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# Asymptotic cut-off point in linear discriminant rule to adjust the misclassification probability for large dimensions

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Abstract. This paper is concerned with the problem of classifying an observation vector into one of two populations $\Pi_1 : N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $\Pi_2 : N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$. Anderson (1973, Ann. Statist.) provided an asymptotic expansion of the distribution for a Studentized linear discriminant function, and proposed a cut-off point in the linear discriminant rule to control one of the two misclassification probabilities. However, as dimension $p$ becomes larger, the precision worsens, which is checked by simulation. Therefore, in this paper we derive an asymptotic expansion of the distribution of a linear discriminant function up to the order $p^{-1}$ as $N_1$, $N_2$, and $p$ tend to infinity together under the condition that $p/(N_1 + N_2 - 2)$ converges to a constant in $(0, 1)$, and $N_1/N_2$ converges to a constant in $(0, \infty)$, where $N_i$ means the size of sample drown from $\Pi_i$ $(i = 1, 2)$. Using the expansion, we provide a cut-off point. A small-scale simulation revealed that our proposed cut-off point has good accuracy.

## 1. Introduction

This paper is concerned with the problem of classifying an observation vector $\boldsymbol{x}$ coming from either $\Pi_1 : N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ or $\Pi_2 : N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$ based on random samples

$$\boldsymbol{x}_{i1}, \dots, \boldsymbol{x}_{iN_i} \sim N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}) \qquad (i = 1, 2).$$

Let

$$W = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)' \boldsymbol{S}^{-1} \left\{ \boldsymbol{x} - \frac{1}{2}(\bar{\boldsymbol{x}}_1 + \bar{\boldsymbol{x}}_2) \right\},$$

where $\bar{\boldsymbol{x}}_1$, $\bar{\boldsymbol{x}}_2$, and $\boldsymbol{S}$ are the sample mean vectors and the pooled sample covariance matrix, respectively, and are defined by

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$$\bar{\boldsymbol{x}}_i = \frac{1}{N_i}\sum_{j=1}^{N_i}\boldsymbol{x}_{ij} \quad (i = 1, 2), \qquad \boldsymbol{S} = \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{N_i}(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)',$$

$$n = N - 2 = N_1 + N_2 - 2.$$

The linear discriminant rule (W-rule) classifies $\boldsymbol{x}$ as $\Pi_1$ if $W > c$ and as $\Pi_2$ if $W < c$ for a constant $c$. This classification causes two types of misclassification. One of those is to allocate $\boldsymbol{x}$ to $\Pi_1$ even though it is actually in $\Pi_2$. The other is that $\boldsymbol{x}$ is classified as $\Pi_2$ although it actually belongs to $\Pi_1$. These probabilities are represented as

$$e(1|2) = P(W > c \mid \boldsymbol{x} \in \Pi_2), \qquad e(2|1) = P(W < c \mid \boldsymbol{x} \in \Pi_1).$$

The distribution of $W$ when $\boldsymbol{x} \in \Pi_1$ is the same as that of $-W$ when $\boldsymbol{x} \in \Pi_2$ by interchanging $N_1$ and $N_2$. This indicates that $e(1|2)$ is obtained from $e(2|1)$ by replacing $(N_1, N_2, c)$ with $(N_2, N_1, -c)$. Thus, in this paper, we only deal with $e(2|1)$.

Generally, it is difficult to provide an analytic expression of $e(2|1)$. Instead, the probability has been studied to derive an asymptotic approximation under large sample asymptotic framework A0:

$$\mathrm{A0} : N_1 \to \infty, \qquad N_2 \to \infty, \qquad N_1/N_2 \to \gamma \in (0, \infty).$$

For a review of the results under A0, see, e.g., Anderson [3], and Fujikoshi et al. [7]. As $p$ becomes large, the accuracy of the approximation worsens. In order to improve it, it has been studied under the high-dimensional asymptotic framework A1:

$$\mathrm{A1} : p \to \infty, \qquad N_1 \to \infty, \qquad N_2 \to \infty, \qquad p/n \to \gamma_0 \in (0, 1),$$
$$\text{and} \qquad N_1/N_2 \to \gamma \in (0, \infty).$$

Raudys [11] derived an asymptotic approximation of the misclassification probability for the case in which $N_1 = N_2$, and Fujikoshi and Seo [6] derived this approximation without assuming that $N_1 = N_2$. Following Lachenbruch [9], for $\boldsymbol{x} \in \Pi_1$, it can be expressed that

$$W = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}\left\{\boldsymbol{x} - \frac{1}{2}(\bar{\boldsymbol{x}}_1 + \bar{\boldsymbol{x}}_2)\right\} = V^{1/2}Z - U, \tag{1}$$

where

$$V = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}\boldsymbol{\Sigma}\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2),$$

$$Z = V^{-1/2}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_1),$$

$$U = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1 - \boldsymbol{\mu}_1) - \frac{1}{2}D^2,$$

and $D^2$ is the squared sample Mahalanobis distance defined by $D^2 = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)$. The normality of $\boldsymbol{x}$ indicates that $Z$ is distributed as the standard normal distribution under the condition that $\bar{\boldsymbol{x}}_1$, $\bar{\boldsymbol{x}}_2$, and $\boldsymbol{S}$ are given. Since the conditional distribution does not depend on $\{\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2, \boldsymbol{S}\}$, $Z$ is independent from $\{U, V\}$. Based on the location and scale mixture representation (1), Fujikoshi [4] provided an error bound for the asymptotic approximation of $e(2|1)$. These results were subsequently reviewed in Fujikoshi et al. [7]. It is noted that the limiting distribution of $W$ under A1 is normal with mean $-u_0 = -\lim_{\mathrm{A1}} E[U]$ and variance $v_0 = \lim_{\mathrm{A1}}\{E[V] + \mathrm{Var}(U)\}$ when $\boldsymbol{x} \sim N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$. The analytic expression for $\mathrm{Var}(U)$ provided by Fujikoshi [4], shows that $\mathrm{Var}(U) \to 0$ under the assumption that the Mahalanobis distance $\Delta = \sqrt{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)}$ converges to a positive constant; thus, it can be abbreviated as $v_0 = \lim_{\mathrm{A1}} E[V]$.

It may be necessary to determine the cut-off point $c$ to adjust the probability of misclassification. Such a cut-off point is needed when one requires one of two misclassification probabilities to take a small value. The result under A0 was obtained from the work of Anderson [1], [2] (as cited in a more recent book by this author [3]). It is based on the asymptotic expansion for the distribution of a Studentized $W$ ($(W - D^2/2)/D$ for $\boldsymbol{x} \in \Pi_1$, $-(W + D^2/2)/D$ for $\boldsymbol{x} \in \Pi_2$) up to terms $n^{-1}$. Fujikoshi and Kanazawa [5] derived an asymptotic expansion for the distribution of a Studentized maximum likelihood classification statistic up to terms $n^{-1}$. Kanazawa [8] used this expansion to propose a cut-off point to control the misclassification probability. Since these cut-off points are asymptotic results under A0, the precision worsens as $p$ becomes large.

The location and scale mixture representation (1) and probability convergences of $\{U, V\}$ implies that the limiting distribution of $(W + u)/\sqrt{v}$ is $N(0,1)$ under A1 when $\boldsymbol{x} \in \Pi_1$, where

$$u = -\frac{n}{2(m-1)}\left\{\Delta^2 + \left(\frac{p}{N_2} - \frac{p}{N_1}\right)\right\},$$

$$v = \frac{n^2(n+1)}{(m-1)(m+1)(m+2)}\left(\Delta^2 + \frac{Np}{N_1 N_2}\right),$$

$m = n - p$. Note that $u = E[U]$, and $v$ is asymptotically equal to $E[V]$ under A1. Since $u$ and $v$ contain the unknown parameter $\Delta^2$, this parameter needs to be estimated to provide a Studentized statistic. A commonly used unbiased estimator of $\Delta^2$ is

$$\widehat{\Delta^2} = \frac{n-p-1}{n}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2) - \frac{pN}{N_1 N_2}. \tag{2}$$

This $\widehat{\Delta^2}$ has consistency under A1. Let $c = -\hat{u} + \sqrt{\hat{v}}x$, where $(\hat{u}, \hat{v})$ is obtained from $(u, v)$ by replacing $\Delta^2$ with $\widehat{\Delta^2}$. From Slutsky's theorem (cf., Rao [10]), the limit of $e(2|1)$ under A1 is given as

$$\lim_{\text{A1}} P\left(\frac{W + \hat{u}}{\sqrt{\hat{v}}} < x \,\middle|\, \boldsymbol{x} \in \Pi_1\right) = \Phi(x),$$

where $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution. In order to improve the accuracy of the asymptotic approximation, we would need to use the asymptotic expansion of the distribution under A1.

In this paper, we derive an asymptotic expansion for the distribution of $(W + \hat{u})/\sqrt{\hat{v}}$ when $\boldsymbol{x} \in \Pi_1$ under A1. Using the expansion, we specify a cut-off point such that one of the two misclassification probabilities takes the presetting value.

This paper is organized as follows. In Section 2, we derive an asymptotic expansion for the distribution of $(W + \hat{u})/\sqrt{\hat{v}}$ under A1 when $\boldsymbol{x} \in \Pi_1$. Based on the expansion, in Section 3 we specify cut-off point $c_h$ such that $e(2|1)$ takes the presetting values. In Section 4, we show the limiting value for $e(1|2)$ for the case in which the cut-off point $c_h$ is used. Section 5 presents the simulation results for the misclassification probability. This paper is concluded in Section 6. The proof of the lemma is given in Appendix.

Hereafter, we denote "$\overset{\mathscr{D}}{=}$" as the equality in distribution, "$\overset{p}{\to}$" as the probability of convergence, and "$\perp\!\!\!\perp$" as the independence.

## 2. Asymptotic expansion for the distribution of a Studentized linear discriminant function under A1

Assume that $\Delta^2$ converges to a positive value as $p \to \infty$. Using Lachenbruch's [9] expression given in (1), we have

$$P\left(\frac{W + \hat{u}}{\sqrt{\hat{v}}} < x \,\middle|\, \boldsymbol{x} \in \Pi_1\right) = E\left[\Phi\left(\frac{\sqrt{\hat{v}}x + U - \hat{u}}{\sqrt{V}}\right)\right]. \tag{3}$$

Note that we use the fact which $(\hat{u}, \hat{v}) \perp\!\!\!\perp Z$ in the above equality. Let

$$\boldsymbol{u}_1 = \left(\frac{1}{N_1} + \frac{1}{N_2}\right)^{-1/2} \boldsymbol{\Sigma}^{-1/2}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2),$$

$$\boldsymbol{u}_2 = \frac{1}{\sqrt{N}} \boldsymbol{\Sigma}^{-1/2}(N_1\bar{\boldsymbol{x}}_1 + N_2\bar{\boldsymbol{x}}_2 - N_1\boldsymbol{\mu}_1 - N_2\boldsymbol{\mu}_2),$$

$$\boldsymbol{B} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}\boldsymbol{\Sigma}^{-1/2}.$$

Then $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{B}$ are independent. In addition, $\boldsymbol{u}_1 \sim N_p((1/N_1 + 1/N_2)^{-1/2}\boldsymbol{\delta}, \boldsymbol{I}_p)$ and $\boldsymbol{u}_2 \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, where $\boldsymbol{\delta} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$. It also holds that $n\boldsymbol{B}$ is distributed as a Wishart distribution with $n$ degrees of freedom and covariance matrix $\boldsymbol{I}_p$, which is denoted as $W_p(n, \boldsymbol{I}_p)$. Substituting them, we have

$$U = -\frac{1}{2}\left(\frac{p}{N_2} - \frac{p}{N_1}\right)\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p} + \frac{p}{\sqrt{N_1N_2}}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_2}{p} - \tau\frac{\boldsymbol{\delta}'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{\sqrt{p}},$$

$$V = \frac{Np}{N_1N_2}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-2}\boldsymbol{u}_1}{p},$$

$$\tau = \sqrt{\frac{pN_2}{NN_1}}.$$

In addition, the unbiased estimator of $\widehat{\Delta^2}$ given as (2) can be written as

$$\widehat{\Delta^2} = \frac{Np}{N_1N_2}\left\{\frac{m-1}{n}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p} - 1\right\}. \tag{4}$$

Replacing $\Delta^2$ in $(u, v)$ with (4), we have

$$\hat{u} = -\frac{n}{2(m-1)}\left\{\frac{Np}{N_1N_2}\frac{m-1}{n}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p} - \frac{Np}{N_1N_2} + \left(\frac{p}{N_2} - \frac{p}{N_1}\right)\right\},$$

$$\hat{v} = \frac{n(n+1)}{(m+1)(m+2)}\frac{Np}{N_1N_2}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p}.$$

Then,

$$\begin{aligned}
&\sqrt{\hat{v}}x + U - \hat{u} \\
&\quad = \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\omega^{-1}\sqrt{\frac{Q_1}{p}}x - \frac{1}{2}\left(\frac{p}{N_2} - \frac{p}{N_1}\right)\frac{Q_1}{p} + \frac{p}{\sqrt{N_1N_2}}\frac{B_2}{p} \\
&\qquad - \tau\frac{B_1}{\sqrt{p}} + \frac{\omega^{-2}}{2}\frac{Q_1}{p} - \frac{n}{2(m-1)}\omega^{-2} + \frac{1}{2}\frac{n}{m-1}\left(\frac{p}{N_2} - \frac{p}{N_1}\right),
\end{aligned} \tag{5}$$

$$V = \omega^{-2}\frac{Q_2}{p}, \tag{6}$$

where $Q_1 = \boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1$, $Q_2 = \boldsymbol{u}_1'\boldsymbol{B}^{-2}\boldsymbol{u}_1$, $B_1 = \boldsymbol{\delta}'\boldsymbol{B}^{-1}\boldsymbol{u}_1$ and $B_2 = \boldsymbol{u}_2'\boldsymbol{B}^{-1}\boldsymbol{u}_1$, $\omega^2 = N_1N_2/(Np)$. The following lemma is used to express $Q_1$, $Q_2$, $B_1$, and $B_2$ as functions of the independent standard normal and chi-squared variables, simultaneously.

LEMMA 1. *Let* $\boldsymbol{v}_1 \sim N_p(\boldsymbol{\delta}, \boldsymbol{I}_p)$, $\boldsymbol{v}_2 \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, $\boldsymbol{A} \sim W_p(n, \boldsymbol{I}_p)$, *and let* $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, *and* $\boldsymbol{A}$ *be independent. Then the following equalities in distribution hold simultaneously:*

$$\boldsymbol{\delta}'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{\Delta}{Y_1}\left(Z_1 + \Delta - \sqrt{\frac{Y_2}{Y_3}}Z_2\right),$$

$$\boldsymbol{v}_2'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \sqrt{\frac{1}{Y_1^2}\left(1 + \frac{Y_2}{Y_3}\right)\{(Z_1 + \Delta)^2 + Z_2^2 + Y_4\}}Z_3,$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{1}{Y_1}\{(Z_1 + \Delta)^2 + Z_2^2 + Y_4\},$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{1}{Y_1^2}\left(1 + \frac{Y_2}{Y_3}\right)\{(Z_1 + \Delta)^2 + Z_2^2 + Y_4\},$$

*where* $\Delta = \sqrt{\boldsymbol{\delta}'\boldsymbol{\delta}}$; $Z_1$, $Z_2$, $Z_3$, *and* $Y_1, \dots, Y_4$ *are independent*, $Z_i \sim N(0, 1)$, $i = 1, 2, 3$, $Y_i \sim \chi^2_{f_i}$, *chi-squared distribution with* $f_i$ *degrees of freedom*, $i = 1, \dots, 4$,

$$f_1 = n - p + 1, \qquad f_2 = p - 1, \qquad f_3 = n - p + 2, \qquad f_4 = p - 2.$$

The proof of Lemma 1 is given in Appendix A. From Lemma 1, we have

$$\frac{Q_1}{p} \stackrel{\mathscr{D}}{=} \frac{n}{f_1}\frac{1}{1 + \sqrt{2/f_1}W_1}S,$$

$$\frac{B_1}{\sqrt{p}} \stackrel{\mathscr{D}}{=} \frac{n}{f_1}\frac{\Delta}{1 + \sqrt{2/f_1}W_1}\left(\frac{Z_1}{\sqrt{p}} + \omega\Delta - \sqrt{\frac{f_2}{f_3}}\sqrt{T}\frac{Z_2}{\sqrt{p}}\right),$$

$$\frac{B_2}{p} \stackrel{\mathscr{D}}{=} \frac{n}{f_1}\frac{1}{1 + \sqrt{2/f_1}W_1}\sqrt{\left(1 + \frac{f_2}{f_3}T\right)S}\frac{Z_3}{\sqrt{p}},$$

$$\frac{Q_2}{p} \stackrel{\mathscr{D}}{=} \frac{n^2}{f_1^2}\frac{1}{(1 + \sqrt{2/f_1}W_1)^2}\left(1 + \frac{f_2}{f_3}T\right)S,$$

where $W_i = \sqrt{f_i/2}(Y_i/f_i - 1)$ for $i = 1, \dots, 4$,

$$S = \left(\frac{Z_1}{\sqrt{p}} + \omega\Delta\right)^2 + \left(\frac{Z_2}{\sqrt{p}}\right)^2 + \frac{p-2}{p}\left(1 + \sqrt{\frac{2}{f_4}}W_4\right), \tag{7}$$

$$T = \frac{1 + \sqrt{2/f_2}W_2}{1 + \sqrt{2/f_3}W_3}. \tag{8}$$

Then (7) is written as

$$S = (1 + \omega^2 \Delta^2) + \frac{1}{\sqrt{p}}\left(2\omega\Delta Z_1 + \sqrt{\frac{2p}{f_4}} W_4\right) + \frac{1}{p}(Z_1^2 + Z_2^2 - 2)$$

$$= s_0 + \frac{1}{\sqrt{p}} S_1 + \frac{1}{p} S_2. \tag{9}$$

Write

$$\frac{1}{1 + \sqrt{2/f_j}\, W_j} = \sum_{k=0}^{4} \left(\frac{2}{f_j}\right)^{k/2} (-W_j)^k - \left(\frac{2}{f_j}\right)^{5/2} \frac{W_j^5}{1 + \sqrt{2/f_j}\, W_j}. \tag{10}$$

Then (8) is

$$T = \left(1 + \sqrt{\frac{2}{f_2}} W_2\right)\left\{1 + \sum_{k=1}^{4}\left(\frac{2}{f_3}\right)^{k/2}(-W_3)^k - \frac{(\sqrt{2/f_3}\, W_3)^5}{1 + \sqrt{2/f_3}\, W_3}\right\}$$

$$= 1 + \frac{1}{\sqrt{p}}\left(\sqrt{\frac{2p}{f_2}} W_2 - \sqrt{\frac{2p}{f_3}} W_3\right) + \frac{1}{p}\left(\frac{2p}{f_3} W_3^2 - \frac{2p}{f_2 f_3} W_2 W_3\right) + r_1$$

$$= 1 + \frac{1}{\sqrt{p}} T_1 + \frac{1}{p} T_2 + r_1, \tag{11}$$

where $r_1 = r_p^{(1)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-3/2}$ times a homogeneous polynomial of degree 3 in the $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-2}$ times a homogeneous polynomial of degree 4, plus a remainder term that is $O(p^{-5/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$.

From (10) for $j = 1$, and (9), we have

$$\frac{Q_1}{p} = \frac{n}{f_1} \frac{1}{1 + \sqrt{2/f_1}\, W_1} S$$

$$= \frac{n}{f_1}\left\{1 + \sum_{k=1}^{4}\left(\frac{2}{f_1}\right)^{k/2}(-W_1)^k - \frac{(\sqrt{2/f_1}\, W_1)^5}{1 + \sqrt{2/f_1}\, W_1}\right\}$$

$$\cdot \left(s_0 + \frac{1}{\sqrt{p}} S_1 + \frac{1}{p} S_2\right)$$

$$= \frac{n}{f_1} s_0 + \frac{1}{\sqrt{p}} \frac{n}{f_1} \left( S_1 - \sqrt{\frac{2p}{f_1}} s_0 W_1 \right)$$

$$+ \frac{1}{p} \frac{n}{f_1} \left( S_2 - \sqrt{\frac{2p}{f_1}} S_1 W_1 + \frac{2p}{f_1} s_0 W_1^2 \right) + r_2$$

$$= q_{1,0} + \frac{1}{\sqrt{p}} Q_{1,1} + \frac{1}{p} Q_{1,2} + r_2, \tag{12}$$

where $r_2 = r_p^{(2)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-3/2}$ times a homogeneous polynomial of degree 3 in the $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-3/2}$ times a homogeneous polynomial of degree 1, plus $p^{-2}$ times a homogeneous polynomial of degree 4, plus $p^{-2}$ times a homogeneous polynomial of degree 2, plus a remainder term that is $O(p^{-5/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$.

From (12), it is written as

$$\sqrt{\frac{Q_1}{p}} = \sqrt{q_{1,0}} \left( 1 + \frac{1}{\sqrt{p}} q_{1,0}^{-1} Q_{1,1} + \frac{1}{p} q_{1,0}^{-1} Q_{1,2} + q_{1,0}^{-1} r_2 \right)^{1/2}$$

$$= \sqrt{q_{1,0}} (1 + x_1)^{1/2},$$

where

$$x_1 = \frac{1}{\sqrt{p}} q_{1,0}^{-1} Q_{1,1} + \frac{1}{p} q_{1,0}^{-1} Q_{1,2} + q_{1,0}^{-1} r_2.$$

Maclaurin series expansion of $(1 + x_1)^{1/2}$ gives

$$\sqrt{\frac{Q_1}{p}} = \sqrt{q_{1,0}} \left\{ 1 + \frac{1}{\sqrt{p}} \frac{Q_{1,1}}{2q_{1,0}} + \frac{1}{p} \left( \frac{Q_{1,2}}{2q_{1,0}} - \frac{Q_{1,1}^2}{8q_{1,0}^2} \right) \right\} + r_3, \tag{13}$$

where $r_3 = r_p^{(3)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-3/2}$ times a homogeneous polynomial of degree 3 in the $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-3/2}$ times a homogeneous polynomial of degree 1, plus $p^{-2}$ times a homogeneous polynomial of degree 4, plus $p^{-2}$ times a homogeneous polynomial of degree 2, plus $p^{-2}$ times a constant that is $O(1)$ as $p \to \infty$ under A1, plus a remainder term that is $O(p^{-5/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$.

From (9) and (11), we have

$$\left(1+\frac{f_2}{f_3}T\right)S=\left\{1+\frac{f_2}{f_3}\left(1+\frac{1}{\sqrt{p}}T_1+\frac{1}{p}T_2+r_1\right)\right\}\left(s_0+\frac{1}{\sqrt{p}}S_1+\frac{1}{p}S_2\right)$$
$$=\left(1+\frac{f_2}{f_3}\right)s_0+\frac{1}{\sqrt{p}}\left\{\left(1+\frac{f_2}{f_3}\right)S_1+\frac{f_2}{f_3}s_0T_1\right\}$$
$$+\frac{1}{p}\left\{\left(1+\frac{f_2}{f_3}\right)S_2+\frac{f_2}{f_3}S_1T_1+\frac{f_2}{f_3}s_0T_2\right\}+r_4, \tag{14}$$

where $r_4=r_p^{(4)}(Z_1,Z_2,Z_3,W_1,W_2,W_3,W_4)$ is a remainder term with the same property as $r_2$. Write

$$\frac{1}{(1+\sqrt{2/f_1}\,W_1)^2}=\sum_{j=0}^{4}\left(-2\sqrt{\frac{2}{f_1}}W_1-\frac{2}{f_1}W_1^2\right)^j-\frac{\left(2\sqrt{\frac{2}{f_1}}W_1+\frac{2}{f_1}W_1^2\right)^5}{1+2\sqrt{2/f_1}\,W_1+(2/f_1)W_1^2}$$

Then, we have

$$\frac{Q_2}{p}=\left(\frac{n}{f_1}\right)^2\frac{1}{(1+\sqrt{2/f_1}\,W_1)^2}\left(1+\frac{f_2}{f_3}T\right)S$$
$$=\left(\frac{n}{f_1}\right)^2\left(1+\frac{f_2}{f_3}\right)s_0+\frac{1}{\sqrt{p}}\left(\frac{n}{f_1}\right)^2$$
$$\times\left\{\left(1+\frac{f_2}{f_3}\right)S_1+\frac{f_2}{f_3}s_0T_1-2\sqrt{\frac{2p}{f_1}}\left(1+\frac{f_2}{f_3}\right)s_0W_1\right\}$$
$$+\frac{1}{p}\left(\frac{n}{f_1}\right)^2\left[\left(1+\frac{f_2}{f_3}\right)S_2+\frac{f_2}{f_3}S_1T_1+\frac{f_2}{f_3}s_0T_2\right.$$
$$\left.-2\sqrt{\frac{2p}{f_1}}\left\{\left(1+\frac{f_2}{f_3}\right)S_1+\frac{f_2}{f_3}s_0T_1\right\}W_1+\frac{6p}{f_1}\left(1+\frac{f_2}{f_3}\right)s_0W_1^2\right]+r_5$$
$$=q_{2,0}+\frac{1}{\sqrt{p}}Q_{2,1}+\frac{1}{p}Q_{2,2}+r_5, \tag{15}$$

where $r_5=r_p^{(5)}(Z_1,Z_2,Z_3,W_1,W_2,W_3,W_4)$ is a remainder term with the same property as $r_2$.

From (14), it can be described that

$$\sqrt{\left(1+\frac{f_2}{f_3}T\right)S}=\sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}\sqrt{1+x_2},$$

where

$$x_2 = \left\{\left(1+\frac{f_2}{f_3}\right)s_0\right\}^{-1}\left[\frac{1}{\sqrt{p}}\left\{\left(1+\frac{f_2}{f_3}\right)S_1 + \frac{f_2}{f_3}s_0 T_1\right\}\right.$$
$$\left. + \frac{1}{p}\left\{\left(1+\frac{f_2}{f_3}\right)S_2 + \frac{f_2}{f_3}S_1 T_1 + \frac{f_2}{f_3}s_0 T_2\right\} + r_4\right].$$

Maclaurin series expansion of $(1+x_2)^{1/2}$ gives

$$\sqrt{\left(1+\frac{f_2}{f_3}T\right)S} = \sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}\left\{1+\frac{1}{2\sqrt{p}}(\tilde{T}_1+\tilde{S}_1)\right\} + r_6, \tag{16}$$

where $\tilde{S}_1 = S_1/s_0$, $\tilde{T}_1 = \{(f_2/f_3)/(1+f_2/f_3)\}T_1$, $r_6 = r_p^{(6)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-1}$ times a homogeneous polynomial of degree 2 in $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-3/2}$ times a homogeneous polynomial of degree 3, plus $p^{-3/2}$ times a homogeneous polynomial of degree 1, plus $p^{-1}$ times a constant that is $O(1)$ as $p \to \infty$ under A1, plus a remainder term that is $O(p^{-2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$. From (10) for $j = 1$, and (16), we have

$$\begin{aligned}
\frac{B_2}{p} &= \frac{n}{f_1}\frac{1}{1+\sqrt{2/f_1}W_1}\sqrt{\left(1+\frac{f_2}{f_3}T\right)S}\frac{Z_3}{\sqrt{p}} \\
&= \frac{1}{\sqrt{p}}\frac{n}{f_1}\sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}\,Z_3 + \frac{1}{p}\frac{n}{f_1}\sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}\left\{\frac{\tilde{T}_1+\tilde{S}_1}{2} - \sqrt{\frac{2p}{f_1}}W_1\right\}Z_3 + r_7 \\
&= \frac{1}{\sqrt{p}}B_{2,1} + \frac{1}{p}B_{2,2} + r_7,
\end{aligned} \tag{17}$$

where $r_7 = r_p^{(7)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_3$.

From (11), it can be described that

$$\sqrt{T} = \sqrt{1+x_3},$$

where

$$x_3 = \frac{1}{\sqrt{p}}T_1 + \frac{1}{p}T_2 + r_1.$$

Maclaurin series expansion of $(1+x_3)^{1/2}$ gives

$$\sqrt{T} = 1 + \frac{1}{2\sqrt{p}}T_1 + \frac{1}{p}\left(\frac{1}{2}T_2 - \frac{1}{8}T_1^2\right) + r_8, \tag{18}$$

where $r_8 = r_p^{(8)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_1$. From (10) for $j = 1$, and (18), we have

$$
\begin{aligned}
\frac{B_1}{\sqrt{p}} &= \frac{n}{f_1}\frac{\Delta}{1+\sqrt{2/f_1}W_1}\left(\frac{Z_1}{\sqrt{p}} + \omega\Delta - \sqrt{\frac{f_2}{f_3}}\sqrt{T}\frac{Z_2}{\sqrt{p}}\right) \\
&= \frac{n}{f_1}\omega\Delta^2 + \frac{1}{\sqrt{p}}\frac{n}{f_1}\left\{\left(Z_1 - \sqrt{\frac{f_2}{f_3}}Z_2\right)\Delta - \sqrt{\frac{2p}{f_1}}\omega\Delta^2 W_1\right\} \\
&\quad + \frac{1}{p}\frac{n}{f_1}\left\{-\frac{\Delta}{2}\sqrt{\frac{f_2}{f_3}}Z_2 T_1 - \sqrt{\frac{2p}{f_1}}\left(Z_1 - \sqrt{\frac{f_2}{f_3}}Z_2\right)\Delta W_1 + \frac{2p}{f_1}\omega\Delta^2 W_1^2\right\} + r_9 \\
&= b_{1,0} + \frac{1}{\sqrt{p}}B_{1,1} + \frac{1}{p}B_{1,2} + r_9, \qquad (19)
\end{aligned}
$$

where $r_9 = r_p^{(9)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_1$.

From the expansions of $Q_1/p$, $\sqrt{Q_1/p}$, $Q_2/p$, $B_2/p$, and $Q_1/\sqrt{p}$, which are given as (12), (13), (15), (17), and (19), respectively, we have

$$
\begin{aligned}
\sqrt{\hat{v}}x + U - \hat{u} &= \sqrt{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0}\,\omega^{-1}x + \frac{1}{\sqrt{p}}U_1 + \frac{1}{p}U_2 + r_{10}, \\
V &= \omega^{-2}\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0 + \frac{\omega^{-2}}{\sqrt{p}}Q_{2,1} + \frac{\omega^{-2}}{p}Q_{2,2} + r_{11},
\end{aligned}
$$

where

$$
\begin{aligned}
U_1 &= AQ_{1,1} - \tau B_{1,1} + \frac{p}{\sqrt{N_1 N_2}}B_{2,1}, \\
U_2 &= AQ_{1,2} - \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}x}{8q_{1,0}^{3/2}}Q_{1,1}^2 - \tau B_{1,2} + \frac{p}{\sqrt{N_1 N_2}}B_{2,2} \\
&\quad + \frac{np}{(m-1)(m+1)}\left[\left(\frac{p}{N_2} - \frac{p}{N_1}\right) - \omega^{-2}\right], \\
A &= \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}}{2\sqrt{q_{1,0}}}x - \frac{1}{2}\left(\frac{p}{N_2} - \frac{p}{N_1}\right) + \frac{\omega^{-2}}{2},
\end{aligned}
$$

$r_{10} = r_p^{(10)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_3$, and $r_{11} = r_p^{(10)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_2$. Write

$$\frac{1}{\sqrt{V}} = \omega\left\{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0\right\}^{-1/2}\frac{1}{\sqrt{1+(1/\sqrt{p})\tilde{Q}_{2,1}+(1/p)\tilde{Q}_{2,2}+\tilde{r}_{11}}}$$

$$= \omega\left\{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0\right\}^{-1/2}\frac{1}{\sqrt{1+x_4}},$$

where

$$x_4 = \frac{1}{\sqrt{p}}\tilde{Q}_{2,1} + \frac{1}{p}\tilde{Q}_{2,2} + \tilde{r}_{11},$$

$$\tilde{Q}_{2,j} = \left\{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0\right\}^{-1} Q_{2,j} \qquad (j = 1, 2),$$

$$\tilde{r}_{11} = \left\{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0\right\}^{-1} \omega^2 r_{11}.$$

Maclaurin series expansion of $(1+x_4)^{-1/2}$ gives

$$\frac{1}{\sqrt{V}} = \omega\left\{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0\right\}^{-1/2} \cdot \left[1 - \frac{1}{\sqrt{p}}\frac{\tilde{Q}_{2,1}}{2} + \frac{1}{p}\left\{\frac{3\tilde{Q}_{2,1}^2}{8} - \frac{\tilde{Q}_{2,2}}{2}\right\} + r_{12}\right], \tag{20}$$

where $r_{12} = r_p^{(12)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_3$. From (18) and (20), we have

$$R = \frac{\sqrt{\hat{v}}x + U - \hat{u}}{\sqrt{V}}$$
$$= x + \frac{1}{\sqrt{p}}\left[-\frac{1}{2}x\tilde{Q}_{2,1} + \tilde{U}_1\right] + \frac{1}{p}\left[x\left(\frac{3}{8}\tilde{Q}_{2,1}^2 - \frac{1}{2}\tilde{Q}_{2,2}\right) - \frac{1}{2}\tilde{U}_1\tilde{Q}_{2,1} + \tilde{U}_2\right] + r_{13}$$
$$= x + \frac{1}{\sqrt{p}}R_1 + \frac{1}{p}R_2 + r_{13}, \tag{21}$$

where $r_{13} = r_p^{(13)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_3$,

$$\tilde{U}_j = \left\{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0\right\}^{-1/2} \omega U_j \qquad (j = 1, 2).$$

The Taylor series expansion of $\Phi(.)$ in (3) up to order 5 gives

$$
\begin{aligned}
&\Phi\left(\frac{\sqrt{\hat{v}}x + U - \hat{u}}{\sqrt{V}}\right) \\
&\quad = \Phi\left(x + \frac{1}{\sqrt{p}}R_1 + \frac{1}{p}R_2 + r_{13}\right) \\
&\quad = \Phi(x) + \sum_{j=1}^{4} \frac{\Phi^{(j)}(x)}{j!}\left[\frac{1}{\sqrt{p}}R_1 + \frac{1}{p}R_2 + r_{13}\right]^j \\
&\qquad + \frac{\Phi^{(5)}\left(x + \theta\left(\frac{1}{\sqrt{p}}R_1 + \frac{1}{p}R_2 + r_{13}\right)\right)}{5!}\left[\frac{1}{\sqrt{p}}R_1 + \frac{1}{p}R_2 + r_{13}\right]^5 \\
&\quad = \Phi(x) + \phi(x)\left\{\frac{1}{\sqrt{p}}R_1 + \frac{1}{p}\left(R_2 - \frac{1}{2}xR_1^2\right)\right\} + \frac{1}{p^{3/2}}r_{14} + \frac{1}{p^2}r_{15} + r_{16} \qquad (22)
\end{aligned}
$$

for some real number $\theta \in (0, 1)$, where $r_{14} = r_p^{(14)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a term consisting of a homogeneous polynomial of degree 3 in $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients (which depend on $x$) are $O(1)$ as $p \to \infty$ under A1, plus a homogeneous polynomial of degree 1, $r_{15} = r_p^{(15)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a term consisting of a homogeneous polynomial of degree 4 in $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients (which depend on $x$) are $O(1)$ as $p \to \infty$ under A1, plus a homogeneous polynomial of degree 2, plus a constant that is $O(1)$ as $p \to \infty$ under A1, and $r_{16} = r_p^{(16)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term that is $O(p^{-5/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, $W_4$, and $x$.

Let $J_p$ be the set of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ such that

$$
\begin{aligned}
|Z_i| &< 2\sqrt{\log p} \qquad (i = 1, 2, 3), \\
|W_i| &< \sqrt{2}\log p \qquad (i = 1, 2, 3, 4).
\end{aligned}
$$

Using the same derivation as given in Appendix of Anderson [2],

$$P(J_p) = 1 - o(p^{-2})$$

under A1. The difference between $E[\Phi(.)]$ and the integral of $\Phi(.)$ times the joint probability density function of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ over $J_p$ is $o(p^{-2})$ under A1, because $0 \le \Phi(.) \le 1$. For the elements of $J_p$,

$$\left|\frac{Z_i}{\sqrt{p}}\right| < 2\sqrt{\frac{\log p}{p}}, \qquad \left|\frac{W_i}{\sqrt{p}}\right| < \sqrt{2}\frac{\log p}{\sqrt{p}}.$$

For a sufficiently large $p$ that satisfies $(\log p)/\sqrt{p} < 1/\sqrt{2}$, it holds that

$$\left|\frac{Z_i}{\sqrt{p}}\right| < 1, \qquad \left|\frac{W_i}{\sqrt{p}}\right| < 1.$$

For such a large $p$, there exists a constant $x_5$, which is $O(1)$ under A1 such that

$$r_p^{(16)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4) < x_5 \left(\frac{\log p}{\sqrt{p}}\right)^5$$

for the element of $J_p$. Hence the integral of this element times the joint density function of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ over $J_p$ is $o(p^{-2})$ under A1. Since the fourth-order absolute moments of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ exist and are bounded, the integral of $r_{15}$ times the joint density function of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ is bounded. Thus, $(1/p^2)E[r_{15}] = O(p^{-2})$.

The differences between $E[R_1]$, $E[R_1^2]$, $E[R_2]$, $E[r_{14}]$, and $E[r_{15}]$ and the integrals over $J_p$ of $R_1$, $R_1^2$, $R_2$, $r_{14}$ and $r_{15}$ times the joint probability density function of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$, respectively, are $o(p^{-2})$ under A1. Thus,

$$\begin{aligned}
&P\left(\frac{W + \hat{u}}{\sqrt{\hat{v}}} < x \,\middle|\, \boldsymbol{x} \in \Pi_1\right) \\
&\quad = E\left[\Phi\left(\frac{\sqrt{\hat{v}}x + U - \hat{u}}{\sqrt{V}}\right)\right] \\
&\quad = \Phi(x) + \phi(x)\left\{\frac{1}{\sqrt{p}}E[R_1] + \frac{1}{p}\left(E[R_2] - \frac{x}{2}E[R_1^2]\right)\right\} + \frac{1}{p^{3/2}}E[r_{14}] + O(p^{-2}) \\
&\quad = \Phi(x) + \frac{1}{p}\phi(x)\left(E[R_2] - \frac{x}{2}E[R_1^2]\right) + O(p^{-2}) \qquad (23)
\end{aligned}$$

because the third-order moments of the elements of $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ are either 0 or $O(p^{-1/2})$ under A1. Note that $R_1$ is represented as the linear combination of $\{Z_1, Z_2, Z_3, W_1, \dots, W_4\}$. Hence $E[R_1] = 0$.

Next, we show the analytic expressions of $E[R_1^2]$ and $E[R_2]$. It can be expressed that

$$E[R_1^2] = \frac{x^2}{4}E[\tilde{Q}_{2,1}^2] + E[\tilde{U}_1^2] - xE[\tilde{Q}_{2,1}\tilde{U}_1].$$

We show the analytic expressions of $E[Q_{2,1}^2]$, $E[U_1^2]$, and $E[Q_{2,1}U_1]$, respectively.

Since $S_1 \perp\!\!\!\perp T_1 \perp\!\!\!\perp W_1$,

$$E[Q_{2,1}^2] = \frac{n^4}{f_1^4}\left[\left(1+\frac{f_2}{f_3}\right)^2 E[S_1^2] + \left(\frac{f_2}{f_3}\right)^2 s_0^2 E[T_1^2] + 4\cdot\frac{2p}{f_1}\left(1+\frac{f_2}{f_3}\right)^2 s_0^2 E[W_1^2]\right].$$

Write

$$S_1^2 = 4\omega^2\Delta^2 Z_1^2 + \frac{2p}{f_4}W_4^2 + 4\omega\Delta\sqrt{\frac{2p}{f_4}}Z_1 W_4.$$

Then

$$E[S_1^2] = 4\omega^2\Delta^2 + \frac{2p}{f_4}.$$

Write

$$T_1^2 = \frac{2p}{f_2}W_2^2 + \frac{2p}{f_3}W_3^2 - 2\frac{2p}{\sqrt{f_2 f_3}}W_2 W_3.$$

Then

$$E[T_1^2] = \frac{2p}{f_2} + \frac{2p}{f_3}.$$

Hence

$$\begin{aligned} E[Q_{2,1}^2] = \frac{n^4}{f_1^4}\Bigg[&\left(1+\frac{f_2}{f_3}\right)^2\left(4\omega^2\Delta^2 + \frac{2p}{f_4}\right) + \left(\frac{f_2}{f_3}\right)^2 (1+\omega^2\Delta^2)^2\left(\frac{2p}{f_2}+\frac{2p}{f_3}\right) \\ &+ \frac{8p}{f_1}\left(1+\frac{f_2}{f_3}\right)^2(1+\omega^2\Delta^2)^2\Bigg]. \end{aligned} \tag{24}$$

It can be described that

$$\begin{aligned} E[U_1^2] = {}& A^2 E[Q_{1,1}^2] + \tau^2 E[B_{1,1}^2] + \frac{p^2}{N_1 N_2}E[B_{2,1}^2] - 2A\tau E[Q_{1,1}B_{1,1}] \\ &+ \frac{2p}{\sqrt{N_1 N_2}}AE[Q_{1,1}B_{2,1}] - \frac{2p}{\sqrt{N_1 N_2}}\tau E[B_{1,1}B_{2,1}]. \end{aligned}$$

Write

$$Q_{1,1}^2 = \frac{n^2}{f_1^2}\left(S_1^2 + \frac{2p}{f_1}s_0^2 W_1^2 - 2\sqrt{\frac{2p}{f_1}}s_0 S_1 W_1\right).$$

Then,

$$
\begin{aligned}
E[Q_{1,1}^2] &= \frac{n^2}{f_1^2}\left[\left(4\omega^2\Delta^2 + \frac{2p}{f_4}\right) + \frac{2p}{f_1}s_0^2\right] \\
&= \frac{n^2}{f_1^2}\left[\left(4\omega^2\Delta^2 + \frac{2p}{f_4}\right) + \frac{2p}{f_1}(1+\omega^2\Delta^2)^2\right]. \qquad (25)
\end{aligned}
$$

Write

$$
B_{2,1}^2 = \frac{n^2}{f_1^2}\left(1 + \frac{f_2}{f_3}\right)s_0 Z_3^2.
$$

Then,

$$
E[B_{2,1}^2] = \frac{n^2}{f_1^2}\left(1 + \frac{f_2}{f_3}\right)s_0 = \frac{n^2}{f_1^2}\left(1 + \frac{f_2}{f_3}\right)(1+\omega^2\Delta^2).
$$

Write

$$
\begin{aligned}
B_{1,1}^2 = \frac{n^2}{f_1^2}\Bigg\{&\left(Z_1^2 + \frac{f_2}{f_3}Z_2^2 - 2\sqrt{\frac{f_2}{f_3}}Z_1Z_2\right)\Delta^2 + \frac{2p}{f_1}\omega^2\Delta^4W_1^2 \\
&- 2\sqrt{\frac{2p}{f_1}}\omega\Delta^2W_1\left(Z_1 - \sqrt{\frac{f_2}{f_3}}Z_2\right)\Delta\Bigg\}.
\end{aligned}
$$

Then,

$$
E[B_{1,1}^2] = \frac{n^2}{f_1^2}\left\{\left(1 + \frac{f_2}{f_3}\right)\Delta^2 + \frac{2p}{f_1}\omega^2\Delta^4\right\}.
$$

From the independence, we have

$$
\begin{aligned}
E[B_{1,1}B_{2,1}] &= E[B_{1,1}]E[B_{2,1}] = 0, \\
E[Q_{1,1}B_{2,1}] &= E[Q_{1,1}]E[B_{2,1}] = 0.
\end{aligned}
$$

It is written as

$$
\begin{aligned}
Q_{1,1}B_{1,1} = \frac{n^2}{f_1^2}&\left(2\omega\Delta Z_1 + \sqrt{\frac{2p}{f_4}}W_4 - \sqrt{\frac{2p}{f_1}}s_0W_1\right) \\
&\cdot\left\{\left(Z_1 - \sqrt{\frac{f_2}{f_3}}Z_2\right)\Delta - \sqrt{\frac{2p}{f_1}}\omega\Delta^2W_1\right\}.
\end{aligned}
$$

Thus, we have

$$\begin{aligned} E[Q_{1,1}B_{1,1}] &= \frac{n^2}{f_1^2}\left(2\omega\Delta^2 + \frac{2p}{f_1}s_0\omega\Delta^2\right) \\ &= \frac{n^2}{f_1^2}\left\{\left(2+\frac{2p}{f_1}\right)\omega\Delta^2 + \frac{2p}{f_1}\omega^3\Delta^4\right\}. \end{aligned}$$

Hence

$$\begin{aligned} E[U_1^2] = \left(\frac{n}{f_1}\right)^2 \Bigg[ & A^2\left\{\left(4\omega^2\Delta^2 + \frac{2p}{f_4}\right) + \frac{2p}{f_1}(1+\omega^2\Delta^2)^2\right\} \\ & - 2\tau A\left\{\left(2+\frac{2p}{f_1}\right)\omega\Delta^2 + \frac{2p}{f_1}\omega^3\Delta^4\right\} + \frac{p^2}{N_1N_2}\left(1+\frac{f_2}{f_3}\right)(1+\omega^2\Delta^2) \\ & + \tau^2\left\{\left(1+\frac{f_2}{f_3}\right)\Delta^2 + \frac{2p}{f_1}\omega^2\Delta^4\right\}\Bigg]. \end{aligned} \tag{26}$$

We also write that

$$\begin{aligned} Q_{2,1} &= \left(1+\frac{f_2}{f_3}\right)S_1 + \frac{f_2}{f_3}s_0T_1 - 2\sqrt{\frac{2p}{f_1}}\left(1+\frac{f_2}{f_3}\right)s_0W_1 \\ &= \left(\frac{n}{f_1}\right)^2\Bigg[2\left(1+\frac{f_2}{f_3}\right)\omega\Delta Z_1 - 2\sqrt{\frac{2p}{f_1}}\left(1+\frac{f_2}{f_3}\right)s_0W_1 + \frac{f_2}{f_3}\sqrt{\frac{2p}{f_2}}s_0W_2 \\ &\qquad - \frac{f_2}{f_3}\sqrt{\frac{2p}{f_2}}s_0W_3 + \left(1+\frac{f_2}{f_3}\right)\sqrt{\frac{2p}{f_4}}W_4\Bigg], \\ U_1 &= AQ_{1,1} - \tau B_{1,1} + \frac{p}{\sqrt{N_1N_2}}B_{2,1} \\ &= \frac{n}{f_1}\Bigg[(2\omega A - \tau)\Delta Z_1 + \tau\sqrt{\frac{f_2}{f_3}}\Delta Z_2 + \frac{p}{\sqrt{N_1N_2}}\sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}Z_3 \\ &\qquad - \sqrt{\frac{2p}{f_1}}(s_0A - \omega\tau\Delta^2)W_1 + \sqrt{\frac{2p}{f_4}}AW_4\Bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} E[Q_{2,1}U_1] &= \left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)\left[2\omega\Delta(2\omega A - \tau)\Delta + \frac{4p}{f_1}s_0(s_0A - \omega\tau\Delta^2) + A\frac{2p}{f_4}\right] \\ &= \left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)\Bigg[\left\{4\omega^2\Delta^2 + \frac{2p}{f_4} + \frac{4p}{f_1}(1+\omega^2\Delta^2)^2\right\}A \\ &\qquad - \frac{2N_2}{N}\Delta^2 - \frac{4p}{f_1}\frac{N_2}{N}(1+\omega^2\Delta^2)\Delta^2\Bigg]. \end{aligned} \tag{27}$$

Next, we show the analytic expression of $E[R_2]$. It can be expressed that

$$E[R_2] = x\left(\frac{3}{8}E[\tilde{Q}_{2,1}^2] - \frac{1}{2}E[\tilde{Q}_{2,2}]\right) - \frac{1}{2}E[\tilde{Q}_{2,1}\tilde{U}_1] + E[\tilde{U}_2].$$

The analytic expression of $E[Q_{2,1}^2]$ and of $E[Q_{2,1}U_1]$ have already been obtained as (28) and (27), respectively. Therefore, we show $E[Q_{2,2}]$ and $E[U_2]$.

Since $S_1 \perp\!\!\!\perp T_1 \perp\!\!\!\perp W_1$, we have

$$E[Q_{2,2}] = \left(\frac{n}{f_1}\right)^2 \left[\left(1 + \frac{f_2}{f_3}\right)E[S_2] + \frac{f_2}{f_3}s_0 E[T_2] + \frac{6p}{f_1}\left(1 + \frac{f_2}{f_3}\right)s_0 E[W_1^2]\right].$$

It is found that $E[S_2] = 0$ and $E[T_2] = 2p/f_3$. Thus,

$$E[Q_{2,2}] = \left(\frac{n}{f_1}\right)^2 (1 + \omega^2 \Delta^2)\left[\frac{2f_2 p}{f_3^2} + \frac{6p}{f_1}\left(1 + \frac{f_2}{f_3}\right)\right]. \tag{28}$$

It can be expressed that

$$\begin{aligned} E[U_2] = {} & AE[Q_{1,2}] - \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}x}{8q_{1,0}^{3/2}}E[Q_{1,1}^2] \\ & - \tau E[B_{1,2}] + \frac{p}{\sqrt{N_1 N_2}}E[B_{2,2}] + \frac{np}{(m-1)(m+1)}\left[\left(\frac{p}{N_2} - \frac{p}{N_1}\right) - \omega^{-2}\right]. \end{aligned}$$

Here, the analytic expression of $E[Q_{1,1}^2]$ is obtained as (25). It is also shown that

$$\begin{aligned} E[Q_{1,2}] &= \frac{n}{f_1}\left[\frac{2p}{f_1}(1 + \omega^2 \Delta^2)\right], \\ E[B_{2,2}] &= 0, \\ E[B_{1,2}] &= \frac{2pn}{f_1^2}\omega \Delta^2. \end{aligned}$$

Hence

$$\begin{aligned} E[U_2] = {} & \frac{2np}{f_1^2}(1 + \omega^2 \Delta^2)A - \frac{B}{8(1 + \omega^2 \Delta^2)}\frac{n}{f_1}\left\{\left(4\omega^2 \Delta^2 + \frac{2p}{f_4}\right) + \frac{2p}{f_1}(1 + \omega^2 \Delta^2)^2\right\} \\ & - \frac{2np}{f_1^2}\omega\tau\Delta^2 + \frac{np}{(m-1)(m+1)}\left\{\left(\frac{p}{N_2} - \frac{p}{N_1}\right) - \omega^{-2}\right\}, \end{aligned} \tag{29}$$

where

$$B = \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}}{\left\{\frac{n}{f_1}(1+\omega^2\Delta^2)\right\}^{1/2}}x.$$

Summarizing these results, we have the following proposition.

Proposition 1. *Assume that $\Delta^2$ converges to a positive constant as $p \to \infty$. Let*

$$\hat{u} = -\frac{n}{2(m-1)}\left\{\widehat{\Delta^2} + \left(\frac{p}{N_2} - \frac{p}{N_1}\right)\right\},$$

$$\hat{v} = \frac{n^2(n+1)}{(m-1)(m+1)(m+2)}\left(\widehat{\Delta^2} + \frac{Np}{N_1N_2}\right),$$

*where $\widehat{\Delta^2}$ is the unbiased estimator for $\Delta^2$ defined as* (2). *Then,*

$$P\left(\frac{W+\hat{u}}{\sqrt{\hat{v}}} < x \,\middle|\, \boldsymbol{x} \in \Pi_1\right) = \Phi(x) + \frac{1}{p}\phi(x)\left(E[R_2] - \frac{x}{2}E[R_1^2]\right) + O(p^{-2})$$

*as $p \to \infty$ under the high-dimensional asymptotic framework* A1*, where*

$$E[R_1^2] = \frac{x^2}{4\zeta^4}E[Q_{2,1}^2] + \frac{\omega^2}{\zeta^2}E[U_1^2] - x\frac{\omega}{\zeta^3}E[Q_{2,1}U_1],$$

$$E[R_2] = x\left(\frac{3}{8\zeta^4}E[Q_{2,1}^2] - \frac{1}{2\zeta^2}E[Q_{2,2}]\right) - \frac{\omega}{2\zeta^3}E[Q_{2,1}U_1] + \frac{\omega}{\zeta}E[U_2],$$

$$\zeta = \sqrt{\frac{n^2(n+1)}{(m+1)^2(m+2)}(1+\omega^2\Delta^2)},$$

$$\omega = \sqrt{\frac{N_1N_2}{Np}}.$$

*Here, the expectations appearing in $E[R_1^2]$ and $E[R_2]$ have the following analytic expressions.*

$$E[Q_{2,1}^2] = \left(\frac{n}{f_1}\right)^4\left[\left(1+\frac{f_2}{f_3}\right)^2\left(4\omega^2\Delta^2 + \frac{2p}{f_4}\right)\right.$$
$$\left. + \left\{\left(\frac{f_2}{f_3}\right)^2\left(\frac{2p}{f_2} + \frac{2p}{f_3}\right) + \frac{8p}{f_1}\left(1+\frac{f_2}{f_3}\right)^2\right\}(1+\omega^2\Delta^2)^2\right],$$

$$
\begin{aligned}
E[U_1^2] = \left(\frac{n}{f_1}\right)^2 \Bigg[ & A^2\left\{4\omega^2\Delta^2 + \frac{2p}{f_4} + \frac{2p}{f_1}(1+\omega^2\Delta^2)^2\right\} \\
& - 2\sqrt{\frac{pN_2}{NN_1}}A\left\{\left(2+\frac{2p}{f_1}\right)\omega\Delta^2 + \frac{2p}{f_1}\omega^3\Delta^4\right\} \\
& + \frac{p^2}{N_1N_2}\left(1+\frac{f_2}{f_3}\right)(1+\omega^2\Delta^2) \\
& + \frac{pN_2}{NN_1}\left\{\left(1+\frac{f_2}{f_3}\right)\Delta^2 + \frac{2p}{f_1}\omega^2\Delta^4\right\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
E[Q_{2,1}U_1] = \left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)\Bigg[ & \left\{4\omega^2\Delta^2 + \frac{2p}{f_4} + \frac{4p}{f_1}(1+\omega^2\Delta^2)^2\right\}A \\
& - \frac{2N_2}{N}\Delta^2 - \frac{4p}{f_1}\frac{N_2}{N}(1+\omega^2\Delta^2)\Delta^2\Bigg],
\end{aligned}
$$

$$
E[Q_{2,2}] = \left(\frac{n}{f_1}\right)^2(1+\omega^2\Delta^2)\left\{\frac{2f_2p}{f_3^2} + \frac{6p}{f_1}\left(1+\frac{f_2}{f_3}\right)\right\},
$$

$$
\begin{aligned}
E[U_2] = & \frac{2np}{f_1^2}(1+\omega^2\Delta^2)A - \frac{B}{8(1+\omega^2\Delta^2)}\frac{n}{f_1} \\
& \times\left\{\left(4\omega^2\Delta^2 + \frac{2p}{f_4}\right) + \frac{2p}{f_1}(1+\omega^2\Delta^2)^2\right\} - \frac{2np}{f_1^2}\frac{N_2}{N}\Delta^2 \\
& + \frac{np}{(m-1)(m+1)}\left\{\left(\frac{p}{N_2} - \frac{p}{N_1}\right) - \omega^{-2}\right\},
\end{aligned}
$$

*where* $f_1 = n - p + 1,\ f_2 = p - 1,\ f_3 = n - p + 2,\ f_4 = p - 2,$

$$
A = \frac{1}{2}\left\{B - \left(\frac{p}{N_2} - \frac{p}{N_1}\right) + \omega^{-2}\right\},
$$

$$
B = \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}}{\left\{\frac{n}{f_1}(1+\omega^2\Delta^2)\right\}^{1/2}}x.
$$

## 3. Asymptotic cut-off point which $e(2|1)$ takes presetting value

In this section, we propose an asymptotic cut-off point that the misclassification probability takes as its presetting value with the error $O(p^{-2})$ under A1.

Let

$$
\tilde{x} = x - \frac{1}{p}\left\{E[R_2] - \frac{x}{2}E[R_1^2]\right\}.
$$

From Proposition 1, we have

$$P\left(\frac{W+\hat{u}}{\sqrt{\hat{v}}} < \tilde{x} \,\middle|\, \boldsymbol{x} \in \Pi_1\right) = \Phi(x) + O(p^{-2})$$

as $p \to \infty$ under A1. We next estimate $E[R_2]$ and $E[R_1^2]$.

From (1), the unbiased estimator of $\Delta^2$ defined as (2) can be written as

$$\widehat{\Delta^2} = \frac{Np}{N_1 N_2}\left[\frac{f_1 - 2}{f_1(1+\sqrt{2/f_1}W_1)}\left\{\left(\frac{Z_1}{\sqrt{p}} + \omega\Delta\right)^2 + \left(\frac{Z_2}{\sqrt{p}}\right)^2 + \frac{p-2}{p}(1+\sqrt{2/f_4}W_4)\right\} - 1\right].$$

From (10), we can write it as

$$\widehat{\Delta^2} = \Delta^2 + r_{17},$$

where $r_{17} = r_p^{(17)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-1/2}$ times a homogeneous polynomial of order 1 in $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-1}$ times a homogeneous polynomial of order 2, plus $p^{-1}$ times a constant that is $O(1)$ as $p \to \infty$ under A1, plus a remainder term that is $O(p^{-3/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$. Let $\hat{\zeta}$, $\widehat{E[Q_{2,1}^2]}$, $\widehat{E[U_1^2]}$, $\widehat{E[Q_{2,1}U_1]}$, $\widehat{E[Q_{2,2}]}$, and $\widehat{E[U_2]}$ be obtained from $\zeta$, $E[Q_{2,1}^2]$, $E[U_1^2]$, $E[Q_{2,1}U_1]$, $E[Q_{2,2}]$, and $E[U_2]$, respectively, by replacing $\Delta^2$ with $\widehat{\Delta^2}$, where $\zeta$, $E[Q_{2,1}^2]$, $E[U_1^2]$, $E[Q_{2,1}U_1]$, $E[Q_{2,2}]$, and $E[U_2]$ are given in Proposition 1. We next evaluate these estimators.

It can be expressed that

$$\begin{aligned}
\hat{\zeta} &= \sqrt{\frac{n^2(n+1)}{(m+1)^2(m+2)}(1+\omega^2\widehat{\Delta^2})} \\
&= \sqrt{\frac{n^2(n+1)}{(m+1)^2(m+2)}\{1+\omega^2(\Delta^2+r_{17})\}} \\
&= \zeta\sqrt{1+r_{18}},
\end{aligned}$$

where $r_{18} = r_p^{(18)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_{17}$. Maclaurin series expansion of $\sqrt{1+r_{18}}$ gives

$$\hat{\zeta} = \zeta + r_{19},$$

where $r_{19} = r_p^{(19)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_{17}$. Maclaurin series expansion of $(1 + \zeta^{-1} r_{19})^{-j}$ gives

$$\frac{1}{\hat{\zeta}^j} = \frac{1}{\zeta^j(1 + \zeta^{-1} r_{19})^j} = \frac{1}{\zeta^j} + r_{20,j}, \tag{30}$$

where $r_{20,j} = r_p^{(20,j)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_{17}$ for $j = 1, 2, 3, 4$. Using the same derivation, we have

$$\widehat{E[Q_{2,1}^2]} = E[Q_{2,1}^2] + r_{21}, \tag{31}$$

$$\widehat{E[U_1^2]} = E[U_1^2] + r_{22}, \tag{32}$$

$$\widehat{E[Q_{2,1} U_1]} = E[Q_{2,1} U_1] + r_{23}, \tag{33}$$

$$\widehat{E[Q_{2,2}]} = E[Q_{2,2}] + r_{24}, \tag{34}$$

$$\widehat{E[U_2]} = E[U_2] + r_{25}, \tag{35}$$

where $r_j = r_p^{(j)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_{17}$ for $j = 21, \ldots, 25$.

From (30) to (35), we find that

$$\widehat{E[R_1^2]} = E[R_1^2] + r_{26}, \tag{36}$$

$$\widehat{E[R_2]} = E[R_2] + r_{27}, \tag{37}$$

where $r_{26} = r_p^{(26)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ and $r_{27} = r_p^{(27)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ are remainder terms with the same property as $r_{17}$.

Let

$$\hat{x} = x - \frac{1}{p}\left\{\widehat{E[R_2]} - \frac{x}{2}\widehat{E[R_1^2]}\right\}.$$

From (36) and (37), it can be expressed that

$$\hat{x} = \tilde{x} + r_{28}, \tag{38}$$

where $r_{28} = r_p^{(28)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-3/2}$ times a homogeneous polynomial of order 1 in $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-2}$ times a homogeneous polynomial of order 2, plus $p^{-2}$ times a constant that is $O(1)$ as $p \to \infty$ under A1, plus a remainder term that is $O(p^{-5/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$. Then,

$$P\left(\frac{W+\hat{u}}{\sqrt{\hat{v}}} < \hat{x} \,\middle|\, \boldsymbol{x} \in \Pi_1\right) = E\left[\Phi\left(\frac{\sqrt{\hat{v}}\hat{x} + U - \hat{u}}{\sqrt{V}}\right)\right]$$
$$= E\left[\Phi\left(\frac{\sqrt{\hat{v}}\tilde{x} + U - \hat{u}}{\sqrt{V}} + \sqrt{\frac{\hat{v}}{V}} r_{28}\right)\right].$$

We proceed to evaluate $\sqrt{\hat{v}/V}$.

Write

$$\sqrt{\frac{\hat{v}}{V}} = \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\,\omega^{-1}\,\frac{\sqrt{Q_1/p}}{\sqrt{V}}.$$

From (13) and (20), we have

$$\sqrt{\frac{\hat{v}}{V}} = \sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\,\omega^{-1}\sqrt{q_{1,0}}\left[1 + \frac{1}{\sqrt{p}}\frac{Q_{1,1}}{2q_{1,0}} + \frac{1}{p}\left(\frac{Q_{1,2}}{2q_{1,0}} - \frac{Q_{1,1}^2}{8q_{1,0}^2}\right) + r_3\right]$$
$$\cdot\,\omega\sqrt{\frac{(m+1)^2(m+2)}{n^2(n+1)}}\,\frac{1}{\sqrt{s_0}}\left[1 - \frac{1}{\sqrt{p}}\frac{\tilde{Q}_{2,1}}{2} + \frac{1}{p}\left\{\frac{3\tilde{Q}_{2,1}^2}{8} - \frac{\tilde{Q}_{2,2}}{2}\right\} + r_{12}\right]$$
$$= 1 + r_{29}, \tag{39}$$

where $r_{29} = r_p^{(29)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term with the same property as $r_{17}$.

From (21) and (39), it can be described that

$$\frac{\sqrt{\hat{v}}\hat{x} + U - \hat{u}}{\sqrt{V}} = \frac{\sqrt{\hat{v}}\tilde{x} + U - \hat{u}}{\sqrt{V}} + \sqrt{\frac{\hat{v}}{V}} r_{28}$$
$$= \tilde{x} + \frac{1}{\sqrt{p}}\tilde{R}_1 + \frac{1}{p}\tilde{R}_2 + r_{30},$$

where $\tilde{R}_j$ is given by $R_j$ by replacing $x$ with $\tilde{x}$ for $j = 1, 2$, $r_{30} = r_p^{(30)}(Z_1, Z_2, Z_3, W_1, W_2, W_3, W_4)$ is a remainder term consisting of $p^{-3/2}$ times a homogeneous polynomial of degree 3 in $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$ of which the coefficients are $O(1)$ as $p \to \infty$ under A1, plus $p^{-3/2}$ times a homogeneous polynomial of degree 1, plus $p^{-2}$ times a homogeneous polynomial of degree 4, plus $p^{-2}$ times a homogeneous polynomial of degree 2, plus $p^{-2}$ times a constant that is $O(1)$ as $p \to \infty$ under A1, plus a remainder term that is $O(p^{-5/2})$ under A1 for fixed $Z_1$, $Z_2$, $Z_3$, $W_1$, $W_2$, $W_3$, and $W_4$.

Using the same derivation from (13) to (23), we have

$$
\begin{aligned}
P\left(\frac{W+\hat{u}}{\sqrt{\hat{v}}} < \hat{x} \,\middle|\, \boldsymbol{x} \in \Pi_1\right) &= E\left[\Phi\left(\frac{\sqrt{\hat{v}}\hat{x} + U - \hat{u}}{\sqrt{V}}\right)\right] \\
&= \Phi(\tilde{x}) + \frac{1}{p}\phi(\tilde{x})\left(E[\tilde{R}_2] - \frac{\tilde{x}}{2}E[\tilde{R}_1^2]\right) + O(p^{-2}) \\
&= \Phi(x) + O(p^{-2}).
\end{aligned}
$$

PROPOSITION 2. *Assume that $\Delta^2$ converges to a positive constant as $p \to \infty$. Set the cut-off point $c_h$ as*

$$
c_h = \sqrt{\hat{v}}\left[z_\alpha - \frac{1}{p}\left\{\widehat{E[R_2]} - \frac{z_\alpha}{2}\widehat{E[R_1^2]}\right\}\right] - \hat{u}, \tag{40}
$$

*where $z_\alpha$ is the $\alpha$ percentile point of the standard normal distribution. Here, $\hat{u}$ and $\hat{v}$ are defined in Proposition* 1, *and $\widehat{E[R_2]}$ and $\widehat{E[R_1^2]}$ are obtained from $E[R_2]$ and $E[R_1^2]$, respectively, which are given in Proposition* 1, *by replacing $\Delta^2$ with $\widehat{\Delta^2}$ given as* (2). *Then*

$$
e(2|1) = \alpha + O(p^{-2}).
$$

*as $p \to \infty$ under the high-dimensional asymptotic framework* A1.

## 4. Limiting value of $e(1|2)$ when the cut-off point $c_h$ is used

We consider the probability $e(1|2)$ for the case in which the cut-off point $c_h$, which is defined as (40), is used.

Following Lachenbruch [9], we have

$$
e(1|2) = P(W > c_h \,|\, \boldsymbol{x} \in \Pi_2) = E\left[\Phi\left(\frac{-c_h + H}{\sqrt{V}}\right)\right],
$$

where

$$
H = -(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_2 - \boldsymbol{\mu}_2) - \frac{1}{2}D^2.
$$

In the above evaluation, we use

$$
-W = V^{1/2}Z_2 - H,
$$

where

$$
Z_2 = V^{-1/2}(\bar{\boldsymbol{x}}_2 - \bar{\boldsymbol{x}}_1)'S^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_2).
$$

When $\boldsymbol{x} \in \Pi_2$, $Z_2$ is distributed as $N(0,1)$ under the condition that $\{\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2, \boldsymbol{S}\}$ is given. Since the conditional distribution does not depend on $\{\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2, \boldsymbol{S}\}$, $Z_2$ is independent from $\{H, V, c_h\}$.

The expectation of $H$ is given as

$$E[H] = -\frac{n}{2(m-1)}\left\{\Delta^2 - \left(\frac{p}{N_2} - \frac{p}{N_1}\right)\right\}.$$

Suppose that $\hat{h}$ is the unbiased estimator of $E[H]$ obtained by replacing $\Delta^2$ with $\widehat{\Delta^2}$. Note that $\mathrm{Var}(H)$ can be obtained as $\mathrm{Var}(U)$ by interchanging $N_1$ and $N_2$. The analytic expression of $\mathrm{Var}(U)$ given by Fujikoshi [4] enables us to show that $\mathrm{Var}(U)$ converges to 0 under A1 and the assumption that $\Delta^2$ converges to a positive constant as $p \to \infty$. Hence $\mathrm{Var}(H)$ converges to 0. From Chebyshev's inequality, we have

$$H - E[H] \xrightarrow{p} 0.$$

Since $\widehat{\Delta^2} - \Delta^2$ converges to 0 in probability,

$$\hat{h} - E[H] \xrightarrow{p} 0.$$

Using Slutsky's theorem, we have

$$H - \hat{h} \xrightarrow{p} 0. \tag{41}$$

In addition, the following probability convergences hold:

$$\hat{u} + \hat{h} + \frac{n}{m}\Delta^2 \xrightarrow{p} 0, \tag{42}$$

$$\frac{\hat{v}}{V} \xrightarrow{p} 1, \tag{43}$$

$$V - \frac{n^3}{m^3}\left(\Delta^2 + \frac{Np}{N_1 N_2}\right) \xrightarrow{p} 0. \tag{44}$$

From (41), (42), (43), and (44), for the cut-off point $c_h$,

$$\lim_{\mathrm{A1}}\left[\frac{-c_h + H}{\sqrt{V}} - \left\{-z_\alpha - \frac{\frac{n}{m}\Delta^2}{\sqrt{\frac{n^3}{m^3}\left(\Delta^2 + \frac{Np}{N_1 N_2}\right)}}\right\}\right] = 0,$$

where $\lim_{\mathrm{A1}}$ represents the limit as $p \to \infty$ under A1. From the continuity and the uniform boundedness of $\Phi(.)$, we have

$$\lim_{\mathrm{A1}} E\left[\Phi\left(\frac{-c_h + H}{\sqrt{V}}\right)\right] = \Phi\left(z_{1-\alpha} - \lim_{\mathrm{A1}}\sqrt{\frac{m}{n}}\frac{\Delta^2}{\sqrt{\Delta^2 + \frac{Np}{N_1 N_2}}}\right).$$

Summarizing the result, we have the following proposition.

PROPOSITION 3. *For the cut-off point $c_h$ defined in* (40),

$$\lim_{\mathrm{A1}} e(1|2) = \Phi\left( z_{1-\alpha} - \lim_{\mathrm{A1}} \sqrt{\frac{m}{n}} \frac{\Delta^2}{\sqrt{\Delta^2 + \frac{Np}{N_1 N_2}}} \right). \tag{45}$$

*under the condition that $\Delta^2$ converges to a positive constant as $p \to \infty$.*

The limiting result (45) indicates that the misclassification probability decreases for $\Delta$, increases for $p$, and decreases for each sample size.

From (45), $e(1|2)$ can be estimated as

$$\Phi\left( z_{1-\alpha} - \sqrt{\frac{m}{n}} \frac{\widehat{\Delta^2}}{\sqrt{\widehat{\Delta^2} + \frac{Np}{N_1 N_2}}} \right). \tag{46}$$

The precision of this estimation is evaluated numerically in Section 5.

## 5. Numerical comparison

We examined the accuracy of the misclassification probability $e(2|1)$ by using the cut-off point $c_h$ given as (40) through simulation. We also calculated the misclassification probability by using the cut-off point $c_{\mathrm{A}}$ obtained by Anderson [2] (which is cited in Ch. 6 of their book [3]). The cut-off point $c_{\mathrm{A}}$ is as follows.

$$c_{\mathrm{A}} = \frac{1}{2} D^2 + D t_{\mathrm{A}},$$

where

$$t_{\mathrm{A}} = z_\alpha - \frac{1}{N_1}\left( \frac{N-1}{D} - \frac{1}{2} z_\alpha \right) + \frac{1}{n}\left\{ \left( p - \frac{3}{4} \right) z_\alpha + \frac{1}{4} z_\alpha^3 \right\}.$$

Without loss of generality we generate samples from $\Pi_1 : N_p(\Delta \boldsymbol{e}_1, \boldsymbol{I}_p)$ and $\Pi_2 : N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, where $\boldsymbol{e}_1 = (1, 0, \dots, 0)'$. The reason is that the statistics $W$ and $D^2$ are invariant with the following transformations:

$$\boldsymbol{x}_{ij}^* = \boldsymbol{\Gamma}' \boldsymbol{\Sigma}^{-1/2} (\boldsymbol{x}_{ij} - \boldsymbol{\mu}_2) \sim N_p(\delta_{i1} \Delta \boldsymbol{e}_1, \boldsymbol{I}_p) \qquad (i = 1, 2, j = 1, \dots, N_i),$$

$$\boldsymbol{x}^* = \boldsymbol{\Gamma}' \boldsymbol{\Sigma}^{-1/2} (\boldsymbol{x} - \boldsymbol{\mu}_2) \sim N_p(\Delta \boldsymbol{e}_1, \boldsymbol{I}_p),$$

where $\boldsymbol{\Gamma}$ is an orthogonal matrix of which the first column is proportional to $\boldsymbol{\delta} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$, and $\delta_{ij}$ denotes the Kronecker delta. We carried out a simulation for the case in which $N_1 = N_2$. The parameters were set to $p = 20$, 30, 40, 50, 60, 70, 80, 90, 100, and $\Delta = 1.05$. The sample sizes of $N_1$ and $N_2$ were set such that $p/n = 1/6$, $3/6$, and $5/6$. As the presetting value, we set

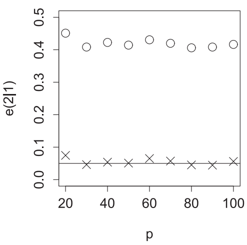


**Fig. 1.** $p/n = 5/6$

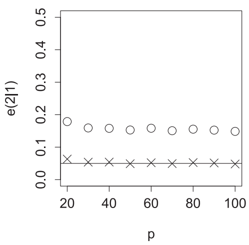


**Fig. 2.** $p/n = 3/6$

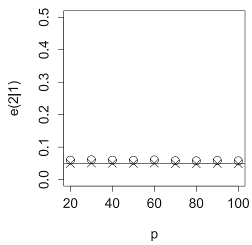


**Fig. 3.** $p/n = 1/6$

$\alpha = 0.05$. The procedure involving the classification of 10,000 testing observations per training data set of $N = N_1 + N_2$ observations constitutes one iterative cycle of the basic Monte Carlo simulation. We calculated the value of the misclassification probability by

$$\frac{\text{number of misclassification}}{10{,}000}. \tag{47}$$

We used 100 iterative cycles. The average of these values is presented in Fig. 1, 2, and 3. The symbols "$\circ$" and "$\times$" represent the values for $c_{\mathrm{A}}$ and $c_h$, respectively. The solid line represents the value of $\alpha$ $(= 0.05)$. We observe from Fig. 1 and Fig. 2 that, although the accuracy for $c_{\mathrm{A}}$ is not acceptable, that for $c_h$ is good. In Fig. 3, there is little difference between $c_h$ and $c_{\mathrm{A}}$.

We also carried out a simulation to assess the precision of estimation for $e(1|2)$ for the case in which $c_h$ is used as the cut-off. We set to $\Delta = 5$. We performed 100 iterations. For each iterative cycle, we generated a training data set, and calculated $c_h$ and (46). The average of the 100 values obtained from (46) is listed in the column "estimation" in Table 1. We also

Table 1. Values of $e(1|2)$

| $p$ | $p/N = 5/6$ simulation | $p/N = 5/6$ estimation | $p/N = 3/6$ simulation | $p/N = 3/6$ estimation | $p/N = 1/6$ simulation | $p/N = 1/6$ estimation |
|---|---|---|---|---|---|---|
| 20 | 0.590 | 0.730 | 0.066 | 0.091 | 0.002 | 0.004 |
| 30 | 0.563 | 0.600 | 0.057 | 0.072 | 0.002 | 0.003 |
| 40 | 0.532 | 0.544 | 0.055 | 0.064 | 0.002 | 0.003 |
| 50 | 0.503 | 0.510 | 0.049 | 0.059 | 0.002 | 0.003 |
| 60 | 0.486 | 0.488 | 0.049 | 0.055 | 0.002 | 0.003 |
| 70 | 0.469 | 0.473 | 0.047 | 0.054 | 0.002 | 0.003 |
| 80 | 0.472 | 0.462 | 0.048 | 0.051 | 0.002 | 0.003 |
| 90 | 0.465 | 0.454 | 0.045 | 0.050 | 0.002 | 0.003 |
| 100 | 0.454 | 0.445 | 0.044 | 0.049 | 0.002 | 0.002 |

generated 10,000 testing observations in each iterative cycle to calculate (47). The average of 100 values of (47) is provided in the column "simulation" in Table 1. We can confirm that the precision of the approximation improves significantly as $p$ and $N$ become large.

## 6. Conclusion

This paper is concerned with a problem determining the cut-off point in linear discriminant analysis, i.e., the point at which one of the two misclassification probabilities takes the presetting value. Our approach was to use an asymptotic expansion of the distribution for a Studentized linear discriminant function under the high-dimensional asymptotic framework A1. The precision of the approximation was demonstrated by carrying out a simulation. The proposed cut-off point was shown to have good accuracy for the case in which $p$ is relatively large compared to the sample sizes.

## Appendix A. Proof of Lemma 1

This section provides the proof of Lemma 1.

PROOF. Let $\boldsymbol{\Gamma}$ be an orthogonal matrix of order $p$ of which the first row is proportional to $\boldsymbol{\delta}'$, and let $\tilde{\boldsymbol{A}} = \boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}'$ and $\boldsymbol{w}_i = \boldsymbol{\Gamma}\boldsymbol{v}_i$, $i = 1, 2$. Then $\tilde{\boldsymbol{A}} \sim W_p(n, \boldsymbol{I}_p)$, $\boldsymbol{w}_1 \sim N_p(\Delta\boldsymbol{e}_1, \boldsymbol{I}_p)$, $\boldsymbol{w}_2 \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$ and $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\tilde{\boldsymbol{A}}$ are independent;

$$\boldsymbol{\delta}'\boldsymbol{A}\boldsymbol{v}_1 = (\boldsymbol{\Gamma}\boldsymbol{\delta})'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-1}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \stackrel{\mathscr{D}}{=} \Delta\boldsymbol{e}_1'\tilde{\boldsymbol{A}}^{-1}\boldsymbol{w}_1,$$

$$\boldsymbol{v}_2'\boldsymbol{A}^{-1}\boldsymbol{v}_1 = (\boldsymbol{\Gamma}\boldsymbol{v}_2)'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-1}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \stackrel{\mathscr{D}}{=} \boldsymbol{w}_2'\tilde{\boldsymbol{A}}^{-1}\boldsymbol{w}_1 \stackrel{\mathscr{D}}{=} \sqrt{\boldsymbol{w}_1'\tilde{\boldsymbol{A}}^{-2}\boldsymbol{w}_1}Z,$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-1}\boldsymbol{v}_1 = (\boldsymbol{\Gamma}\boldsymbol{v}_1)'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-1}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \stackrel{\mathscr{D}}{=} \boldsymbol{w}_1'\tilde{\boldsymbol{A}}^{-1}\boldsymbol{w}_1,$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 = (\boldsymbol{\Gamma}\boldsymbol{v}_1)'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-2}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \stackrel{\mathscr{D}}{=} \boldsymbol{w}_1'\tilde{\boldsymbol{A}}^{-2}\boldsymbol{w}_1,$$

where $\boldsymbol{e}_i$ denotes a fundamental vector with 1 in the $i$-th position, $Z \sim N(0, 1)$, and $Z$ and $\{\tilde{\boldsymbol{A}}, \boldsymbol{w}_1\}$ are independent. By using reflection matrix (Householder matrix) $\boldsymbol{H}$ between $\boldsymbol{e}_1$ and $(1/\sqrt{\boldsymbol{w}_1'\boldsymbol{w}_1})\boldsymbol{w}_1$,

$$\boldsymbol{\delta}'\boldsymbol{A}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \Delta\sqrt{\boldsymbol{w}_1'\boldsymbol{w}_1}(\boldsymbol{H}\boldsymbol{e}_1)'(\boldsymbol{H}\tilde{\boldsymbol{A}}\boldsymbol{H}')^{-1}\{\boldsymbol{H}(1/\sqrt{\boldsymbol{w}_1'\boldsymbol{w}_1})\boldsymbol{w}_1\} = \Delta\boldsymbol{w}_1'(\boldsymbol{H}\tilde{\boldsymbol{A}}\boldsymbol{H}')^{-1}\boldsymbol{e}_1.$$

In addition,

$$\boldsymbol{v}_1'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \boldsymbol{w}_1'\tilde{\boldsymbol{A}}^{-1}\boldsymbol{w}_1 = \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'(\boldsymbol{H}\tilde{\boldsymbol{A}}\boldsymbol{H}')^{-1}\boldsymbol{e}_1,$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \boldsymbol{w}_1'\tilde{\boldsymbol{A}}^{-2}\boldsymbol{w}_1 = \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'(\boldsymbol{H}\tilde{\boldsymbol{A}}\boldsymbol{H}')^{-2}\boldsymbol{e}_1.$$

Given $\boldsymbol{w}_1$, $\boldsymbol{C} \equiv \boldsymbol{H}\tilde{\boldsymbol{A}}\boldsymbol{H}' \sim W_p(n, \boldsymbol{I}_p)$; thus, $\boldsymbol{C}$ and $\boldsymbol{w}_1$ are independent. Partition

$$\boldsymbol{C} = \begin{pmatrix} c_{11} & \boldsymbol{c}_{21}' \\ \boldsymbol{c}_{21} & \boldsymbol{C}_{22} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w}_1 = \begin{pmatrix} w_{11} \\ \boldsymbol{w}_{21} \end{pmatrix}.$$

It can be expressed that

$$\boldsymbol{\delta}'\boldsymbol{A}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \Delta \boldsymbol{w}_1'\boldsymbol{C}^{-1}\boldsymbol{e}_1 = \frac{\Delta}{c_{11\cdot 2}}(w_{11} - \boldsymbol{w}_{21}'\boldsymbol{C}_{22}^{-1}\boldsymbol{c}_{21}),$$

where $c_{11\cdot 2} = c_{11} - \boldsymbol{c}_{21}'\boldsymbol{C}_{22}^{-1}\boldsymbol{c}_{21}$. In addition,

$$\boldsymbol{v}_1'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'\boldsymbol{C}^{-1}\boldsymbol{e}_1 = \frac{\boldsymbol{w}_1'\boldsymbol{w}_1}{c_{11\cdot 2}},$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'\boldsymbol{C}^{-2}\boldsymbol{e}_1 = \frac{\boldsymbol{w}_1'\boldsymbol{w}_1}{c_{11\cdot 2}^2}(1 + \boldsymbol{c}_{21}'\boldsymbol{C}_{22}^{-1}\boldsymbol{c}_{21}).$$

It is noted that $\boldsymbol{x} \equiv \boldsymbol{C}_{22}^{-1/2}\boldsymbol{c}_{21} \sim N_{p-1}(\boldsymbol{0}, \boldsymbol{I}_p)$, $\boldsymbol{D} \equiv \boldsymbol{C}_{22} \sim W_{p-1}(n, \boldsymbol{I}_{p-1})$, and $\boldsymbol{x}$ and $\boldsymbol{D}$ are independent; thus, $w_{11}$, $\boldsymbol{w}_{21}$, $\boldsymbol{x}$, $\boldsymbol{D}$ and $c_{11\cdot 2}$ are independent. Using these results, we have

$$\boldsymbol{\delta}'\boldsymbol{A}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{\Delta}{c_{11\cdot 2}}(w_{11} - \boldsymbol{w}_{21}'\boldsymbol{D}^{-1/2}\boldsymbol{x}) \qquad \text{and} \qquad \boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{\boldsymbol{w}_{21}'\boldsymbol{w}_{21}}{c_{11\cdot 2}^2}(1 + \boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x}).$$

Let $\boldsymbol{G}$ be an orthogonal matrix of order $p-1$ of which the first row is proportional to $\boldsymbol{x}'\boldsymbol{D}^{-1/2}$. Given $\boldsymbol{x}$ and $\boldsymbol{D}$, $\boldsymbol{y} \equiv \boldsymbol{G}\boldsymbol{w}_{21} \sim N_{p-1}(\boldsymbol{0}, \boldsymbol{I}_{p-1})$, and it is found that $w_{11}$, $c_{11\cdot 2}$, $\boldsymbol{x}$, $\boldsymbol{D}$, and $\boldsymbol{y}$ are independent. Partitioning $\boldsymbol{y} = (y_1 \boldsymbol{y}_2')'$, we have

$$\boldsymbol{\delta}'\boldsymbol{A}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{\Delta}{c_{11\cdot 2}}\{w_{11} - (\boldsymbol{G}\boldsymbol{w}_{21})'(\boldsymbol{G}\boldsymbol{D}^{-1/2}\boldsymbol{x})\} \stackrel{\mathscr{D}}{=} \frac{\Delta}{c_{11\cdot 2}}(w_{11} - \sqrt{\boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x}}y_1),$$

$$\boldsymbol{v}_1'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \stackrel{\mathscr{D}}{=} \frac{w_{11}^2 + (\boldsymbol{G}\boldsymbol{w}_{21})'(\boldsymbol{G}\boldsymbol{w}_{21})}{c_{11\cdot 2}} \stackrel{\mathscr{D}}{=} \frac{1}{c_{11\cdot 2}}(w_{11}^2 + y_1^2 + \boldsymbol{y}_2'\boldsymbol{y}_2),$$

$$\begin{aligned} \boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 &\stackrel{\mathscr{D}}{=} \frac{w_{11}^2 + (\boldsymbol{G}\boldsymbol{w}_{21})'(\boldsymbol{G}\boldsymbol{w}_{21})}{c_{11\cdot 2}^2}(1 + \boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x}) \\ &\stackrel{\mathscr{D}}{=} \frac{1}{c_{11\cdot 2}^2}(1 + \boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x})(w_{11}^2 + y_1^2 + \boldsymbol{y}_2'\boldsymbol{y}_2). \end{aligned}$$

The assertion of the lemma is followed from this result. □

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