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A small generating set for the twist subgroup of the mapping class group of a non-orientable surface by Dehn twists

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ABSTRACT. We give a small generating set for the twist subgroup of the mapping class group of a non-orientable surface by Dehn twists. The difference between the number of the generators and a lower bound of numbers of generators for the twist subgroup by Dehn twists is one. The lower bounds is obtained from an argument of Hirose [5].

1. Introduction

Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 0$ boundary components, and put $\Sigma_g = \Sigma_{g,0}$. The *mapping class group* $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the boundary pointwise. Dehn [2] proved that $\mathcal{M}(\Sigma_g)$ is generated by $2g(g-1)$ Dehn twists. The generating set includes Dehn twists along separating simple closed curves. Mumford [12] showed that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along non-separating simple closed curves, and Lickorish [10] gave a finite generating set for $\mathcal{M}(\Sigma_g)$ by $3g-1$ Dehn twists along non-separating simple closed curves. For $n = 1$, $\mathcal{M}(\Sigma_{g,1})$ is also generated by $3g-1$ Dehn twists along non-separating simple closed curves (see the proof of Theorem 4.13 in [4]). After that, Humphries [6] proved that $\mathcal{M}(\Sigma_{g,n})$ is generated by a subset of Lickorish's generating set whose cardinality is $2g+1$ for $g \geq 2$ and $n \in \{0,1\}$, and he also proved that the generating set is minimal among the generating sets for $\mathcal{M}(\Sigma_{g,n})$ consisting of Dehn twists. A small generating set for $\mathcal{M}(\Sigma_{g,n})$ by Dehn twists is very useful for the study of group structures of $\mathcal{M}(\Sigma_{g,n})$. For example, Humphries' generating set for $\mathcal{M}(\Sigma_{g,n})$ is used for the studies of torsion generators for $\mathcal{M}(\Sigma_g)$ [8] and generators for the Torelli group of $\Sigma_{g,1}$ [7].

Let $N_{g,n}$ be a compact connected non-orientable surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The surface $N_g = N_{g,0}$ is a connected sum of g real projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the

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group of isotopy classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $n \in \{0,1\}$, $\mathcal{M}(N_{1,n})$ is the trivial group (see [3, Theorem 3.4]). For $g \geq 2$, Lickorish proved that $\mathcal{M}(N_g)$ is not generated by Dehn twists in [9], and $\mathcal{M}(N_{g,n})$ is generated by Dehn twists and a "Y-homeomorphism" in [9, 11]. The Y-homeomorphism is introduced by Lickorish in [9]. Lickorish [9] also showed that $\mathcal{M}(N_2)$ is generated by a single Dehn twist and a Y-homeomorphism. In general, Chillingworth [1] gave a finite generating set for $\mathcal{M}(N_g)$ which consists of $\frac{3g-5}{2}$ (resp. $\frac{3g-6}{2}$) Dehn twists and a Y-homeomorphism for odd (resp. even) g. After that, Szepietowski [16] proved that $\mathcal{M}(N_g)$ is generated by a subset of Chillingworth's generating set which consists of g Dehn twists and a Y-homeomorphism, and Hirose [5] showed that the generating set is minimal among the generating sets for $\mathcal{M}(N_g)$ consisting of Dehn twists and Y-homeomorphisms. Theorem 4.1 shows that the generating sets in Stukow's finite presentation for $\mathcal{M}(N_{g,1})$ in [14] is also minimal among the generating sets consisting of Dehn twists and Y-homeomorphisms. Szepietowski's generating set for $\mathcal{M}(N_g)$ is used for the studies of torsion generators for $\mathcal{M}(N_g)$ [16] and generators for the level 2 mapping class group of N_g [17].

The *twist subgroup* $\mathcal{T}(N_{g,n})$ *of* $\mathcal{M}(N_{g,n})$ is the subgroup of $\mathcal{M}(N_{g,n})$ generated by all Dehn twists. Note that $\mathcal{T}(N_{g,n})$ is an index 2 subgroup of $\mathcal{M}(N_{g,n})$ (see [11] and [13, Corollary 6.4]). In particular, $\mathcal{T}(N_{g,n})$ is finitely generated. Chillingworth [1] showed that $\mathcal{T}(N_g)$ is generated by a single Dehn twist for $g = 2$, two Dehn twists for $g = 3$, $\frac{3g-1}{2}$ Dehn twists for the other odd g and $\frac{3g}{2}$ Dehn twists for the other even g. By an argument as in [6], we can reduce the number of Chillingworth's generators to $g+2$ for odd $g > 3$ and $g+3$ for even $g > 3$. For $n \in \{0,1\}$, Stukow [15] gave a finite presentation for $\mathcal{T}(N_{g,n})$ whose generators are $g+2$ Dehn twists essentially by relations of the presentation (see the proof of Theorem 3.1). A small generating set for $\mathcal{T}(N_{g,n})$ by Dehn twists is also useful for the study of generators for $\mathcal{T}(N_{g,n})$ and its subgroups.

In this paper we proved that $\mathcal{T}(N_{g,n})$ is generated by $g+1$ Dehn twists for $g \geq 4$ (Theorem 3.1). The generating set is a proper subset of the generating set of Stukow's finite presentation in [15]. By applying Hirose's argument in [5], we show that if a family of Dehn twists generates $\mathcal{T}(N_{g,n})$ then its cardinality is at least g (Theorem 3.3). The author does not know whether the generating set for $\mathcal{T}(N_{g,n})$ in Theorem 3.1 is minimal among the generating sets for $\mathcal{T}(N_{g,n})$ consisting of Dehn twists or not.

2. Preliminaries

For a two-sided simple closed curve γ on $N_{g,n}$, we take an orientation of the regular neighborhood of γ in $N_{g,n}$. Then we denote by t_γ the right-handed

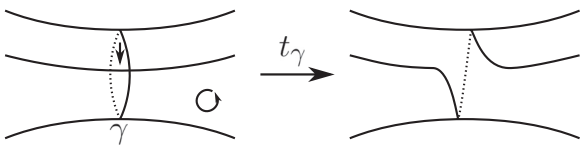

Fig. 1. The right-handed Dehn twist t_γ along a two-sided simple closed curve γ on $N_{g,n}$.

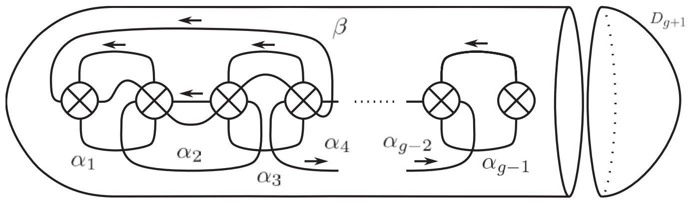

Fig. 2. Simple closed curves $\alpha_1, \dots, \alpha_{g-1}$ and β on $N_{g,n}$.

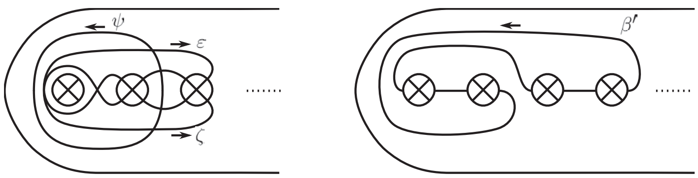

Fig. 3. Simple closed curves ε, ζ, ψ and β' on $N_{g,n}$.

Dehn twist along γ with respect to the orientation. In particular, for a given explicit two-sided simple closed curve, an arrow on a side of the simple closed curve indicates the direction of the Dehn twist (see Figure 1).

Let $e_i : D \hookrightarrow \Sigma_0$ for $i = 1, 2, \dots, g+1$ be smooth embeddings of the unit disk D into a 2-sphere Σ_0 such that $D_i = e_i(D)$ and D_j are disjoint for distinct $1 \leq i, j \leq g+1$. Then we take a model of N_g (resp. $N_{g,1}$) as the surface obtained from $\Sigma_0 - \mathrm{int}(D_1 \sqcup \cdots \sqcup D_g)$ (resp. $\Sigma_0 - \mathrm{int}(D_1 \sqcup \cdots \sqcup D_{g+1})$) by identifying antipodal points of the boundary components of $D_1, \dots, D_g$ and we indicate the identification of ∂D_i by the x-mark as in Figure 2.

For $n \in \{0, 1\}$, we denote by $\alpha_1, \dots, \alpha_{g-1}$ and β two-sided simple closed curves on $N_{g,n}$ as in Figure 2, and denote by β', ε, ζ and ψ two-sided simple closed curves on $N_{g,n}$ as in Figure 3. Then we set $a_i = t_{\alpha_i}$ $(i = 1, \dots, g-1)$, $b = t_\beta$, $e = t_\varepsilon$, $f = t_\zeta$, $h = t_\psi$ and $c = t_{\beta'}$.

3. Main result

The main theorem in this paper is as follows.

THEOREM 3.1. *For* $g \geq 4$ *and* $n \in \{0, 1\}$, $\mathscr{T}(N_{g,n})$ *is generated by* $a_1, \dots, a_{g-1}$, b *and* e. *In particular,* $\mathscr{T}(N_{g,n})$ *is generated by* $g+1$ *Dehn twists along non-separating simple closed curves.*

PROOF. Assume $g \geq 4$ and $n \in \{0, 1\}$. Stukow's presentation for $\mathscr{T}(N_{g,n})$ in [15] has the following generating set:

- $X = \{a_1, \dots, a_{g-1}, b, e, f, h, c\}$ for odd g and $n = 1$, or $g = 4$ and $n = 1$,
- $X' = X \cup \{b_0, b_1, \dots, b_{(g-2)/2}, \bar{b}_{(g-6)/2}, \bar{b}_{(g-4)/2}, \bar{b}_{(g-2)/2}\}$ for even $g \geq 6$ and $n = 1$,
- $X \cup \{\rho\}$ for odd g and $n = 0$,
- $X \cup \{\bar{\rho}\}$ for $g = 4$ and $n = 0$,
- $X' \cup \{\bar{\rho}\}$ for even $g \geq 6$ and $n = 0$.

In the above generating sets, $b_0, b_1, \dots, b_{(g-2)/2}, \bar{b}_{(g-6)/2}, \bar{b}_{(g-4)/2}, \bar{b}_{(g-2)/2}$, ρ and $\bar{\rho}$ are products of elements in X by the relations

(A7) $b_0 = a_1,\ b_1 = b$ for even $g \geq 6$,

(A8) $b_{i+1} = (b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}b_i)^5(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6}$ for $1 \leq i \leq \frac{g-4}{2}$ and even $g \geq 6$,

$(\overline{\text{A7a}})$ $\bar{b}_0 = a_1^{-1},\ \bar{b}_1 = c$ for $g = 6$,

$(\overline{\text{A7b}})$ $\bar{b}_1 = c$ for $g = 8$,

$(\overline{\text{A7c}})$ $\bar{b}_i = z_{g-1}b_iz_{g-1}^{-1}$ for $i = \frac{g-6}{2}, \frac{g-4}{2}$, $i \geq 2$ and even $g \geq 6$, where $z_{g-1} = (a_{g-2}a_{g-1}a_{g-3}a_{g-2} \dots a_3a_4e^{-1}a_3a_1^{-1}e^{-1})(a_2^{-1}a_1^{-1} \dots a_{g-2}^{-1}a_{g-3}^{-1}a_{g-1}^{-1}a_{g-2}^{-1})$,

$(\overline{\text{A8a}})$ $\bar{b}_2 = (\bar{b}_0e^{-1}a_3a_4a_5\bar{b}_1)^5(\bar{b}_0e^{-1}a_3a_4a_5)^{-6}$ for $g = 6$,

$(\overline{\text{A8b}})$ $\bar{b}_{(g-2)/2} = (\bar{b}_{(g-6)/2}a_{g-4}a_{g-3}a_{g-2}a_{g-1}\bar{b}_{(g-4)/2})^5(\bar{b}_{(g-6)/2}a_{g-4}a_{g-3}a_{g-2} \cdot a_{g-1})^{-6}$ for even $g \geq 8$,

(C1a) $(a_1a_2 \dots a_{g-1})^g = \rho$ for odd g and $n = 0$,

$(\overline{\text{C4}})$ $(\bar{\rho}a_2a_3 \dots a_{g-1})^{g-1} = 1$ for even $g \geq 4$ and $n = 0$

by Theorems 2.1, 2.2, 3.1 and 3.2 of [15]. Thus $\mathscr{T}(N_{g,n})$ is generated by X. By the relation $(\overline{\text{B2}_1})$ in Theorem 3.1 of [15], h is a product of elements in $X - \{h\}$, and by the relation $(\overline{\text{B6}_1})$ in Theorem 3.1 of [15], c is a product of $a_1, \dots, a_{g-1}$, b, e and f.

Finally, we can check that $a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1}(\varepsilon) = \zeta$ and the orientation of a regular neighborhood of $a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1}(\varepsilon)$ is different from one of ζ as in Figure 4. Hence, we have $f = (a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1})e^{-1}(a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1})^{-1}$. Therefore, $\mathscr{T}(N_{g,n})$ is generated by $a_1, \dots, a_{g-1}$, b and e.

REMARK 3.2. *The regular neighborhood* $\mathscr{N}$ *of the union of* $\alpha_1, \dots, \alpha_{g-1}$ *is an orientable subsurface of* $N_{g,n}$ *and* $\{a_1, \dots, a_{g-1}, b\}$ *is the minimal generating set for* $\mathscr{M}(\mathscr{N})$ *by Dehn twists which is given by Humphries* [6]. *Remark that* $N_{g,n} - \operatorname{int}\mathscr{N}$ *is not a disjoint union of disks, and an element of the subgroup of* $\mathscr{T}(N_{g,n})$ *which is generated by* $a_1, \dots, a_{g-1}$ *and* b *is represented by a diffeomorphism of* $N_{g,n}$ *whose restriction to* $N_{g,n} - \operatorname{int}\mathscr{N}$ *is the identity map. However,* e *does not fix* $N_{g,n} - \operatorname{int}\mathscr{N}$ *up to ambient isotopies of* $N_{g,n}$. *Hence* $\mathscr{T}(N_{g,n})$ *is*

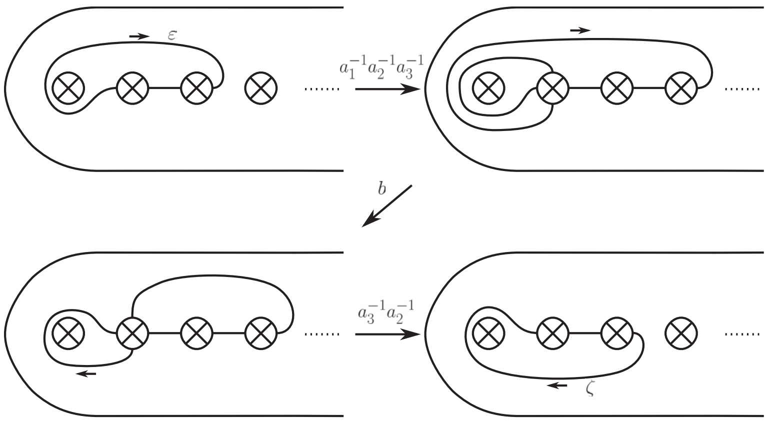

Fig. 4. Proving that $a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1}(\varepsilon) = \zeta$.

not generated by $a_1, \dots, a_{g-1}$ *and* b. *Define* $X_0 = \{\alpha_1, \dots, \alpha_{g-1}, b, \varepsilon\}$. *For* $x_0 \in \{\alpha_4, \dots, \alpha_{g-1}, \varepsilon\}$, *the complement* $N_{g,n} - \bigcup_{x \in X_0 \setminus \{x_0\}} x$ *has a non-disk component. Thus* $\mathscr{T}(N_{g,n})$ *is not generated by* $X_0 - \{x_0\}$ *for* $x_0 \in \{\alpha_4, \dots, \alpha_{g-1}, \varepsilon\}$.

By applying Hirose's argument in [5] to $\mathscr{T}(N_{g,n})$ for $g \geq 4$ and $n \in \{0, 1\}$, we have the following proposition.

Theorem 3.3. *Let* $g \geq 4$ *and* $n \in \{0, 1\}$. *Then the minimum number of generators for* $\mathscr{T}(N_{g,n})$ *by Dehn twists is at least* g.

We prove Theorem 3.3 in Section 4. By Theorem 3.3, the minimum number of generators for $\mathscr{T}(N_{g,n})$ by Dehn twists is at least g for $g \geq 4$ and $n \in \{0, 1\}$, and the difference between the number of the generators for $\mathscr{T}(N_{g,n})$ in Theorem 3.1 and the lower bound of numbers of generators for $\mathscr{T}(N_{g,n})$ by Dehn twists given by Theorem 3.3 is one.

Finally we raise the following problem.

Problem 3.4. *Determine which of* g *and* $g+1$ *is the minimum number of generators for* $\mathscr{T}(N_{g,n})$ *by Dehn twists when* $g \geq 4$ *and* $n \in \{0, 1\}$.

4. Proof of Theorem 3.3

In this section, we give a proof of Theorem 3.3. Assume that $g \geq 4$ and $n \in \{0, 1\}$ throughout this section. First, we have the following theorem.

Theorem 4.1. *If Dehn twists* $t_{\gamma_1}, \dots, t_{\gamma_k}$ *and Y-homeomorphisms* $Y_1, \dots, Y_l$ *generate* $\mathscr{M}(N_{g,n})$, *then* $k \geq g$ *and* $l \geq 1$.

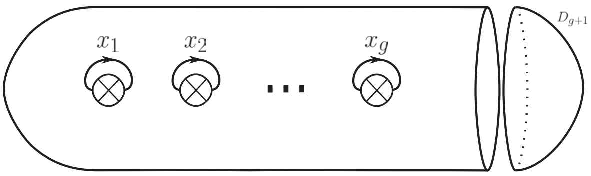

Fig. 5. A basis $\{x_1, x_2, \dots, x_g\}$ for $H_1(N_{g,n}; \mathbb{Z}_2)$.

Hirose proved Theorem 4.1 for $n = 0$ in Theorem 2 of [5], and we can prove Theorem 4.1 for $n = 1$ by a parallel argument of his.

To prove Theorem 3.3, we apply the proof of Theorem 2 in [5] and Theorem 4.1 to $\mathcal{T}(N_{g,n})$ for $g \geq 4$ and $n \in \{0, 1\}$. Put $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for an integer $m \geq 2$. Let $w_1 : H_1(N_{g,n}; \mathbb{Z}_2) \to \mathbb{Z}_2$ be the first Stiefel-Whitney class and $H_1^+(N_{g,n}; \mathbb{Z}_2)$ the kernel of w_1. Hence $H_1^+(N_{g,n}; \mathbb{Z}_2)$ is a $g-1$ dimensional $\mathbb{Z}_2$-vector space and $H_1^+(N_{g,n}; \mathbb{Z}_2)$ is generated by the homology classes of two-sided simple closed curves on $N_{g,n}$. We take a basis $\{x_1, x_2, \dots, x_g\}$ for $H_1(N_{g,n}; \mathbb{Z}_2)$ as in Figure 5. We denote $[\gamma]$ the homology class in $H_1(N_{g,n}; \mathbb{Z}_2)$ represented by a simple closed curve γ on $N_{g,n}$. For $y \in H_1(N_{g,n}; \mathbb{Z}_2)$, we define an isomorphism τ_y on $H_1(N_{g,n}; \mathbb{Z}_2)$ by $\tau_y(x) = x + (x, y)y$, where (x, y) is the mod-2 intersection number of x and y. Note that $(t_\gamma)_* = \tau_{[\gamma]}$ for a two-sided simple closed curve γ on $N_{g,n}$. A two-sided simple closed curve γ on $N_{g,n}$ is *admissible* if γ is non-separating and $N_{g,n} - \gamma$ is non-orientable.

LEMMA 4.2. *If* $t_{\gamma_1}, \dots, t_{\gamma_k}$ *generate* $\mathcal{T}(N_{g,n})$, *then* $[\gamma_1], \dots, [\gamma_k]$ *generate* $H_1^+(N_{g,n}; \mathbb{Z}_2)$. *In particular,* $k \geq g - 1$.

PROOF. This can be proved by the following argument similar to that in the proof of Lemma 6 in [5]. Since $t_{\gamma_1}, \dots, t_{\gamma_k}$ generate $\mathcal{T}(N_{g,n})$, there exists $i \in \{1, \dots, k\}$ such that γ_i is admissible. In fact, by Lemma 4 in [5], if Dehn twists along non-admissible simple closed curves generate $\mathcal{T}(N_{g,n})$, then any isomorphism on $H_1(N_{g,n}; \mathbb{Z}_2)$ induced by an element of $\mathcal{T}(N_{g,n})$ is a power of $\tau_{x_1 + \cdots + x_g}$. Without loss of generality we can assume that γ_1 is admissible. For any $x \in H_1^+(N_{g,n}; \mathbb{Z}_2)$, we can write $x = x_{i_1} + x_{i_2} + \cdots + x_{i_{2l}}$. Then there exist admissible simple closed curves $\delta_1, \delta_2, \dots, \delta_l$ on $N_{g,n}$ such that $x = [\delta_1] + \cdots + [\delta_l]$. By Lemma 7.2 in [13], there exist $\phi_j \in \mathcal{T}(N_{g,n})$ $(j = 1, \dots, l)$ such that $\phi_j(\gamma_1) = \delta_j$. Thus we have $x = (\phi_1)_*([\gamma_1]) + \cdots + (\phi_l)_*([\gamma_1])$. By the assumption, each ϕ_j is a product of $t_{\gamma_1}, \dots, t_{\gamma_k}$. Since $\tau_{[\gamma_i]}([\gamma_{i'}]) = [\gamma_{i'}] + ([\gamma_{i'}], [\gamma_i])[\gamma_i]$, x is a sum of $[\gamma_1], \dots, [\gamma_k]$.

Let $2\times : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ be the injective homomorphism defined by $2 \times [m] = [2m] \in \mathbb{Z}_4$. A map $q : H_1(N_{g,n}; \mathbb{Z}_2) \to \mathbb{Z}_4$ is a $\mathbb{Z}_4$-*quadratic form* if $q(x + y) = q(x) + q(y) + 2 \times (x, y)$ for any $x, y \in H_1(N_{g,n}; \mathbb{Z}_2)$. The next lemma follows directly from the proof of Lemma 7 in [5].

Lemma 4.3. *For any $\mathbb{Z}_4$-quadratic form $q : H_1(N_{g,n}; \mathbb{Z}_2) \to \mathbb{Z}_4$, there exists an element ϕ of $\mathscr{T}(N_{g,n})$ such that $q \circ \phi \neq \phi$.*

Proof (Proof of Theorem 3.3). Suppose that $t_{\gamma_1}, \dots, t_{\gamma_k}$ generate $\mathscr{T}(N_{g,1})$. By Lemma 4.2, we have $k \geq g - 1$. We assume that $k = g - 1$. Then, by Lemma 8 in [5], there exists a $\mathbb{Z}_4$-quadratic form $q : H_1(N_{g,n}; \mathbb{Z}_2) \to \mathbb{Z}_4$ such that $q \circ t_{\gamma_i} = q$ for any $i = 1, \dots, g - 1$. This is a contradiction to Lemma 4.3. Therefore, we have $k \geq g$.

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