpha \Delta_j g(t,x) = \int_{\mathbb{R}^d} P_j(t,x-y)\Delta_j g(y)dy,$$

where

$$P_j(t,x) = 2^{j\beta}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\hat{\Psi}(2^{-j}\xi)e^{-t|\xi|^\alpha}\, d\xi.$$

By the triangle inequality and Lemma 1

$$A(t) = \left\|\left(\sum_{j=1}^{\infty} |S_j(t)|^q\right)^{1/q}\right\|_p \le \left\|\sum_{j=1}^{\infty} |S_j(t)|\right\|_p$$

$$\le \sum_{j=1}^{\infty} 2^{j\beta} e^{-ct2^{j\alpha}} \|\Delta_j g\|_p.$$

By the triangle inequality

$$\|A\|_{L^r([0,T])} \lesssim \sum_{j=1}^{\infty} 2^{j(\beta-\alpha/r)}\|\Delta_j g\|_p.$$

If $\beta < \alpha/r$ and $q \le p$, then by the same estimate in (13)

$$\sum_{j=1}^{\infty} 2^{j(\beta-\alpha/r)}\|\Delta_j g\|_p \lesssim \|g\|_{F_\sigma^{p,q}(\mathbb{R}^d)}.$$

This completes the proof of (i), and hence the proof of Theorem 5. □

Remark 6. *In* (18)*, the operator norm depends on* T*, but it is unavoidable even for the case* $p = r = 2$. *Indeed, Plancherel's theorem yields*

$$\int_0^T \int_{\mathbb{R}^d} |S^\alpha g(t,x)|^2 dx dt = \int_{\mathbb{R}^d} \left(\int_0^T e^{-2t|\xi|^\alpha} \, dt \right) |\hat{g}(\xi)|^2 d\xi.$$

By the same reasoning in Remark 3*, we choose a suitable* g *so that we obtain the following lower bound*

$$\|S^\alpha\|_{L^2(\mathbb{R}^d) \to L^2 L^2(Q)} \gtrsim T^{1/2}.$$

If we consider the homogeneous Triebel–Lizorkin space, the operator norm in (ii) *does not depend on* T*, i.e. for* $1 \le p \le r < \infty$

$$\|S^\alpha g\|_{L^r \dot{F}^{p,p}_{\sigma+\alpha/r}(Q)} \lesssim \|g\|_{\dot{F}^{p,p}_\sigma(\mathbb{R}^d)},$$

which can be easily checked by a minor modification of the proof of (ii).

We now turn to the proof of Theorem 6.

Proof (Proof of Theorem 6.). We may assume to prove the theorem with assuming $\sigma = 0$. Again we divide its proof into three steps.

- (i) We use the notations in the proof of Theorem 5 (i). By (18), it suffices to show

$$A := \left(\int_0^T \left[\sum_{j=1}^\infty \left(\int_{\mathbb{R}^d} |S_j(t,x)|^p dx \right)^{q/p} \right]^{r/q} dt \right)^{1/r} \le C \|g\|_{B^{p,q}_\sigma(\mathbb{R}^d)}.$$

By Lemma 1 and Young's inequality we have

$$\|S_j(t,\cdot)\|_p \lesssim 2^{j\beta} e^{-ct2^{j\alpha}} \|\Delta_j g\|_p.$$

So, by Minkowski's inequality we have under the assumption $r \ge q$ as follows:

$$\begin{aligned} A &\le \left(\sum_{j=1}^\infty \left(\int_0^T \left[\int_{\mathbb{R}^d} |S_j(t,x)|^p dx \right]^{r/p} dt \right)^{q/r} \right)^{1/q} \\ &\le \left(\sum_{j=1}^\infty \left(\int_0^T \{ 2^{\beta j} e^{-ct2^{\alpha j}} \|\Delta_j g\|_p \}^r dt \right)^{q/r} \right)^{1/q} \\ &= \left(\sum_{j=1}^\infty \left(\int_0^T 2^{r\beta j} e^{-rct2^{\alpha j}} \, dt \right)^{q/r} \|\Delta_j g\|_p^q \right)^{1/q} \end{aligned}$$

$$\lesssim \left(\sum_{j=1}^{\infty} 2^{q(\beta-\alpha/r)j} \|\Delta_j g\|_p^q \right)^{1/q}$$

$$\lesssim \left(\sum_{j=1}^{\infty} \|\Delta_j g\|_p^q \right)^{1/q} \lesssim \|g\|_{B_\sigma^{p,q}(\mathbb{R}^d)}.$$

This finishes the proof of (i).

- Next we check (iii). This follows from (i) when $p = q$ and $\beta = \alpha/r$.
- Finally we shall show (ii). By (18), it suffices to show

$$A := \left(\int_0^T \left[\sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^d} |S_j(t,x)|^p dx \right)^{q/p} \right]^{r/q} dt \right)^{1/r} \le C\|g\|_{B_\sigma^{p,q}(\mathbb{R}^d)}.$$

We first prove the case $r = 1$. For $q \ge 1$, we have

$$\begin{aligned} A &\le \int_0^T \left[\sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^d} |S_j(t,x)|^p dx \right)^{1/p} \right] dt \\ &\lesssim \sum_{j=1}^{\infty} \int_0^T \{2^{\beta j} e^{-ct2^{\alpha j}} \|\Delta_j g\|_p\} dt \\ &= \sum_{j=1}^{\infty} \left(\int_0^T 2^{\beta j} e^{-ct2^{\alpha j}}\, dt \right) \|\Delta_j g\|_p \\ &\lesssim \sum_{j=1}^{\infty} 2^{(\beta-\alpha)j} \|\Delta_j g\|_p \\ &\le \left(\sum_{j=1}^{\infty} 2^{-q'(\alpha-\beta)j} \right)^{1/q'} \left(\sum_{j=1}^{\infty} \|\Delta_j g(\cdot)\|_{L^p(\mathbb{R}^d)}^q \right)^{1/q} \\ &\lesssim \|g\|_{B_\sigma^{p,q}(\mathbb{R}^d)}. \end{aligned}$$

This shows that if $\beta_0 < \alpha$, we have

$$\|S^\alpha g\|_{L^1 B_{\sigma+\beta_0}^{p,q}(Q)} \lesssim \max(1,T)\|g\|_{B_\sigma^{p,q}(\mathbb{R}^d)}, \tag{19}$$

which shows (ii) in the case $r = 1$ at the same time.

On the other hand, by (i) we know that if $\beta_1 < \alpha/q$, we have

$$\|S^\alpha g\|_{L^q B_{\sigma+\beta_1}^{p,q}(Q)} \lesssim \max(1, T^{1/q})\|g\|_{B_\sigma^{p,q}(\mathbb{R}^d)}. \tag{20}$$

Now, if $1 < r < q$ and $\beta < \alpha/r$, we see that for $\theta = \frac{\frac{1}{r} - \frac{1}{q}}{1 - \frac{1}{q}}$ we get

$$\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{1}$$

and we can find $\beta_0 < \alpha$ and $\beta_1 < \alpha/q$ such that

$$\beta = (1-\theta)\beta_0 + \theta\beta_1.$$

Hence by interpolation between (19) and (20) we have the desired estimate

$$\|S^\alpha g\|_{L^r B^{p,q}_{\sigma+\beta}(Q)} \lesssim (1 + T^{1/r}) \|g\|_{B^{p,q}_\sigma(\mathbb{R}^d)}.$$

This finishes the proof of (ii), and hence the proof of Theorem 6. □

REMARK 7. *If we consider the homogeneous Besov space, the operator norm in* (iii) *does not depend on* T, *i.e. for* $1 \le p \le r < \infty$ *and* $0 < \alpha < \infty$

$$\|S^\alpha g\|_{L^r \dot{B}^{p,p}_{\sigma+\alpha/r}(Q)} \lesssim \|g\|_{\dot{B}^{p,p}_\sigma(\mathbb{R}^d)},$$

which can be easily checked by a minor modification of the proof of (iii).

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