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Stable extendibility and extendibility of vector bundles over lens spaces

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ABSTRACT. Firstly, we obtain conditions for stable extendibility and extendibility of complex vector bundles over the $(2n+1)$-dimensional standard lens space $L^n(p)$ mod p, where p is a prime. Secondly, we prove that the complexification $c(\tau_n(p))$ of the tangent bundle $\tau_n(p)$ $(= \tau(L^n(p)))$ of $L^n(p)$ is extendible to $L^{2n+1}(p)$ if p is a prime, and is not stably extendible to $L^{2n+2}(p)$ if p is an odd prime and $n \geq 2p-2$. Thirdly, we show, for some odd prime p and positive integers n and m with $m > n$, that $\tau(L^n(p))$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.

1. Introduction

Let $\mathbb{F}$ denote either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. Let A be a subspace of a space X. A t-dimensional $\mathbb{F}$-vector bundle α over A is said to be stably extendible (respectively extendible) to X if and only if there is a t-dimensional $\mathbb{F}$-vector bundle over X whose restriction to A is stably equivalent (respectively equivalent) to α (cf. [3] and [9]). For simplicity, we use the same letter for an $\mathbb{F}$-vector bundle and its equivalence class and k for the k-dimensional trivial $\mathbb{F}$-bundle.

For an integer p with $p > 1$, let $L^n(p)$ $(= S^{2n+1}/(\mathbb{Z}/p))$ be the $(2n+1)$-dimensional standard lens space mod p. Then, we obtain conditions for stable extendibility and extendibility of a $\mathbb{C}$-vector bundle over $L^n(p)$ in the following theorem.

THEOREM 1. *Let p be a prime and α a t-dimensional $\mathbb{C}$-vector bundle over $L^n(p)$ which is stably equivalent to a sum of s non-trivial $\mathbb{C}$-line bundles. Then the following hold.*

(1) *α is stably extendible to $L^m(p)$ for every $m > n$ if $s \leq t$.*

(2) *α is extendible to $L^t(p)$ if $n \leq t \leq s$.*

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If $t < s$, the conclusion of Theorem 1(1) does not hold in general. In fact, for $t = 2n+1$ and $s = 2n+2$, there exists a t-dimensional $\mathbb{C}$-vector bundle over $L^n(p)$ which is stably equivalent to a sum of s non-trivial $\mathbb{C}$-line bundles and is not stably extendible to $L^{2n+2}(p)$. Such $\mathbb{C}$-vector bundle is given in the latter part of the following theorem.

Let $c(\alpha)$ be the complexification of an $\mathbb{R}$-vector bundle α, and $\tau_n(p)$ $(= \tau(L^n(p)))$ denote the tangent bundle of $L^n(p)$.

THEOREM 2. *The complexification $c(\tau_n(p))$ of the tangent bundle $\tau_n(p)$ is extendible to $L^{2n+1}(p)$ if p is a prime, and is not stably extendible to $L^{2n+2}(p)$ if p is an odd prime and $n \geq 2p-2$.*

Furthermore, we show, for some odd prime p and positive integers n and m with $m > n$, that $\tau_n(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.

For $n > p$, we have the following.

THEOREM 3. *Let p be an odd prime and n an integer with $n > p$. Then $\tau_n(p)$ is stably extendible to $L^{2n+1}(p)$ but is not extendible to $L^{2n+1}(p)$.*

The next theorem for $n \leq p$ is an explicit statement of the fact that remarked in Section 1 of [4].

THEOREM 4. *Let p be an odd prime.*

(1) *Let n be an integer with $p-3 \leq n \leq p$ and $n \neq 0, 1$ and 3, and m an integer with $m > n$. Then $\tau_n(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.*

(2) *Let $p \equiv \pm 1 \pmod{12}$ and m an integer with $m > 2$. Then $\tau_2(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.*

COROLLARY 1. *Let p be a prime with $p \geq 5$ and m an integer with $m > p$. Then $\tau_p(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.*

COROLLARY 2. *Let p be an odd prime and m an integer with $m > p-1$. Then $\tau_{p-1}(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.*

COROLLARY 3. *Let p be a prime with $p \geq 7$ and m an integer with $m > p-2$. Then $\tau_{p-2}(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.*

COROLLARY 4. *Let p be a prime with $p \geq 5$ and m an integer with $m > p-3$. Then $\tau_{p-3}(p)$ is stably extendible to $L^m(p)$ but is not extendible to $L^m(p)$.*

This paper is organized as follows. After preparing some known results, we prove Theorem 1 in Section 2. Using some known facts, we study stable extendibility of $c(\tau_4(3))$ and $c(\tau_5(3))$ in Section 2, and prove Theorem 2 in Section 3. Using several conditions for stable extendibility and extendibility, we give proofs of Theorems 3–4 and Corollaries 1–4 in Section 5.

2. Proof of Theorem 1

Let $\mathbf{C}P^n$ $(= S^{2n+1}/S^1)$ denote the complex projective space of complex dimension n and μ_n stand for the canonical $\mathbf{C}$-line bundle over $\mathbf{C}P^n$. Then we define $\eta_n = \pi^*(\mu_n)$, the bundle induced by the natural projection $\pi : L^n(p) \to \mathbf{C}P^n$ from μ_n, and $\sigma_n = \eta_n - 1$ $(\in \tilde{K}(L^n(p)))$. We call η_n the canonical $\mathbf{C}$-line bundle over $L^n(p)$. The structure of the ring $\tilde{K}(L^n(p))$ is determined in [5] as follows.

Theorem 2.1 ([5, Theorem 1]). *Let p be a prime and n a positive integer. Let $n = s(p-1) + r$, where s and r are integers with $0 \le r < p-1$. Then*

$$\tilde{K}(L^n(p)) \cong (\mathbf{Z}/p^{s+1})^r + (\mathbf{Z}/p^s)^{p-r-1}.$$

(Here, $(\mathbf{Z}/q)^k$ denotes the direct sum of k-copies of $\mathbf{Z}/q$.) The first r summands are generated by $\sigma_n^1, \sigma_n^2, \dots, \sigma_n^r$, and the last $p-r-1$ summands by $\sigma_n^{r+1}, \sigma_n^{r+2}, \dots, \sigma_n^{p-1}$. Moreover, the ring structure of $\tilde{K}(L^n(p))$ is given by the relations:

$$(\sigma_n + 1)^p (= \eta_n^p) = 1 \qquad \text{and} \qquad \sigma_n^{n+1} = 0.$$

For a real number x, let $\langle x \rangle$ denote the smallest integer q with $x \le q$.

Theorem 2.2 ([2, Theorem 1.2, p. 99]). *Let X be a finite dimensional CW-complex and ζ an s-dimensional $\mathbf{C}$-vector bundle over X. If $t = \langle \{(\dim X) - 1\}/2 \rangle \le s$, then there exists a t-dimensional $\mathbf{C}$-vector bundle γ over X such that $\zeta = \gamma \oplus (s-t)$. (Here, $\oplus$ denotes the Whitney sum.)*

Theorem 2.3 ([8, Theorem 2.3]). *Let Y be a subcomplex of a finite dimensional CW-complex X and α a $\mathbf{C}$-vector bundle over Y such that* $\dim \alpha \ge \langle (\dim Y)/2 \rangle$. *Then α is extendible to X if and only if α is stably extendible to X.*

Using Theorems 2.1, 2.2 and 2.3, we prove Theorem 1.

Proof of Theorem 1. By Theorem 2.1, there exist non-negative integers $a_1, a_2, \dots, a_{p-1}$ such that

$$\alpha - t = \sum_{1 \le j \le p-1} a_j \eta_n^j - s \ \ (\in \tilde{K}(L^n(p))),$$

where $\sum_{1 \le j \le p-1} a_j = s$.

(1) Let m be any integer with $m > n$ and $i : L^n(p) \to L^m(p)$ be the standard inclusion. Then, if $s \le t$, for the non-negative integers $a_1, a_2, \dots, a_{p-1}$ with $\sum_{1 \le j \le p-1} a_j = s$, a $\mathbf{C}$-vector bundle

$$\beta = \sum_{1 \le j \le p-1} a_j \eta_m^j \oplus (t - s)$$

over $L^m(p)$ is t-dimensional and, for the induced homomorphism $i^* : K(L^m(p)) \to K(L^n(p))$,

$$i^*(\beta) = \sum_{1 \le j \le p-1} a_j \eta_n^j \oplus (t - s) = \alpha,$$

since $i^*(\eta_m) = \eta_n$ and $i^*(t - s) = t - s$. Hence α is stably extendible to $L^m(p)$.

(2) Let $n \le t \le s$. If $n = t$, the conclusion is trivial. So we may assume $n < t \le s$. Setting $X = L^t(p)$ and $\zeta = \sum_{1 \le j \le p-1} a_j \eta_t^j$, where $\sum_{1 \le j \le p-1} a_j = s$, in Theorem 2.2, we see that there exists a t-dimensional $\mathbf{C}$-vector bundle γ over $L^t(p)$ such that

$$\sum_{1 \le j \le p-1} a_j \eta_t^j = \gamma \oplus (s - t).$$

Let $i : L^n(p) \to L^t(p)$ be the standard inclusion. Then, applying the induced homomorphism $i^* : K(L^t(p)) \to K(L^n(p))$ to the both sides of the above equality, we have

$$\sum_{1 \le j \le p-1} a_j \eta_n^j = i^*(\gamma) \oplus (s - t).$$

So $\alpha - t = \sum_{1 \le j \le p-1} a_j \eta_n^j - s = i^*(\gamma) - t$ in $\tilde{K}(L^n(p))$. Thus α is stably extendible to $L^t(p)$. Setting $X = L^t(p)$ and $Y = L^n(p)$ in Theorem 2.3, we have $\dim \alpha = t \ge n + 1 = \langle (\dim L^n(p))/2 \rangle$. Hence α is extendible to $L^t(p)$. $\square$

3. Stable extendibility of $c(\tau_4(3))$ and $c(\tau_5(3))$

We recall some known facts for the proofs.

FACT 3.1. *Let $c : KO(X) \to K(X)$, $r : K(X) \to KO(X)$ and $t : K(X) \to K(X)$ be the complexfication, the real restriction and the complex conjugation, respectively. Then they are natural with respect to maps and satisfy: $rc = 2$ and $cr = 1 + t$. In particular, for the canonical $\mathbf{C}$-line bundle η_n over $L^n(p)$, $cr(\eta_n) = \eta_n + \eta_n^{-1} = \eta_n + \eta_n^{p-1}$.*

FACT 3.2. *For the tangent bundle $\tau_n(p)$ of $L^n(p)$, $\tau_n(p) \oplus 1 = (n + 1)r(\eta_n)$.*

Fact 3.3. *The total Chern class* $C(\eta_n^i)$ *of* η_n^i *is given by* $C(\eta_n^i) = 1 + iz_n$, *where* $z_n = C_1(\eta_n)$, *the first Chern class of* η_n, *is the generator of* $H^2(L^n(p); \mathbf{Z})$ $(\cong \mathbf{Z}/p)$.

Fact 3.4. *Let* p *be a prime and let* $a = \sum_{0 \le i \le m} a(i)p^i$ *and* $b = \sum_{0 \le i \le m} b(i)p^i$ $(0 \le a(i) < p,\ 0 \le b(i) < p)$. *Then*

$$\binom{b}{a} \equiv \prod_{0 \le i \le m} \binom{b(i)}{a(i)} \pmod{p}.$$

We prove results on stable extendibility of $c(\tau_4(3))$ and $c(\tau_5(3))$. The method is similar to that of Theorem 8 in [1].

Theorem 3.1. $c(\tau_4(3))$ *is not stably extendible to* $L^{10}(3)$.

Proof. Suppose that there exists a 9-dimensional $\mathbf{C}$-vector bundle β over $L^{10}(3)$ satisfying $i^*(\beta) = c(\tau_4(3))$, where $i : L^4(3) \to L^{10}(3)$ is the standard inclusion. According to Theorem 2.1, there exist integers a and b such that

$$\beta - 9 = a\sigma_{10} + b\sigma_{10}^2 \in \tilde{K}(L^{10}(3)) \ \ (\cong \mathbf{Z}/3^5 + \mathbf{Z}/3^5).$$

Applying the induced homomorphism $i^* : \tilde{K}(L^{10}(3)) \to \tilde{K}(L^4(3))$ to the both sides of the above equality, we obtain

$$i^*(\beta - 9) = a\sigma_4 + b\sigma_4^2 \in \tilde{K}(L^4(3)) \ \ (\cong \mathbf{Z}/9 + \mathbf{Z}/9).$$

Using Facts 3.2 and 3.1, we have

$$\begin{aligned} i^*(\beta - 9) &= c(\tau_4(3)) - 9 = c(\tau_4(3) \oplus 1) - 10 \\ &= c(5r(\eta_4)) - 10 = 5cr(\eta_4) - 10 = 5(\eta_4 + \eta_4^2) - 10 \\ &= 15(\eta_4 - 1) + 5(\eta_4 - 1)^2 = 15\sigma_4 + 5\sigma_4^2. \end{aligned}$$

Since σ_4 and σ_4^2 are of order 9 by Theorem 2.1, $a = 9x + 6$ and $b = 9y + 5$ for some integers x and y. So

$$\begin{aligned} \beta - 9 &= (9x + 6)(\eta_{10} - 1) + (9y + 5)(\eta_{10} - 1)^2 \\ &= (9x - 18y - 4)\eta_{10} + (9y + 5)\eta_{10}^2 + 9y - 9x - 1. \end{aligned}$$

Define $A = 9x - 18y - 4$ $(= 9(x - 2y - 1) + 5)$ and $B = 9y + 5$. Since we may take integers a and b with $a \ge 2b \ge 0$, we may consider that x and y satisfy inequalities: $A \ge 0$ and $B \ge 0$. Now, by Fact 3.3, the total Chern class of β is given by

$$C(\beta) = C(\eta_{10})^A C(\eta_{10}^2)^B = (1 + z_{10})^A (1 + 2z_{10})^B = (1 + z_{10})^A (1 - z_{10})^B.$$

Hence, the 10-th Chern class of β is given as follows.

$$C_{10}(\beta) = \sum_{i+j=10} \binom{A}{i}\binom{B}{j}(-1)^j z_{10}^{10}.$$

Here, by Fact 3.4, we have

$$\begin{aligned}
\binom{A}{i} \equiv \binom{B}{i} &\equiv 0 \pmod 3 \qquad \text{for } i = 6,7,8,\\
&\equiv 1 \pmod 3 \qquad \text{for } i = 0,2,3,5,\\
&\equiv 2 \pmod 3 \qquad \text{for } i = 1,4,
\end{aligned}$$

$$\binom{A}{9} \equiv x - 2y - 1 \pmod{3}, \qquad \binom{B}{9} \equiv y \pmod{3},$$

$$\binom{A}{10} \equiv 2(x - 2y - 1) \pmod{3}, \qquad \binom{B}{10} \equiv 2y \pmod{3}.$$

Therefore

$$\begin{aligned}
C_{10}(\beta) &= \left\{\binom{A}{0}\binom{B}{10} - \binom{A}{1}\binom{B}{9} - \binom{A}{5}\binom{B}{5} - \binom{A}{9}\binom{B}{1} + \binom{A}{10}\binom{B}{0}\right\} z_{10}^{10}\\
&= \{2y - 2y - 1 - 2(x - 2y - 1) + 2(x - 2y - 1)\} z_{10}^{10} = -z_{10}^{10} \neq 0.
\end{aligned}$$

On the other hand, $C_{10}(\beta) = 0$ since β is 9-dimensional. This is a contradiction. □

THEOREM 3.2. *$c(\tau_5(3))$ is not stably extendible to $L^{12}(3)$.*

PROOF. Suppose that there exists an 11-dimensional $\mathbb{C}$-vector bundle β over $L^{12}(3)$ satisfying $i^*(\beta) = c(\tau_5(3))$, where $i : L^5(3) \to L^{12}(3)$ is the standard inclusion. According to Theorem 2.1, there exist integers a and b such that

$$\beta - 11 = a\sigma_{12} + b\sigma_{12}^2 \in \tilde{K}(L^{12}(3)) \ \ (\cong \mathbb{Z}/3^6 + \mathbb{Z}/3^6).$$

Applying the induced homomorphism $i^* : \tilde{K}(L^{12}(3)) \to \tilde{K}(L^5(3))$ to the both sides of the above equality, we obtain

$$i^*(\beta - 11) = a\sigma_5 + b\sigma_5^2 \in \tilde{K}(L^5(3)) \ \ (\cong \mathbb{Z}/27 + \mathbb{Z}/9).$$

Using Facts 3.2 and 3.1, we have

$$\begin{aligned}
i^*(\beta - 11) &= c(\tau_5(3)) - 11 = c(\tau_5(3) \oplus 1) - 12\\
&= c(6r(\eta_5)) - 12 = 6cr(\eta_5) - 12 = 6(\eta_5 + \eta_5^2) - 12\\
&= 18(\eta_5 - 1) + 6(\eta_5 - 1)^2 = 18\sigma_5 + 6\sigma_5^2.
\end{aligned}$$

Since σ_5 is of order 27 and σ_5^2 are of order 9 by Theorem 2.1, $a = 27x + 18$ and $b = 9y + 6$ for some integers x and y. So

$$\begin{aligned}\beta - 11 &= (27x + 18)(\eta_{12} - 1) + (9y + 6)(\eta_{12} - 1)^2 \\ &= (27x - 18y + 6)\eta_{12} + (9y + 6)\eta_{12}^2 + 9y - 27x - 12.\end{aligned}$$

Define $A = 27x - 18y + 6$ $(= 9(3x - 2y) + 6)$ and $B = 9y + 6$. Since we may take integers a and b with $a \geq 2b \geq 0$, we may consider that x and y satisfy inequalities: $A \geq 0$ and $B \geq 0$. Now, by Fact 3.3, the total Chern class of β is given by

$$C(\beta) = C(\eta_{12})^A C(\eta_{12}^2)^B = (1 + z_{12})^A (1 + 2z_{12})^B = (1 + z_{12})^A (1 - z_{12})^B.$$

Hence, the 12-th Chern class of β is given as follows.

$$C_{12}(\beta) = \sum_{i+j=12} \binom{A}{i}\binom{B}{j}(-1)^j z_{12}^{12}.$$

Here, by Fact 3.4, we have

$$\begin{aligned}\binom{A}{i} \equiv \binom{B}{i} &\equiv 0 \pmod 3 \qquad \text{for } i = 1, 2, 4, 5, 7, 8, 10, 11, \\ &\equiv 1 \pmod 3 \qquad \text{for } i = 0, 6, \\ &\equiv 2 \pmod 3 \qquad \text{for } i = 3,\end{aligned}$$

$$\binom{A}{9} \equiv \binom{B}{9} \equiv y \pmod 3, \qquad \binom{A}{12} \equiv \binom{B}{12} \equiv 2y \pmod 3.$$

Therefore

$$\begin{aligned}C_{12}(\beta) &= \left\{\binom{A}{0}\binom{B}{12} - \binom{A}{3}\binom{B}{9} + \binom{A}{6}\binom{B}{6} - \binom{A}{9}\binom{B}{3} + \binom{A}{12}\binom{B}{0}\right\} z_{12}^{12} \\ &= (2y - 2y + 1 - 2y + 2y) z_{12}^{12} = z_{12}^{12} \neq 0.\end{aligned}$$

On the other hand, $C_{12}(\beta) = 0$ since β is 11-dimensional. This is a contradiction. $\square$

4. Proof of Theorem 2

For a real number x, let $[x]$ denote the largest integer q with $q \leq x$. Then, for the proof of the latter part of Theorem 2, we use the following.

THEOREM 4.1 ([7, Theorem 4.5]). *Let p be a prime and α a t-dimensional $\mathbb{C}$-vector bundle over $L^n(p)$ which is stably equivalent to a sum of s non-trivial*

$\mathbf{C}$-line bundles where $t < s < p^{[n/(p-1)]}$. Then $n < s$ and α is not stably extendible to $L^s(p)$.

To apply Theorem 4.1, the next lemma is useful.

LEMMA 4.2. (1) $2n+2 < 3^{[n/2]}$ *if and only if* $n \geq 6$.
(2) *For* $p \geq 5$, $2n+2 < p^{[n/(p-1)]}$ *if and only if* $n \geq 2p-2$.

PROOF. Since (1) is clear, we prove (2).

If $n \geq 2p-2$, we may set $n = a(p-1)+b$, where a and b are integers with $a \geq 2$ and $0 \leq b < p-1$. Then

$$p^{[n/(p-1)]} - (2n+2) = p^a - 2a(p-1) - 2b - 2 \geq p^a - 2(p-1)a - 2(p-1).$$

For each integer $a \geq 2$, define $f(a) = p^a - 2(p-1)a - 2(p-1)$. Then, for $p \geq 5$, $f(2) = (p-3)^2 - 3 > 0$ and $f(a+1) - f(a) = (p^a - 2)(p-1) > 0$. We therefore have $f(a) > 0$ for every integer $a \geq 2$. Since $f(a) \leq p^{[n/(p-1)]} - (2n+2)$, we have $2n+2 < p^{[n/(p-1)]}$. Thus the "if" part of (2) is proved. In case $n < 2p-2$,

$$\begin{aligned} 2n+2-p^{[n/(p-1)]} &= 2n+1 > 0 \quad \text{if } 1 \leq n < p-1, \\ &= p > 0 \quad \text{if } n = p-1, \quad \text{and} \\ &> p > 0 \quad \text{if } p-1 < n < 2p-2. \end{aligned}$$

We therefore have $2n+2 > p^{[n/(p-1)]}$. Thus the "only if" part of (2) is proved. □

PROOF OF THEOREM 2. By Facts 3.2 and 3.1,

$$c(\tau_n(p) \oplus 1) = c((n+1)r(\eta_n)) = (n+1)(\eta_n + \eta_n^{p-1}).$$

Put $\alpha = c(\tau_n(p))$, $t = 2n+1$ and $s = 2n+2$ in Theorem 1(2). Then the former part follows immediately from Theorem 1(2). The latter part is proved as follows. Using Theorem 4.1, we have the results for $p = 3$ and $n \geq 6$ by Lemma 4.2(1), and for $p \geq 5$ and $n \geq 2p-2$ by Lemma 4.2(2). For $p = 3$ and $n = 4, 5$, we have the results by Theorems 3.1, 3.2, respectively. □

5. Proofs of Theorems 3–4 and Corollaries 1–4

We recall some known results on stable extendibility and extendibility of $\tau_n(p)$ for the proofs of Theorems 3–4 and Corollaries 1–4.

THEOREM 5.1 ([4, Theorem 1.2]). *Let p be an odd prime and n an integer with $n > p$. Then $\tau_n(p)$ is stably extendible to $L^{2n+1}(p)$ and is not stably extendible to $L^{2n+2}(p)$.*

Theorem 5.2 ([4, Theorem 1.3]). *Let p be an odd prime.*

(1) *Let n be an integer with $p-3 \le n \le p$. Then $\tau_n(p)$ is stably extendible to $L^m(p)$ for every $m > n$.*

(2) *If $p \equiv \pm 1 \pmod{12}$, $\tau_2(p)$ is stably extendible to $L^m(p)$ for every $m > 2$.*

Theorem 5.3 ([6, Theorems 5.1 and 5.3]). *Let p be an integer with $p > 1$. Then the following three conditions are equivalent to one another:*

(i) *$\tau_n(p)$ is extendible to $L^m(p)$ for every $m > n$.*

(ii) *$\tau_n(p)$ is extendible to $L^{n+1}(p)$.*

(iii) *$n = 0, 1$ or 3.*

Corollary 5.4. *Let p be an integer with $p \ge 2$ and $p \ne 3$ and m an integer with $m > p$. Then $\tau_p(p)$ is not extendible to $L^m(p)$.*

Proof. Suppose that $\tau_p(p)$ is extendible to $L^m(p)$. Then, by the implication (ii) $\Rightarrow$ (iii) of Theorem 5.3, we have $p = 0, 1$ or 3, since $L^{p+1}(p) \subset L^m(p)$. This contradicts to the assumption. $\square$

Corollary 5.5. *Let p be an integer with $p \ge 3$ and $p \ne 4$ and m an integer with $m > p-1$. Then $\tau_{p-1}(p)$ is not extendible to $L^m(p)$.*

Proof. Suppose that $\tau_{p-1}(p)$ is extendible to $L^m(p)$. Then, by the implication (ii) $\Rightarrow$ (iii) of Theorem 5.3, we have $p-1 = 0, 1$ or 3, that is, $p = 1, 2$ or 4, since $L^p(p) \subset L^m(p)$. This contradicts to the assumption. $\square$

Similarly, we have

Corollary 5.6. *Let p be an integer with $p \ge 4$ and $p \ne 5$ and m an integer with $m > p-2$. Then $\tau_{p-2}(p)$ is not extendible to $L^m(p)$.*

Corollary 5.7. *Let p be an integer with $p \ge 5$ and $p \ne 6$ and m an integer with $m > p-3$. Then $\tau_{p-3}(p)$ is not extendible to $L^m(p)$.*

Proof of Theorem 3. The former part is equal to that of Theorem 5.1. The latter part is proved as follows. By the assumption $n > p$, we see that $n \ne 0, 1$ and 3. Hence the implication (ii) $\Rightarrow$ (iii) of Theorem 5.3 shows that $\tau_n(p)$ is not extendible to $L^{n+1}(p)$. Thus the latter part holds, since $L^{n+1}(p) \subset L^{2n+1}(p)$. $\square$

Proof of Theorem 4. (1) The former part is a consequence of Theorem 5.2(1).

The latter part follows from the implication (ii) $\Rightarrow$ (iii) of Theorem 5.3, since $L^{n+1}(p) \subset L^m(p)$.

(2) The former part is a consequence of Theorem 5.2(2).

The latter part follows from the implication (ii) $\Rightarrow$ (iii) of Theorem 5.3, since $L^3(p) \subset L^m(p)$. $\square$

PROOF OF COROLLARIES 1–4. Using Theorem 5.2(1), we can prove these corollaries by Corollaries 5.4–5.7, respectively. $\square$

For $p = 11, 13$ and 17, additional results are obtained (cf. [4, Lemma 1.4]). Combining these results with Theorem 5.3, we have results similar to those in Theorem 4.

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