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Bifurcation analysis of a diffusion-ODE model with Turing instability and hysteresis

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Abstract. This paper is devoted to the existence and (in)stability of nonconstant steady-states in a system of a semilinear parabolic equation coupled to an ODE, which is a simplified version of a receptor-ligand model of pattern formation. In the neighborhood of a constant steady-state, we construct spatially heterogeneous steady-states by applying the bifurcation theory. We also study the structure of the spectrum of the linearized operator and show that bifurcating steady-states are unstable against high wave number disturbances. In addition, we consider the global behavior of the bifurcating branches of nonconstant steady-states. These are quite different from classical reaction-diffusion systems where all species diffuse.

1. Introduction

In this paper we continue the study of a diffusion-ODE model with Turing instability and hysteresis started in [4]. In the seminal paper [16] in 1952, Turing proposed the idea of Diffusion-Driven Instability (DDI, for short), which asserts that in a system of two reacting chemicals, different diffusion rates may lead to a destabilization of a spatially uniform stationary state under spatially heterogeneous disturbance, thereby leading to a spontaneous formation of a nontrivial spatial structure, i.e., pattern. Since then many mathematical models based on DDI have been proposed to explain pattern formation in the natural world. However, not all patterns are formed as a result of DDI. There are phenomena which involve interactions among diffusive substances and non-diffusive substances. For instance, [12] proposed a model system consisting of free receptors, bound receptors, biochemical and degradative enzyme. Free and bound receptors are located on the cell surface and hence do not diffuse, while the biochemical and the enzyme both diffuse. This model contains spatially dependent coefficients. Later, [5] and [6] generalized

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the receptor-based model to models with constant coefficients, which produce patterns as a result of the existence of hysteresis in the quasi-stationary states. Recently in [4], we considered systems of reaction-diffusion equations coupled with ordinary differential equations, similar to the models proposed in [5], which exhibit DDI and hysteresis. We proved the existence and stability of stable *far from equilibrium* steady-states with jump discontinuity. On the other hand, instability of all continuous steady-states in such systems has been shown in [7].

In this paper we study the following system proposed in [4]:

$$\frac{\partial u}{\partial t} = -u - uv + m_1 \frac{u^2}{1+ku^2} \qquad \text{for } x \in (0,l),\ t>0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} - \mu_3 v - uv + m_2 \frac{u^2}{1+ku^2} \qquad \text{for } x \in (0,l),\ t>0, \tag{1.2}$$

$$\frac{\partial v}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,l) = 0 \qquad \text{for } x = 0,l,\ t>0, \tag{1.3}$$

$$(u(0,x), v(0,x)) = (u_0(x), v_0(x)) \in C^0(\bar{I}) \times (C^2(I) \cap C(\bar{I})), \tag{1.4}$$

with positive initial data. This is a quasi-steady state approximation of the system

$$\begin{cases} \dfrac{\partial u_f}{\partial t} = -\nu_1 u_f - \beta u_f v + \theta_1 \dfrac{u_f^2}{1+\kappa u_f^2} + \alpha u_b, \\ \delta \dfrac{\partial u_b}{\partial t} = -\nu_2 u_b + \beta u_f v - \alpha u_b, \\ \dfrac{\partial v}{\partial t} = D\dfrac{\partial^2 v}{\partial x^2} - \nu_3 v - \beta u_f v + \theta_2 \dfrac{u_f^2}{1+\kappa u_f^2} + \alpha u_b \end{cases}$$

where u_f and u_b denote the density of free receptors and bound receptors, respectively; v is the concentration of ligand which is diffusive. Equations (1.1)–(1.4) are obtained formally by letting $\delta \downarrow 0$ and appropriate scaling of the independent and dependent variables.

In order to understand the global-in-time behavior of solutions of the initial-boundary value problem (1.1)–(1.4) it is crucial to know not only the stability of discontinuous steady-states but also *how unstable* the steady-states in the neighborhood of the constant solution are.

To explain the novelty in the model dynamics, let us recall the Turing scenario: Starting from an almost uniform initial state, DDI augments the amplitude of the disturbance of specific wave numbers and the solution is forced to leave the neighborhood of the constant steady state, eventually being attracted by a stable steady state with, presumably, the same wave numbers.

Numerical simulations suggest that the final Turing pattern usually does not depend on the initial data. Rather, the initial disturbance seems to only trigger the emergence of pattern. We are still far from the complete understanding of the phenomenon.

The purpose of this paper is to investigate the existence of nonconstant (continuous) steady states in a neighborhood of the constant steady state (Theorem 3.6) and their *degree of instability* (Lemma 3.3 and Proposition 3.9). Unlike the reaction-diffusion systems with all diffusing species, the constant steady state of (1.1)–(1.4) which undergoes DDI is unstable against any disturbances of sufficiently large wave number. This is quite a contrast to the standard case where only disturbances of finitely many wave numbers can cause the instability.

Moreover, we consider the global structure of the set of *continuous* steady states of (1.1)–(1.4) by taking advantage of the fact that finding such steady states is reduced to finding solutions of the boundary value problem (3.24) for a single equation. It is to be emphasized that, for some range of the parameter k, the branch $\mathscr{C}_n = \{(v, D) \mid v'(x)$ has exactly $n-1$ zeros in $(0, l)\}$ does not continue to the neighborhood of $D = 0$ and hence it does not contain solutions which exhibit a concentration phenomenon such as boundary layer (Theorem 4.2).

This paper is organized as follows: Section 2 provides conditions for the existence of positive constant steady states in our model system (2.1)–(2.3). In section 3, we perform the bifurcation analysis near the constant steady states. In section 4, we focus on the stationary boundary value problem for a single equation. Particular attention is given to the construction of the monotone increasing steady states and their dependence on the diffusion coefficient and the initial data.

2. Constant steady-states

Consider the system of equations

$$\frac{\partial u}{\partial t} = f(u, v) \qquad \text{for } x \in (0, l),\ t > 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + g(u, v) \qquad \text{for } x \in (0, l),\ t > 0, \tag{2.2}$$

$$\frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(l, t) = 0 \qquad \text{for } t > 0, \tag{2.3}$$

where

$$f(u, v) = -u - uv + m_1\frac{u^2}{1 + ku^2}, \tag{2.4}$$

and

$$g(u,v) = -\mu_3 v - uv + m_2 \frac{u^2}{1+ku^2}. \tag{2.5}$$

and k, m_1, m_2, μ_3 and D are positive constants.

To start with, we search for a constant steady states of system (2.1)–(2.3). Note that $f(u,v) = 0$ is equivalent to the conditions $u = 0$ or

$$-1 - v + \frac{m_1 u}{1+ku^2} = 0. \tag{2.6}$$

Lemma 2.1. *For arbitrary positive parameters m_1, m_2, k and μ_3, the trivial solution $(u,v) = (0,0)$ is a spatially homogeneous steady-state of* (2.1)–(2.3).

Proof. If $u = 0$ then the equation $g(u,v) = 0$ reduces to $-\mu_3 v = 0$, i.e., $v = 0$. Hence $(0,0)$ is a unique solution of $f(u,v) = g(u,v) = 0$ with the property $u = 0$.

In what follows, we therefore assume that $u \neq 0$, hence, the stationary equation $g(u,v) = 0$ implies

$$v = \frac{m_2}{\mu_3 + u} \frac{u^2}{1+ku^2}. \tag{2.7}$$

Putting (2.6) and (2.7) together yields

$$\frac{m_1 u}{1+ku^2} - 1 = \frac{m_2}{\mu_3 + u} \frac{u^2}{1+ku^2}. \tag{2.8}$$

Positive solutions of (2.8) satisfy the cubic equation $(ku^2+1)(u+\mu_3) + m_2 u^2 - m_1 u(u+\mu_3) = 0$, that is,

$$ku^3 + (\mu_3 k + m_2 - m_1)u^2 + (1 - \mu_3 m_1)u + \mu_3 = 0. \tag{2.9}$$

When k is small, the limiting equation

$$(m_2 - m_1)u^2 + (1 - \mu_3 m_1)u + \mu_3 = 0 \tag{2.10}$$

plays an essential role. If $m_2 \neq m_1$ then this quadratic equation has two roots β_1 and β_2:

$$\begin{aligned} \beta_1 &= \frac{\mu_3 m_1 - 1 - \sqrt{(\mu_3 m_1 - 1)^2 - 4\mu_3(m_2 - m_1)}}{2(m_2 - m_1)}, \\ \beta_2 &= \frac{\mu_3 m_1 - 1 + \sqrt{(\mu_3 m_1 - 1)^2 - 4\mu_3(m_2 - m_1)}}{2(m_2 - m_1)}, \end{aligned} \tag{2.11}$$

which are real if and only if $(\mu_3 m_1 + 1)^2 \geq 4\mu_3 m_2$. Note that if $(\mu_3 m_1 + 1)^2 < 4\mu_3 m_2$ and $m_2 - m_1 \geq 0$ then the left-hand side of (2.10) is positive for all $u \in \mathbb{R}$, and hence the left-hand side of (2.9) is positive for all $u \geq 0$ whenever $k > 0$. This means that (2.9) has no positive roots for any $k > 0$ if $(\mu_3 m_1 + 1)^2 < 4\mu_3 m_2$ and $m_2 - m_1 \geq 0$. Assume that $(\mu_3 m_1 + 1)^2 \geq 4\mu_3 m_2$ and $m_2 \neq m_1$. Then, we have the following cases:

(i) $\beta_2 < 0 < \beta_1$ if $m_2 < m_1$;

(ii) $\beta_1 \leq \beta_2 < 0$ if $m_2 > m_1$ and $\mu_3 m_1 < 1$;

(iii) $0 < \beta_1 \leq \beta_2$ if $m_2 > m_1$ and $\mu_3 m_1 > 1$.

On the other hand, if $m_2 = m_1$ and $\mu_3 m_1 > 1$, then (2.10) has a unique root $\beta_0 > 0$:

$$\beta_0 = \frac{\mu_3}{\mu_3 m_1 - 1}. \tag{2.12}$$

The following proposition refines the result in Lemma 3.5 of [4].

Proposition 2.2. (A) *Let* $m_2 \geq m_1$ *be satisfied. If* $\mu_3 m_1 \leq 1$ *or* $(\mu_3 m_1 + 1)^2 < 4\mu_3 m_2$, *then equation* (2.9) *has no positive roots for any* $k > 0$.

(B) *If* $m_2 < m_1$, *then for* $k > 0$ *sufficiently small* (2.9) *has three roots* $\alpha_1 < 0 < \alpha_2 < \alpha_3$ *such that*

$$\alpha_1 = \beta_1 + \frac{(\beta_1 + \mu_3)\beta_1^2}{\mu_3 m_1 - 1 + 2(m_1 - m_2)\beta_1} k + O(k^2),$$

$$\alpha_2 = \beta_2 + \frac{(\beta_2 + \mu_3)\beta_2^2}{\mu_3 m_1 - 1 + 2(m_1 - m_2)\beta_2} k + O(k^2), \qquad \text{and}$$

$$\alpha_3 = \frac{m_1 - m_2}{k} + \frac{\mu_3 m_2 - 1}{m_1 - m_2} + O(k) \qquad \text{as } k \downarrow 0.$$

(C) *If* $m_2 = m_1$ *and* $\mu_3 m_1 > 1$, *then for* $k > 0$ *sufficiently small* (2.9) *has three roots* $\alpha_1 < 0 < \alpha_2 < \alpha_3$ *such that*

$$\alpha_1 = -\frac{\sqrt{\mu_3 m_1 - 1}}{\sqrt{k}} - \frac{\mu_3^2 m_1}{2(\mu_3 m_1 - 1)} + O(\sqrt{k}),$$

$$\alpha_2 = \beta_0 + \frac{\mu_3^4 m_1}{(\mu_3 m_1 - 1)^4} k + O(k^2),$$

$$\alpha_3 = \frac{\sqrt{\mu_3 m_1 - 1}}{\sqrt{k}} - \frac{\mu_3^2 m_1}{2(\mu_3 m_1 - 1)} + O(\sqrt{k}) \qquad \text{as } k \downarrow 0.$$

(D) *If* $m_2 > m_1$, $\mu_3 m_1 > 1$ *and* $(\mu_3 m_1 + 1)^2 \geq 4\mu_3 m_2$, *then for* $k > 0$ *sufficiently small* (2.9) *has three roots* $\alpha_1 < 0 < \alpha_2 < \alpha_3$ *such that*

$$\alpha_1 = -\frac{m_2 - m_1}{k} - \frac{\mu_3 m_1 - 2}{m_2 - m_1} + O(k),$$

$$\alpha_2 = \beta_1 + \frac{(\beta_1 + \mu_3)\beta_1^2}{\mu_3 m_1 - 1 + 2(m_1 - m_2)\beta_1} k + O(k^2),$$

$$\alpha_3 = \beta_2 + \frac{(\beta_2 + \mu_3)\beta_2^2}{\mu_3 m_1 - 1 + 2(m_1 - m_2)\beta_2} k + O(k^2) \qquad \text{as } k \downarrow 0.$$

Proof. Let $Q(u)$ denote the left-hand side of (2.9). Then its discriminant Δ_Q is given by

$$\begin{aligned}\Delta_Q = {} & (\mu_3 k + m_2 - m_1)^2(1 - \mu_3 m_1)^2 - 4k(1 - \mu_3 m_1)^3 - 4(\mu_3 k + m_2 - m_1)^3 \mu_3 \\ & - 27k^2\mu_3^2 + 18k(\mu_3 k + m_2 - m_1)(1 - \mu_3 m_1)\mu_3. \end{aligned} \tag{2.13}$$

It is well-known that (i) if $\Delta_Q > 0$, then (2.9) has three distinct real roots, (ii) if $\Delta_Q = 0$, then (2.9) has either one real triple root or one real simple root and one real double root, and (iii) if $\Delta_Q < 0$, then (2.9) has only one real root. Since $Q(0) > 0$ and $Q(u) \to -\infty$ as $u \to -\infty$, there is at least one negative root, *i.e.*, $\alpha_1 < 0$. Assume that $\Delta_Q \geq 0$. We note that the other two roots $\alpha_2 \leq \alpha_3$ are of the same sign because of $Q(0) > 0$. If $Q'(0) \leq 0$, then $Q'(u) = 0$ has a negative root and a positive root; hence $0 < \alpha_2 \leq \alpha_3$. If $Q'(0) > 0$ and $Q''(0) \geq 0$, then $Q'(u)$ does not vanish for $u > 0$, and hence $\alpha_2 \leq \alpha_3 < 0$. Finally, if $Q'(0) > 0$ and $Q''(0) < 0$, then $Q'(u) = 0$ has two positive roots, implying $0 < \alpha_2 \leq \alpha_3$. Notice also that $Q'(0) < 0$ if and only if $\mu_3 m_1 > 1$. We therefore conclude that (2.9) has positive roots if and only if either (a) $\Delta_Q \geq 0$ and $\mu_3 m_1 \geq 1$ or (b) $\Delta_Q \geq 0$, $\mu_3 m_1 < 1$ and $\mu_3 k + m_2 - m_1 < 0$. Case (b) occurs only when $m_1 > m_2$.

Let us regard the discriminant as a function of $k \in (-\infty, +\infty)$ and denote it by $\Delta_Q(k)$. Notice that $\Delta_Q(k) \to -\infty$ as $k \to +\infty$. If $\Delta_Q(0) = (m_2 - m_1)^2\{(1 - \mu_3 m_1)^2 - 4(m_2 - m_1)\mu_3\} > 0$ then there exists a positive number k^* such that $\Delta_Q(k) > 0$ for all $0 < k < k^*$ and $\Delta_Q(k^*) = 0$. It is easy to see that $\Delta_Q(0) > 0$ if and only if one of the following (1) and (2) is satisfied: (1) $m_1 > m_2$ or (2) $m_2 > m_1$ and $(1 + \mu_3 m_1)^2 > 4\mu_3 m_2$. Therefore, $Q(u) = 0$ has two positive roots $\alpha_2 < \alpha_3$ if one of the following conditions 1°)–3°) is satisfied:

1°) $m_1 < m_2$, $\mu_3 m_1 \geq 1$, $(1 + \mu_3 m_1)^2 > 4\mu_3 m_1$ and $0 < k < k^*$,

2°) $m_1 > m_2$, $\mu_3 m_1 \geq 1$ and $0 < k < k^*$,

3°) $m_1 > m_2$ and $0 < k < \min\{k^*, (m_1 - m_2)/\mu_3\}$.

When $k > 0$ is close to 0, we can construct two roots near β_1, β_2 or β_0 by the standard perturbation arguments. Moreover, by putting $u = \phi(k)/k$ with $\phi(0) \neq 0$ in the case $m_2 \neq m_1$ and $u = \psi(k)/\sqrt{k}$ with $\psi(0) \neq 0$ in the case

$m_2 = m_1$, we find easily the leading terms of $\phi(k)$ and $\psi(k)$ as $k \downarrow 0$. We omit the detail.

Proposition 2.3. *Let α_1, α_2, α_3 be three roots of equation* (2.9). *Put*

$$\kappa_1 = \frac{\mu_3 m_1 - 2 - \sqrt{(\mu_3 m_1 - 2)^2 + 8\mu_3(m_1 - m_2)}}{4\mu_3},$$

$$\kappa_2 = \frac{\mu_3 m_1 - 2 + \sqrt{(\mu_3 m_1 - 2)^2 + 8\mu_3(m_1 - m_2)}}{4\mu_3}.$$

(i) *If $m_2 < m_1$, then $\kappa_1 < 0 < \kappa_2$ and* (2.9) *has three real roots such that $\alpha_1 < 0 < \alpha_2 < 1/\sqrt{k} < \alpha_3$ for $0 < k < \kappa_2^2$.*

(ii) *If $m_2 = m_1$ and $\mu_3 m_1 > 2$, then* (2.9) *has three real roots such that $\alpha_1 < 0 < \alpha_2 < 1/\sqrt{k} < \alpha_3$ for $0 < k < (\mu_3 m_1 - 2)/(2\mu_3)$.*

(iii) *Let $m_2 > m_1$ and $\mu_3 > \mu^*$, where*

$$\mu^* = \frac{2}{m^2}(2m_2 - m_1 + 2\sqrt{(m_2 - m_1)m_1}).$$

Then $0 < \kappa_1 < \kappa_2$, and (2.9) *has three real roots such that $\alpha_1 < 0 < \alpha_2 < 1/\sqrt{k} < \alpha_3$ for $\kappa_1 < \sqrt{k} < \kappa_2$.*

Proof. The assertions are verified by finding conditions for $Q(1/\sqrt{k})$ to be negative. Indeed, since

$$Q(1/\sqrt{k}) = \frac{1}{k}(m_2 - m_1 + (2 - \mu_3 m_1)\sqrt{k} + 2\mu_3 k),$$

we have only to examine the sign of the quadratic function $2\mu_3 x^2 - (\mu_3 m_1 - 2)x + m_2 - m_1$. Since the reasoning is elementary we omit the detail.

In what follows we assume one of the conditions stated in (B), (C) and (D) of Proposition 2.2 and in (i), (ii) and (iii) of Proposition 2.3. Then we have two positive roots $0 < \alpha_2 < \alpha_3$ of equation (2.9) and define $\underline{u} = \alpha_2$ and $\bar{u} = \alpha_3$. Hence, putting

$$\underline{v} = \frac{m_2 \underline{u}^2}{(\mu_3 + \underline{u})(1 + k\underline{u}^2)}, \qquad \bar{v} = \frac{m_2 \bar{u}^2}{(\mu_3 + \bar{u})(1 + k\bar{u}^2)}, \tag{2.14}$$

we conclude that system (2.1)–(2.3) has a spatially homogeneous steady state $(0, 0)$ and exactly two positive homogeneous steady states $(\underline{u}, \underline{v})$, $(\bar{u}, \bar{v})$ when one of the conditions (B), (C), (D) of Proposition 2.2 and (i), (ii), (iii) of Proposition 2.3 is satisfied.

The existence of spatially homogeneous states is illustrated in Figure 1.

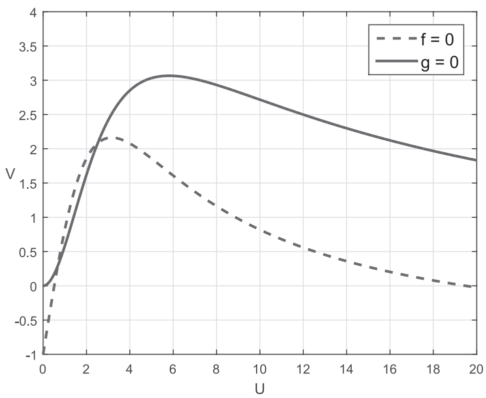

Fig. 1. Plot of the nullclines of $f(u,v)=0$ and $g(u,v)=0$.

3. Bifurcation analysis

In this section we formulate the stationary problem for (2.1)–(2.3)

$$\begin{cases} f(u,v)=0, \\ D\dfrac{d^2v}{dx^2}+g(u,v)=0, \\ \dfrac{dv}{dx}(0)=\dfrac{dv}{dx}(l)=0 \end{cases} \tag{3.1}$$

in an abstract setting and apply the classical theorem on bifurcation from simple eigenvalues (see Proposition 3.5 below). Then we proceed to studying the spectral properties of the linearized operator around the bifurcating solutions by applying the theorem on perturbation of K-simple eigenvalues (see Definition 3.7 and Lemma 3.8, and references [2], [9], [13]).

3.1. Spectrum of linearized operator around constant solutions. Let $C^0([0,l])$ denote the Banach space of all continuous functions on the interval $[0,l]$ equipped with the maximum norm: $\|u\|_\infty=\max_{0\le x\le l}|u(x)|$. Let $C_N^2([0,l])$ denote the space of all twice continuously differentiable functions $v(x)$ on $[0,l]$ satisfying homogeneous Neumann boundary conditions: $v'(0)=v'(l)=0$. Let

$$X=C^0([0,l])\times C_N^2([0,l]) \qquad \text{and} \qquad Y=C^0([0,l])\times C^0([0,l]).$$

These are Banach spaces with respective norms

$$\|(u,v)\|_X=\|u\|_\infty+\|v\|_\infty+\|v'\|_\infty+\|v''\|_\infty \qquad \text{and} \qquad \|(u,v)\|_Y=\|u\|_\infty+\|v\|_\infty.$$

We sometimes denote a point $(u, v) \in X$ by U. Let $\mathscr{U}$ be an open set of X defined by

$$\mathscr{U} = \{U = (u, v) \in X \mid u(x) + 1 > 0 \text{ for all } 0 \le x \le l\}.$$

We define a mapping $F(U, D)$ from $\mathscr{U} \times (0, +\infty)$ into Y by

$$F(U, D) = (f(u, v), Dv'' + g(u, v)) \qquad \text{for } U = (u, v). \tag{3.2}$$

Since $f(u, v)$ and $g(u, v)$ are real analytic, we can prove that F is an analytic mapping from $\mathscr{U} \times (0, +\infty)$ into Y and that the Fréchet (partial) derivative with respect to $U = (u, v)$ of F at (U_*, D_*), $U_* = (u_*, v_*)$, is given by

$$\partial_U F(U_*, D_*) = \begin{pmatrix} f_u^\star & f_v^\star \\ g_u^\star & D_* d^2/dx^2 + g_v^\star \end{pmatrix}, \tag{3.3}$$

where

$$f_u^\star = f_u(u_*, v_*), \qquad f_v^\star = f_v(u_*, v_*), \qquad g_u^\star = g_u(u_*, v_*), \qquad g_v^\star = g_v(u_*, v_*).$$

Recall that

$$\begin{cases} f_u(u, v) = -1 - v + \dfrac{2m_1 u}{(1 + ku^2)^2}, \quad f_v(u, v) = -u, \\ g_u(u, v) = -v + \dfrac{2m_2 u}{(1 + ku^2)^2}, \quad g_v(u, v) = -\mu_3 - u. \end{cases} \tag{3.4}$$

The Jacobi matrix J at U_* of the kinetic system is given by

$$J = \begin{pmatrix} f_u^\star & f_v^\star \\ g_u^\star & g_v^\star \end{pmatrix} \tag{3.5}$$

and plays an important role in what follows.

For $j = 0, 1, 2, \ldots,$ let

$$\ell_j = \left(\frac{\pi j}{l}\right)^2. \tag{3.6}$$

Then ℓ_j is an eigenvalue of $-d^2/dx^2$ under homogeneous Neumann boundary conditions, and $\cos(\pi jx/l)$ is an eigenfunction belonging to ℓ_j. Moreover, $\{\cos(\pi jx/l)\}_{j=0}^{\infty}$ form a basis of $L^2(0, l)$.

Lemma 3.1. *Suppose that $U_* = (u_*, v_*)$ is a constant steady-state of* (2.1)–(2.3). *Let $\mathscr{L}_*$ denote the linearized operator $\partial_U F(U_*, D_*) : X \to Y$. Then the spectrum of $\mathscr{L}_*$ consists of the eigenvalues $\{\lambda_j\}_{j=0}^{\infty} \cup \{\mu_j\}_{j=0}^{\infty}$, with* $\mathrm{Re}\,\lambda_j \le \mathrm{Re}\,\mu_j$, *of finite multiplicity and the point $\lambda = f_u(u_*, v_*)$ which is an eigenvalue of infinite* *multiplicity if* $f_v(u_*, v_*)g_u(u_*, v_*) = 0$ *or is in the continuous spectrum if* $f_v(u_*, v_*)g_u(u_*, v_*) \neq 0$. *Furthermore,*

$$\lambda_j = -D_*\ell_j - (f_u^\star - \operatorname{tr} J) + O(1/\ell_j),$$
$$\mu_j = f_u^\star + O(1/\ell_j)$$

as $j \to \infty$.

For the definition of point spectrum and continuous spectrum, see [3], VII.5.1, p. 580.

Proof. For $\lambda \in \mathbb{C}$ and $(\sigma, \tau) \in Y$ we consider the nonhomogeneous problem

$$f_u^\star\phi + f_v^\star\psi = \lambda\phi + \sigma, \tag{3.7}$$
$$D_*\psi'' + g_u^\star\phi + g_v^\star\psi = \lambda\psi + \tau, \tag{3.8}$$
$$\psi'(0) = \psi'(l) = 0. \tag{3.9}$$

If $\lambda \neq f_u^\star$, then from (3.7) we have

$$\phi = \frac{\sigma - f_v^\star\psi}{f_u^\star - \lambda}, \tag{3.10}$$

and hence (3.8) reduces to

$$D_*\psi'' + \frac{(f_u^\star - \lambda)(g_v^\star - \lambda) - f_v^\star g_u^\star}{f_u^\star - \lambda}\psi = \tau - \frac{g_u^\star\sigma}{f_u^\star - \lambda}. \tag{3.11}$$

This nonhomogeneous Sturm-Liouville problem has a unique solution if and only if

$$\frac{(f_u^\star - \lambda)(g_v^\star - \lambda) - f_v^\star g_u^\star}{(f_u^\star - \lambda)D_*} \notin \{\ell_j\}_{j=0}^\infty. \tag{3.12}$$

Moreover, if (3.12) holds, then the unique solution ψ of (3.11) satisfies the estimate $\|\psi\|_\infty + \|\psi'\|_\infty + \|\psi''\|_\infty \leq C(\|\sigma\|_\infty + \|\tau\|_\infty)$ for some constant $C > 0$ independent of (σ, τ). Then, (3.10) determines ϕ uniquely and $\|\phi\|_\infty \leq C_1(\|\sigma\|_\infty + \|\tau\|_\infty)$ for an appropriate positive constant C_1. Hence, $\mathscr{L}_* - \lambda$ has a bounded inverse $(\mathscr{L}_* - \lambda)^{-1}$ if (3.12) is satisfied.

Assume now that (3.12) is violated, i.e., λ satisfies the characteristic equation

$$\lambda^2 - (\operatorname{tr} J - D_*\ell_j)\lambda + \det J - D_*\ell_j f_u^\star = 0 \tag{3.13}$$

for some $j \geq 0$. This equation has two roots λ_j and μ_j:

$$\begin{cases} \lambda_j = \dfrac{1}{2}\{\operatorname{tr} J - D_*\ell_j - \sqrt{(\operatorname{tr} J - D_*\ell_j)^2 - 4(\det J - D_*\ell_j f_u^\star)}\}, \\ \mu_j = \dfrac{1}{2}\{\operatorname{tr} J - D_*\ell_j + \sqrt{(\operatorname{tr} J - D_*\ell_j)^2 - 4(\det J - D_*\ell_j f_u^\star)}\}. \end{cases} \tag{3.14}$$

It is straightforward to check that λ_j and μ_j are indeed eigenvalues of $\mathscr{L}_*$ and

$$\begin{aligned} (\phi_{j,-}, \psi_{j,-}) &= \left(-\frac{f_v^\star}{f_u^\star - \lambda_j} \cos\frac{\pi j x}{l}, \cos\frac{\pi j x}{l}\right) \qquad \text{and} \\ (\phi_{j,+}, \psi_{j,+}) &= \left(-\frac{f_v^\star}{f_u^\star - \mu_j} \cos\frac{\pi j x}{l}, \cos\frac{\pi j x}{l}\right) \end{aligned} \tag{3.15}$$

are eigenvectors belonging to λ_j and μ_j, respectively.

We now prove that $\lambda = f_u^\star$ is in the spectrum of $\mathscr{L}_*$. We distinguish the two cases (i) $f_v^\star g_u^\star = 0$ and (ii) $f_v^\star g_u^\star \neq 0$.

First we treat the case (i). Suppose $f_v^\star = 0$. Then, (3.7) with $\sigma = 0$ is satisfied for any ψ. If in addition, $g_u^\star \neq 0$, then (3.8) with $\tau = 0$ determines ϕ uniquely for each $\psi \in C_N^2([0,l])$. If $g_u^\star = 0$, then we find that (3.8) with $\tau = 0$ is satisfied for any $\phi \in C^0([0,l])$ and $\psi = 0$. Therefore, (3.7)–(3.9) with $\sigma = \tau = 0$ has a nontrivial solution of the form $(\phi, 0)$, so that $\lambda = f_u^\star$ is an eigenvalue of $\mathscr{L}_*$ of infinite multiplicity. Let us turn to the case $f_v^\star \neq 0$. Then $g_u^\star = 0$, and hence (3.7) with $\sigma = 0$ implies $\psi = 0$. Thus, (3.8) with $\tau = 0$ reduces to $g_u^\star \phi = 0$. Since $g_u^\star = 0$, ϕ is arbitrary. Therefore, $\ker(\mathscr{L}_* - f_u^\star) \supset \{(\phi, 0) \mid \phi \in C^0([0,l])\}$, showing $\dim \ker(\mathscr{L}_* - f_u^\star) = \infty$.

Second, we turn to handle the case (ii). It is easy to check that $\mathscr{L}_* - f_u^\star$ is injective. Let $\mathfrak{R}$ denote the range of $\mathscr{L}_* - f_u^\star$. We prove that $\mathfrak{R} = C_N^2([0,l]) \times C^0([0,l])$. To this end, we consider the nonhomogeneous problem (3.7)–(3.9) with $\lambda = f_u^\star$. From (3.7) it follows that $\psi = \sigma/f_v^\star \in C^0([0,l])$. But, for (3.8) to have a solution $\psi \in C_N^2([0,l])$, it is necessary that $\sigma \in C_N^2([0,l])$. Hence, $\mathfrak{R} \subset C_N^2([0,l]) \times C^0([0,l])$. Conversely, if $(\sigma, \tau) \in C_N^2([0,l]) \times C^0([0,l])$, then $\phi = [\tau + \{f_u^\star \sigma - (D_*\sigma'' + g_v^\star \sigma)\}/f_u^\star]/g_u^\star$ and $\psi = \sigma/f_v^\star$ satisfy (3.7)–(3.9) with $\lambda = f_u^\star$. We thus obtain $\mathfrak{R} = C_N^2([0,l]) \times C^0([0,l])$, which is a proper subset of $C^0([0,l]) \times C^0([0,l])$. Finally, we observe that $C_N^2([0,l])$ is dense in $C^0([0,l])$. This seems to be a standard fact, but for completeness, we sketch a proof: Approximate $\sigma \in C^0([0,l])$ by the solution u_ε of the bundary value problem $\varepsilon^2 u'' - u + \sigma = 0$ in $(0,l)$ and $u'(0) = u'(l) = 0$, where ε is a positive number. Since u_ε is expressed by using the Green's function, it is straghtforward to verify that $u_\varepsilon(x)$ converges to $\sigma(x)$ uniformly as $\varepsilon \downarrow 0$ (for details, see the proof of Lemma 2.9 in [14]). Therefore, $\mathfrak{R}$ is dense in

$C^0([0,l]) \times C^0([0,l])$, and we conclude that $\lambda = f_u^\star$ is in the continuous spectrum.

The asymptotic behavior of λ_j and μ_j as $j \to \infty$ is easily obtained from the formula (3.14).

To know the distribution of eigenvalues we need the following

Lemma 3.2. (a) *If* $U_* = (0,0)$, *then* $f_u^\star < 0$, $\operatorname{tr} J < 0$ *and* $\det J > 0$.

(b) *Assume that one of the conditions* (B), (C), (D) *of Proposition* 2.2 *or* (i), (ii), (iii) *of Proposition* 2.3 *is satisfied. Let* $U_* = (\underline{u}, \underline{v})$. *Then* $f_u^\star > 0$ *and* $\det J < 0$.

(c) *Suppose that one of the conditions* (B), (C), (D) *of Proposition* 2.2 *is satisfied. Let* $U_* = (\bar{u}, \bar{v})$. *Then* $f_u^\star > 0$ *and* $\det J > 0$. *Moreover, if* $m_1^2 < m_2$, *then* $\operatorname{tr} J < 0$, *provided that* $k > 0$ *is sufficiently small.*

(d) *Assume that one of the conditions* (i), (ii), (iii) *of Proposition* 2.3 *is satisfied. Let* $U_* = (\bar{u}, \bar{v})$. *Then* $f_u^\star < 0$, $\operatorname{tr} J < 0$ *and* $\det J > 0$.

Proof. For $U_* = (0,0)$ the assertion is obvious.

Letting $Q(u)$ be the left hand side of (2.9) as in Section 2, we first prove that

$$\det J = -\frac{f_v^\star}{1 + ku_*^2} Q'(u_*). \tag{3.16}$$

To show this we observe that $v(u) = m_1 u/(1 + ku^2) - 1$ satisfies $f(u, v(u)) = 0$, hence $f_u(u, v(u)) + f_v(u, v(u))v'(u) = 0$. Therefore, for $U_* = (u_*, v_*) \neq (0,0)$, we have

$$v'(u_*) = -\frac{f_u(u_*, v(u_*))}{f_v(u_*, v(u_*))} = -\frac{f_u^\star}{f_v^\star}. \tag{3.17}$$

On the other hand, it is straightforward to see that $g(u, v(u)) = Q(u)/(1 + ku^2)$. Hence,

$$g_u(u, v(u)) + g_v(u, v(u))v'(u) = Q'(u)/(1 + ku^2) - 2kuQ(u)/(1 + ku^2)^2.$$

By virtue of $Q(u_*) = 0$ and (3.17), we obtain

$$g_u^\star - g_v^\star f_u^\star / f_v^\star = Q'(u_*)/(1 + ku_*^2),$$

which results in (3.16).

From (3.4) and (2.6) it follows that

$$f_u^\star = -\frac{m_1 u_*}{1 + ku_*^2} + \frac{2m_1 u_*}{(1 + ku_*^2)^2} = \frac{m_1 u_*}{(1 + ku_*^2)^2}(1 - ku_*^2).$$

Hence, $f_u^\star > 0$ if $u_* < 1/\sqrt{k}$ and $f_u^\star < 0$ if $u_* > 1/\sqrt{k}$. Since $g_v^\star < 0$, we conclude that $\operatorname{tr} J < 0$ if $u_* > 1/\sqrt{k}$.

Notice that $0 < \underline{u} < 1/\sqrt{k}$ in all cases stated in Propositions 2.2 and 2.3. Hence, $f_u(\underline{u}, \underline{v}) > 0$. Moreover, $Q'(\underline{u}) < 0$ because it is the intermediate root of $Q(u) = 0$. Consequently $\det J < 0$ for $U_* = (\underline{u}, \underline{v})$ by (3.16). Therefore, (b) is proved.

Assertion (c) is the same as Lemma 3.9 of [4]. Hence we have proved all assertions of the lemma.

Lemma 3.3. *Let $U_* = (u_*, v_*)$ be a constant steady-state of* (2.1)–(2.3) *and let $\mathscr{L}_*$ denote the linearized operator $\partial_U F(U_*, D_*)$. For $j = 1, 2, 3, \ldots$, define*

$$D_j = \frac{\det J}{f_u^\star \ell_j} \qquad \text{if } f_u^\star \neq 0.$$

(I) *If $U_* = (0,0)$, then the spectrum of $\mathscr{L}_*$ is contained in the left half plane $\{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda < 0\}$ for all $D_* > 0$.*

(II) *If $U_* = (\underline{u}, \underline{v})$, then D_j is negative for all $j = 1, 2, 3, \ldots$. Moreover, $\lambda_j < 0 < \mu_j$ for all $j = 0, 1, 2, \ldots$.*

(III) *If $U_* = (\bar{u}, \bar{v})$ such that $\bar{u} > 1/\sqrt{k}$, then D_j is negative for all $j = 1, 2, 3, \ldots$. Moreover, the spectrum of $\mathscr{L}_*$ is contained in the left-half plane for all $D_* > 0$.*

(IV) *If $U_* = (\bar{u}, \bar{v})$ such that $0 < \bar{u} < 1/\sqrt{k}$, then D_j is positive for all $j \geq 1$. If $D_* > D_1$ then $\lambda_n < 0 < \mu_n$ for all $n \geq 0$. If $D_* = D_j$ for some $j \geq 1$, then $\operatorname{Re} \lambda_n \leq \operatorname{Re} \mu_n < 0$ for $1 \leq n \leq j-1$, $\lambda_j < 0 = \mu_j$, and $\lambda_n < 0 < \mu_n$ for $n \geq j+1$. If $D_{j+1} < D_* < D_j$, then $\operatorname{Re} \lambda_n \leq \operatorname{Re} \mu_n < 0$ for $1 \leq n \leq j$, while $\lambda_n < 0 < \mu_n$ for any $n \geq j+1$.*

Proof. First, we observe that $\operatorname{Re} \lambda_n \leq \operatorname{Re} \mu_n < 0$ if and only if

$$\operatorname{tr} J - D_* \ell_n < 0 \qquad \text{and} \qquad \det J - D_* f_u^\star \ell_n > 0.$$

Moreover, if $\det J - D_* f_u^\star \ell_n < 0$, then $\lambda_n < 0 < \mu_n$. Combining these observations with Lemma 3.2, we can prove all assertions easily and we omit the detail.

Definition 3.4. Let $U_* = (u_*, v_*)$ be a constant steady-state of (2.1)–(2.3). Define the *stability index* $\operatorname{Ind}_S(U_*, D_*)$ of (U_*, D_*) by

$$\operatorname{Ind}_S(U_*, D_*) = \#\{\mu_n \mid \operatorname{Re} \mu_n < 0\},$$

where $\#A$ stands for the number of distinct elements of a countable set A.

If $U_* = (\bar{u}, \bar{v})$ with $0 < \bar{u} < 1/\sqrt{k}$, then $\operatorname{Ind}_S(U_*, D_*) = j$ if $D_{j+1} < D_* < D_j$. This formula is valid for $j \geq 0$ if we understand $D_0 = +\infty$. We observe that U_* is linearly stable against disturbances of wave number $\leq \mathrm{Ind}_S(U_*, D_*)$, but disturbance of wave number $> \mathrm{Ind}_S(U_*, D_*)$ grows.

3.2. Bifurcation. If $U_* = (u_*, v_*)$ is a constant solution of (3.1), then

$$F(U_*, D) = 0 \qquad \text{for all } D > 0.$$

If $\partial_U F(U_*, D_*)$ is an isomorphism from X to Y, then by the implicit function theorem we see that $F^{-1}(0) \cap \mathscr{V}_0 = \{(U_*, D) \mid (U_*, D) \in \mathscr{V}_0\}$ where $\mathscr{V}_0$ is a neighborhood of (U_*, D_*) in $X \times \mathbb{R}$. Therefore, for (U_*, D_*) to be a bifurcation point it is necessary that $\mathscr{L}_* = \partial_U F(U_*, D_*)$ does not have a bounded inverse. The following proposition is a restatement of Theorems 1.7 and 1.18 of [1] in our notation.

Proposition 3.5 (Bifurcation from a Simple Eigenvalue). *Let X, Y, $\mathscr{V}$ and F be as above. For $D_* > 0$, let L_* denote the Fréchet derivative $\partial_U F(U_*, D_*)$ and $L_1 = \partial_U \partial_D F(U_*, D_*)$. Assume that the following conditions hold:*

(1) $\ker(L_*)$ *is one-dimensional, spanned by* Φ_0*;*
(2) $\mathrm{range}(L_*)$ *has co-dimension* 1*; i.e.* $\dim[Y/\mathrm{range}(L_*)] = 1$*;*
(3) $L_1 \Phi_0 \notin \mathrm{range}(L_*)$.

Let Z be any closed subspace of X such that $X = \mathrm{span}\{\Phi_0\} \oplus Z$ (i.e. any $U \in X$ can be uniquely written as $U = \alpha\Phi_0 + V$, $\alpha \in \mathbb{R}$, $V \in Z$). Then there is a $\delta > 0$, a neighborhood $\mathscr{V}$ of (U_*, D_*) in $X \times \mathbb{R}$ and a smooth curve $(\Psi, D) : (-\delta, \delta) \to Z \times \mathbb{R}$ such that $\Psi(0) = 0$, $D(0) = D_*$ and $F^{-1}(0) \cap \mathscr{V} = \{(U_* + s(\Phi_0 + \Psi(s)), D(s)) \mid |s| < \delta\} \cup \{(U_*, D) \mid (U_*, D) \in \mathscr{V}\}$.

Let us now condsider when $\mathscr{L}_*$ satisfies the assumptions of Proposition 3.5. By Lemma 3.3, we see that $\mathscr{L}_*$ is not invertible only when $U_* = (\bar{u}, \bar{v})$ with $0 < \bar{u} < 1/\sqrt{k}$ and $D_* = D_j$ for some $j \geq 1$. From Lemma 3.1 it follows that $\ker \partial_U F(U_*, D_j)$ is spanned by $\Phi_0 = (\phi_{j,+}, \psi_{j,+})$ (see (3.15)). To find range $\partial_U F(U_*, D_*)$, we consider the nonhomogeneous problem (3.7)–(3.9) with $D_* = D_j$ and $\lambda = \mu_j = 0$. Then (3.11) reduces to

$$f_u^\star D_j \psi'' + \det J\psi = f_u^\star \tau - g_u^\star \sigma. \tag{3.18}$$

Since $f_u^\star D_j \psi_{j,+}'' + \det J\psi_{j,+} = 0$, this equation has a solution if and only if

$$\int_0^l (f_u^\star \tau - g_u^\star \sigma)\psi_{j,+}\, dx = 0. \tag{3.19}$$

If (3.18) has a solution ψ, then (3.10) gives us ϕ and we obtain a solution of (3.7)–(3.9) with $D_* = D_j$, $\lambda = \mu_j = 0$. Therefore, $(\sigma, \tau) \in \mathrm{range}\, \partial_U F(U_*, D_j)$ if and only if (3.19) is satisfied.

Let us expand $\sigma = \sum_{n=0}^{\infty} \sigma_n \cos(\pi n x/l)$ and $\tau = \sum_{n=0}^{\infty} \tau_n \cos(\pi n x/l)$. Since $\psi_{j,+} = \cos(\pi j x/l)$, (3.19) is equivalent to the condition

$$f_u^{\star} \tau_j - g_u^{\star} \sigma_j = 0.$$

Hence, codim range $\partial_U F(U_*, D_j) = 1$.

Note that

$$\partial_U \partial_D F(U_*, D_j) = \begin{pmatrix} 0 & 0 \\ 0 & d^2/dx^2 \end{pmatrix}.$$

Hence, $\partial_U \partial_D F(U_*, D_j)\Phi_0 = (0, \psi_{j,+}'')$. Then

$$\int_0^l (f_u^{\star} \psi_{j,+}'' - g_u^{\star} \cdot 0)\psi_{j,+}\, dx = -f_u^{\star} \ell_j \int_0^l \cos^2 \frac{\pi j x}{l}\, dx \neq 0.$$

This means that condition (3) of Proposition 3.5 is satisfied.

Consequently, we have a one-parameter family of non-constant solutions of $F(U, D) = 0$:

Theorem 3.6. *Let $\overline{U} = (\bar{u}, \bar{v})$ be a constant steady state of* (2.1)–(2.3) *such that* $0 < \bar{u} < 1/\sqrt{k}$. *For each positive integer j, there exists a $\delta > 0$ such that* (3.1) *has a one-parameter family of nonconstant solutions $\{(U(s), D_j + D(s))\}_{|s|<\delta}$ of the form*

$$u(x, s) = \bar{u} + s(\phi_{j,+}(x) + \phi(x, s)),$$

$$v(x, s) = \bar{v} + s(\psi_{j,+}(x) + \psi(x, s))),$$

$$\phi(x, 0) \equiv 0, \qquad \psi(x, 0) \equiv 0, \qquad D(0) = 0.$$

Moreover, in a small neighborhood of $(\overline{U}, D_j)$ in $X \times \mathbb{R}$, there is no solutions other than $\{(U(s), D_j + D(s))\}_{|s|<\delta} \cup \{(\overline{U}, D)\}_{|D - D_j| < \delta_0}$, where $\delta_0 > 0$.

3.3. Behavior of critical eigenvalue. For simplicity, let $\mathscr{L}_D$ denote the linearized operator $\partial_U F(\overline{U}, D)$ and $\mathscr{L}(s)$ denote $\partial_U F(\overline{U} + s(\Phi_0 + \Psi(s)), D_j + D(s))$. Obviously $\mathscr{L}(0) = \mathscr{L}_{D_j}$ has 0 as an eigenvalue. Recall that $\mathscr{L}_D$ has $\mu_j(D)$ as an eigenvalue such that $\mu_j(D_j) = 0$, where we write $\mu_j(D)$ in order to emphasize the dependence on D. From Lemma 3.3 (IV) it follows that $\mu_j(D) < 0$ if $D < D_j$, $\mu_j(D_j) = 0$, and $\mu_j(D) > 0$ if $D > D_j$. This means that $\overline{U}$ becomes more unstable as D increases over D_j.

In this subsection we study the behavior of the eigenvalue $\mu(s)$ of $\mathscr{L}(s)$ such that $\mu(0) = 0$. To be rigorous, we recall the notion of a K-simple eigenvalue and its perturbation theory ([2]):

Definition 3.7. Let X, Y be Banach spaces. Let L, K be bounded linear operators from X into Y. We say that $\mu \in \mathbb{R}$ is a *K-simple eigenvalue of L* if

(i) $\dim \ker(L - \mu K) = \operatorname{codim} \operatorname{range}(L - \mu K) = 1$,

(ii) if $\ker(L - \mu K)$ is spanned by x_0, then $Kx_0 \notin \operatorname{range}(L - \mu K)$.

In this terminology, we have seen above that $\mathscr{L}_{D_j}$ has 0 as an $\mathscr{L}_1$-simple eigenvalue, which makes $(\overline{U}, D_j)$ a bifurcation point. Moreover, 0 is also an i-simple eigenvalue of $\mathscr{L}_{D_j}$, where i is the inclusion mapping $X \subset Y$. Indeed, by (3.15) we see that the left-hand side of (3.19) with $(\sigma, \tau) = \Phi_0 = (-f_v^*/f_u^*, 1)\psi_{j,+}$ is equal to

$$\left(f_u^\star + \frac{f_v^\star g_u^\star}{f_u^\star}\right) \int_0^l \cos^2 \frac{\pi j x}{l}\, dx.$$

Note that $(f_u^\star)^2 + f_v^\star g_u^\star = f_u^\star \operatorname{tr} J - \det J < 0$ since $f_u^\star > 0$, $\operatorname{tr} J < 0$ and $\det J > 0$. Therefore, (3.19) does not hold, yielding $\Phi_0 \notin \operatorname{range} \mathscr{L}_{D_j}$.

Therefore, by Corollay 1.13 of [2] we obtain an i-simple eigenvalue $\gamma(D)$ of $\mathscr{L}_D$ near $D = D_j$ and an i-simple eigenvalue $\mu(s)$ of $\mathscr{L}(s)$ near $s = 0$. Due to the uniqueness, we see that $\gamma(D) \equiv \mu_j(D)$. Now we apply Theorem 1.16 of [2] and obtain

$$\lim_{s \to 0, \mu(s) \neq 0} \frac{-sD'(s)\mu_j'(D_j)}{\mu(s)} = 1. \tag{3.20}$$

From (3.13) we have

$$2\mu_j(D)\mu_j'(D) + \ell_j \mu_j(D) - (\operatorname{tr} J - D\ell_j)\mu_j'(D) - \ell_j f_u^\star = 0.$$

Hence $\mu_j(D_j) = 0$ implies

$$\mu_j'(D_j) = -\ell_j f_u^\star / (\operatorname{tr} J - D_j \ell_j) > 0.$$

Therefore, we obtain the following

Lemma 3.8. *Let $\mu(s)$ be the i-simple eigenvalue of $\mathscr{L}(s)$. Then $\mu(s) < 0$ if $sD'(s) > 0$, while $\mu(s) > 0$ if $sD'(s) < 0$.*

To compute $D(s)$ we take a shorter way.

Define a function $u = p(v)$ by

$$p(v) = \frac{m_1 - \sqrt{m_1^2 - 4k(1+v)^2}}{2k(1+v)} \tag{3.21}$$

for $0 \le v \le v_M$, where

$$v_M = \frac{m_1}{2\sqrt{k}} - 1. \tag{3.22}$$

Put

$$h(v) = g(p(v), v) \qquad \text{for } v \in (0, v_M). \tag{3.23}$$

Since $u = p(v)$ solves (2.6) for each $v \in [0, v_M]$, we see that if v solves the boundary value problem

$$\begin{cases} Dv'' + h(v) = 0 & \text{for } 0 < x < l, \\ v'(0) = v'(l) = 0. \end{cases} \tag{3.24}$$

then $(p(v), v)$ is a solution of (3.1), as long as $0 \le v(x) < v_M$ is satisfied for $0 \le x \le l$. We also note that (u, v) is a solution of (3.1) in a small neighborhood of $(\bar{u}, \bar{v})$ if and only if v is a solution of (3.24) and $u = p(v)$.

We expand v and D around $\bar{v}$ and D_j as follows:

$$v(x) = \bar{v} + sv_1(x) + s^2 v_2(x) + s^3 v_3(x) + \cdots,$$

$$D(s) = D_j + sd_1 + s^2 d_2 + s^3 d_3 + \cdots.$$

We substitute these expressions in (3.24) and equate each coefficient of s^m to zero $(m = 0, 1, 2, \ldots)$, obtaining

$$\begin{cases} d_1 v_1'' + h'(\bar{v}) v_1 = 0, \\ D_j v_2'' + d_1 v_1'' + h'(\bar{v}) v_2 + \dfrac{1}{2} h'' v_1^2 = 0, \\ D_j v_3'' + d_1 v_2'' + d_2 v_1'' + h'(\bar{v}) v_3 + h''(\bar{v}) v_1 v_2 + \dfrac{1}{6} h'''(\bar{v}) v_1^3 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{3.25}$$

The first equation is satisfied if we take $v_1 = \psi_{j,+}(x) = \cos(\pi j x / l)$. Let $L_0 = D_j d^2/dx^2 + h'(\bar{v})$. Then the second equation reads

$$L_0 v_2 + d_1 v_1'' + \frac{1}{2} h''(\bar{v}) v_1^2 = 0. \tag{3.26}$$

This is solvable if and only if

$$\int_0^l \left(d_1 v_1'' + \frac{1}{2} h''(\bar{v}) v_1^2 \right) \psi_{j,+} \, dx = 0. \tag{3.27}$$

Since $v_1'' = -\ell_j \psi_{j,+}$, we have

$$-\ell_j d_1 \int_0^l \cos^2 \frac{\pi j x}{l} \, dx + \frac{1}{2} h''(\bar{v}) \int_0^l \cos^3 \frac{\pi j x}{l} \, dx = 0,$$

which yields

$$d_1 = 0. \tag{3.28}$$

Then (3.26) reduces to $L_0 v_2 + 2^{-1} h''(\bar{v}) v_1^2 = 0$. In view of

$$v_1^2 = \cos^2(\pi j x/l) = \frac{1}{2}\left(1 + \cos\frac{2\pi j x}{l}\right),$$

we find that $v_2(x)$ is of the following form:

$$v_2(x) = \xi_2 + \eta_2 \cos\frac{2\pi j x}{l}, \qquad (\xi_2, \eta_2 \in \mathbb{R}).$$

Substituting this in the equation, we obtain

$$-\ell_{2j} D_j \eta_2 \cos\frac{2\pi j x}{l} + h'(\bar{v})\left(\xi_2 + \eta_2 \cos\frac{2\pi j x}{l}\right) + \frac{h''(\bar{v})}{4}\left(1 + \cos\frac{2\pi j x}{l}\right) = 0.$$

Hence,

$$\xi_2 = -\frac{h''(\bar{v})}{4h'(\bar{v})}, \qquad \eta_2 = -\frac{h''(\bar{v})}{4\{-D_j \ell_{2j} + h'(\bar{v})\}}.$$

Here, we observe that (i) differentiation of $f(p(v), v) = 0$ with respect to v yields $p'(v) = -f_v(p(v), v)/f_u(p(v), v)$ and (ii) $h'(v) = g_u(p(v), v) p'(v) + g_v(p(v), v)$. Therefore,

$$h'(v) = \frac{f_u(p(v), v) g_v(p(v), v) - f_v(p(v), v) g_u(p(v), v)}{f_u(p(v), v)}.$$

Hence,

$$h'(\bar{v}) = \frac{\det J}{f_u^\star}. \tag{3.29}$$

From this and $D_j \ell_j = \det J / f_u^\star$ we see that $h'(\bar{v}) - D_j \ell_{2j} = -3 \det J / f_u^\star = -3h'(\bar{v}) < 0$ and η_2 is well-defined. We now have a formula for v_2:

$$v_2(x) = -\frac{h''(\bar{v})}{4h'(\bar{v})} + \frac{h''(\bar{v})}{12h'(\bar{v})} \cos\frac{2\pi j x}{l}. \tag{3.30}$$

Now the third equation of (3.25) becomes

$$L_0 v_3 + d_2 v_1'' + h''(\bar{v}) v_1 v_2 + \frac{1}{6} h'''(\bar{v}) v_1^3 = 0.$$

The solvability condition for this equation is as follows:

$$d_2 \int_0^l v_1'' v_1 \, dx + h''(\bar{v}) \int_0^l v_1^2 v_2 \, dx + \frac{h'''(\bar{v})}{6} \int_0^l v_1^4 \, dx = 0. \tag{3.31}$$

We observe

$$\int_0^l v_1'' v_1 \, dx = -\frac{l}{2}\ell_j, \qquad \int_0^l v_1^2 v_2 \, dx = \left(\frac{\xi_2}{2} + \frac{\eta_2}{4}\right) l, \qquad \int_0^l v_1^4 \, dx = \frac{3}{8} l.$$

Also, from (3.30) it follows that

$$2\xi_2 + \eta_2 = -\frac{5h''(\bar{v})}{12h'(\bar{v})}.$$

Therefore, substituting these together in (3.31), we obtain

$$d_2 = \frac{1}{24\ell_j}(3h'''(\bar{v}) - 5h''(\bar{v})^2). \tag{3.32}$$

We conclude this section with the following

Proposition 3.9. *Let the assumptions of Proposition* 2.2 *be satisfied and let the steady state* $\bar{U} = (\bar{u}, \bar{v})$ *of* (2.1)–(2.3) *satisfy* $0 < \bar{u} < 1/\sqrt{k}$. *Let* $\{(U(s), D(s))\}_{|s|<\delta}$ *be the branch of nonconstant solutions bifurcating from* $(\bar{U}, D_j)$. *Denote* $\partial_U F(U(s), D(s))$ *by* $\mathscr{L}(s)$ *and let* $\mu(s)$ *be the i-simple eigenvalue of* $\mathscr{L}(s)$ *such that* $\mu(0) = 0$. *Then,* $\mu(s) > 0$ *if* $3h'''(\bar{v}) < 5h''(\bar{v})^2$, *while* $\mu(s) < 0$ *if* $3h'''(\bar{v}) > 5h''(\bar{v})^2$.

Proof. Since $sD'(s) = s(d_1 + 2sd_2 + O(s^2)) = 2s^2 d_2 + O(s^3)$, we have $sD'(s) > 0$ if $d_2 > 0$, whereas $sD'(s) < 0$ if $d_2 < 0$, providede that $|s|$ is sufficiently small. Combining this observation with Lemma 3.8 and (3.32), we obtain the assertion of the proposition.

4. Boundary value problem for a single equation

In this section we describe the classical method to construct all solutions of the boundary value problem for a single equation obtained from the system of two stationary algebra-differential equations. Recall that $h(v)$ is defined as a Hölder continuous function on the interval $0 \le v \le v_M$ by (3.23), and it is smooth on the interval $0 \le v < v_M$. By a solution of (3.24) we mean a pair $(v, D) \in C_N^2([0, l]) \times (0, \infty)$ for which (3.24) is satisfied and $0 \le v(x) < v_M$ for $x \in [0, l]$.

To begin with, we remark that any nonconstant solution of (3.24) is obtained once we have all of (strictly) monotone increasing solutions. Indeed, first assume that $(v_1(x), D)$ is a monotone decreasing solution. Then $(v_1(l - x), D)$ is a monotone increasing solution. Therefore, there exists a monotone increasing solution $(v_0(x), D)$ such that $v_0(x) = v_1(l - x)$. Hence $v_1(x) = v_0(l - x)$. Next, we assume that $(v_2(x), D)$ is a solution of (3.24) for which there is an $x_1 \in (0, l)$ such taht $v_2'(x_1) = 0$. Then there exist x_0 and x_1

such that $0 \le x_0 < x_1 < x_2 \le l$, $v_2'(x_0) = v_2'(x_2) = 0$, $x_2'(x) \ne 0$ for $x \in (x_0, x_1) \cup (x_1, x_2)$. Suppose that $v_2''(x_1) = 0$. Then $h(v_2(x_1)) = 0$, which implies that $v_2(x_1) = \underline{v}$ or $v_2(x_1) = \bar{v}$. Since $v_2'(x_1) = 0$, the uniqueness of solution of the initial value problem for $v'' = -h(v)/D$ yields that $v_2(x) \equiv \underline{v}$ or $v_2(x) \equiv \bar{v}$, which is not possible. Hence, $v_2''(x_1) \ne 0$. We consider the case $v_2''(x_1) > 0$. Then $v_2'(x) < 0$ for $x \in (x_0, x_1)$ and $v_2'(x) > 0$ for $x \in (x_1, x_2)$. Let us put $w(t) = v_2(x_1 + t)$ for $x_0 - x_1 \le t \le x_2 - x_1$. Then $w''(t) = v_2''(x_1 + t) = -h(v_2(x_1 + t))/D = -h(w(t))/D$, and $w(0) = v_2(x_1)$, $w'(0) = v_2'(x_1) = 0$. Notice also that $z(t) = w(-t)$ satisfies $z''(t) = -h(z(t))/D$, $z(0) = v_2(x_1)$ and $z'(0) = 0$. The uniqueness of solution of the initial value problem implies that $w(t) \equiv z(t)$ on $[0, \min\{x_2 - x_1, x_1 - x_0\}]$. We know that $w'(x_2 - x_1) = z'(x_1 - x_0) = 0$, $w'(t) > 0$ for $0 < t < x_2 - x_1$ and $z'(t) > 0$ for $0 < t < x_1 - x_0$. Hence, we must have $x_2 - x_1 = x_1 - x_0$. Therefore, we find that $v_2(x_1 - t) = v_2(x_1 + t)$ for $t \in [0, x_1 - x_0]$. If $x_2 < l$, then there is an x_3 such that $x_2 < x_3 \le l$, $v_2'(x) < 0$ for $x \in (x_2, x_3)$ and $v_2'(x_3) = 0$. By the same reasoning as above we find that $v_2(x_2 - t) = v_2(x_2 + t)$ for $t \in [0, x_3 - x_2]$ and $x_2 - x_1 = x_3 - x_2$. Repeating this process, we conclude that $l = n(x_2 - x_1)$ for some integer $n \ge 2$, and $v_2(x)$ is a periodic function. Therefore, $v_2(x)$ is monotone on $[0, l/n]$. We may assume that $v_2'(x) > 0$ on $(0, l/n)$. We define $V(x) = v_2(nx/l)$ for $0 \le x \le l/n$. Then $(V(x), Dl^2/n^2)$ is a monotone increasing solution of (3.24). In this way, we obtain any solution of (3.24) by making use of a monotone increasing solution for some $D > 0$.

4.1. Monotone increasing solutions. Multiplying both sides of

$$Dv'' + h(v) = 0 \tag{4.1}$$

by v', we obtain

$$Dv'v'' + h(v)v' = 0.$$

Let us define

$$H(v) = \int_{\bar{v}}^{v} h(s)ds \qquad \text{for } 0 \le v \le v_M. \tag{4.2}$$

Then

$$\left(\frac{D}{2}(v')^2 + H(v)\right)' = 0,$$

so that, by virtue of $v'(0) = 0$,

$$\frac{D}{2}v'(x)^2 + H(v(x)) = H(a), \qquad \text{where } a = v(0).$$

Therefore

$$v'(x)^2 = \frac{2}{D}(H(a) - H(v(x))). \tag{4.3}$$

This is meaningful only if $H(a) \geq H(v(x))$. If $v(x)$ is monotone increasing, then $v(x) \geq a$ for $x \geq 0$. Hence we require that $H(a) \geq H(v)$ for $v \geq a$. Recall that

$$h(v)\begin{cases} > 0 & \text{if } 0 \leq v < \underline{v}, \\ = 0 & \text{if } v = \underline{v}, \\ < 0 & \text{if } \underline{v} < v < \bar{v}, \\ = 0 & \text{if } v = \bar{v}, \\ > 0 & \text{if } \bar{v} < v \leq v_M. \end{cases}$$

Therefore, $H(v)$ achieves a local maximum at $v = \underline{v}$, is monotone decreasing in the interval $\underline{v} < v < \bar{v}$, is monotone increasing in the interval $\bar{v} < v \leq v_M$. Consequently the only possible choice for a is $a \in [\underline{v}, \bar{v}]$.

Since we are looking for monotone increasing solutions,

$$v'(x) = \frac{1}{\sqrt{D}}\sqrt{2(H(a) - H(v(x)))}. \tag{4.4}$$

By the boundary condition at $x = l$, it holds $H(a) = H(v(l))$. Let $b = v(l)$. Then it is required that there exists a $b \geq a$ such that $H(b) = H(a)$, which may restrict the range of a. We shall discuss this later.

Now equation (4.4) can be integrated as

$$\int_a^v \frac{dw}{\sqrt{2(H(a) - H(w))}} = \frac{x}{\sqrt{D}}. \tag{4.5}$$

In particular, putting $x = l$ results in

$$\sqrt{D} = l \Bigg/ \int_a^{b(a)} \frac{dw}{\sqrt{2(H(a) - H(w))}}, \tag{4.6}$$

which defines a function $D = D(a)$. Equation (4.5) defines $v = v(x; a)$ as the inverse function and it is a monotone increasing solution of the boundary value problem (3.24) for $D = D(a)$.

Let us consider the domain of the functions $D(a)$ and $v(x; a)$.

(A) $H(\underline{v}) \leq H(v_M)$. In this case, for each $a \in [\underline{v}, \bar{v}]$ there exists a unique $b = b(a)$ such that $H(b) = H(a)$. Hence, $D(a)$ and $v(x, a)$ are defined for $a \in [\underline{v}, \bar{v}]$.

We may state this situation in terms of b as follows. There is a unique $v^\star \in (\bar{v}, v_M]$ such that $H(v^\star) = H(\underline{v})$ and, for each $b \in [\bar{v}, v^\star]$, the equation $H(a) = H(b)$ has a unique solution $a = a(b) \in [\underline{v}, \bar{v}]$. Hence, $D(b)$ and $v(x, b)$ are defined for $b \in [\bar{v}, v^\star]$. We show the case more clearly in Figure 2 and Figure 3, where $k_c = 0.021$.

(B) $H(\underline{v}) > H(v_M)$. In this case there is a unique $v_m \in (\underline{v}, \bar{v})$ such that $H(v_m) = H(v_M)$ and the equation $H(b) = H(a)$ has a unique solution $b = b(a) \in [\bar{v}, v_M]$ if $a \in [v_m, \bar{v}]$. Hence, $D(a)$ and $v(x, a)$ are defined for $a \in (v_m, \bar{v}]$. This case is described by Figure 4.

We explore the cases using numerical calculations for parameter values $m_1 = 2.00$, $m_2 = 5.05$ and $\mu_3 = 6.90$. It holds

(1) if $k = 0.02$, then $\underline{u} = 0.64$, $\underline{v} = 0.27$, $\bar{u} = 3.29$, $\bar{v} = 4.40$, $H(\underline{v}) = 14.92$ and $H(v_M) = 15.74$. Therefore $H(\underline{v}) < H(v_M)$, and we obtain case (A);

(2) if $k = 0.03$, then $\underline{u} = 0.65$, $\underline{v} = 0.28$, $\bar{u} = 3.17$, $\bar{v} = 3.87$, $H(\underline{v}) = 11.91$ and $H(v_M) = 4.51$. Thus $H(\underline{v}) > H(v_M)$, and we obtain case (B).

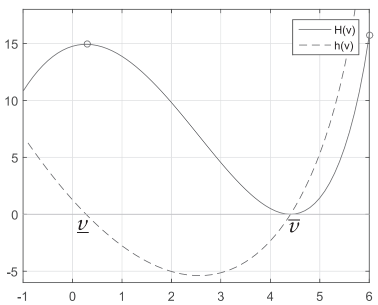

Fig. 2. Case (A) with $H(\underline{v}) < H(v_M)$, $k < k_c$.

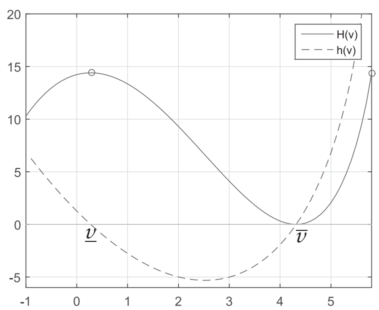

Fig. 3. Case (A) with $H(\underline{v}) = H(v_M)$, $k = k_c$.

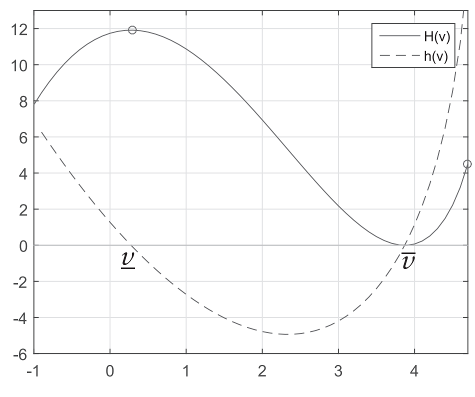

Fig. 4. Case (B) with $H(\underline{v}) > H(v_M)$, $k > k_c$.

Hence, both Case (A) and Case (B) can be observed under suitable parameter values.

For simplicity we put

$$a_m = \begin{cases} \underline{v} & \text{if } H(\underline{v}) \le H(v_M), \\ v_m & \text{if } H(\underline{v}) > H(v_M). \end{cases}$$

We define

$$\mathscr{C}_{1,+} = \{(v(x,a), D(a)) \mid a_m < a \le \bar{v}\},$$

$$\mathscr{C}_{1,-} = \{(v(l-x,a), D(a)) \mid a_m < a \le \bar{v}\},$$

$$\mathscr{C}_1 = \mathscr{C}_{1,+} \cup \mathscr{C}_{1,-}.$$

These are the branches of monotone increasing solutions, monotone decreasing solutions and monotone solutions, respectively.

4.2. Boundary layer in monotone solutions. In this subsection we consider the asymptotic behavior of monotone increasing solutions $(v(x,a), D(a))$ as $a \to \bar{v}$ and as $a \to a_m$, where a_m is a critical value defined above.

Theorem 4.1. *As $a \uparrow \bar{v}$,* (i) *$v(x,a) \to \bar{v}$ uniformly on $[0,l]$ and* (ii) *$D(a) \to D_1 = h'(\bar{v})/(\pi/l)^2$.*

Theorem 4.2. *It holds that*

1) *if $H(\underline{v}) \le H(v_M)$, then $v(x,a)$ develops a boundary layer at $x = l$, as $a \downarrow \underline{v}$, namely,*
   1a) *$v(x,a) \to \underline{v}$ locally uniformly in $[0,l)$, whereas $v(l,a) \to b(\underline{v})$;*
   1b) *$D(a) \to 0$;*
2) *if $H(\underline{v}) > H(v_M)$, then there exists a unique $v_m \in (\underline{v}, \bar{v})$ such that $H(v_m) = H(v_M)$ and as $a \downarrow v_m$,*
   2a) *$v(x,a) \to v(x,v_m)$ uniformly on $[0,l]$;*
   2b) *$D(a) \to D_c$, where D_c is a positive number;*

Remark. In Case 2), the limit $v(x, v_M)$ is twice continuously differentiable on $[0,l]$ and satisfies $v(l, v_M) = v_M$ and $v'(l, v_M) = 0$; hence it is a solution of (3.24). However, the linearized operator $L = Dd^2/dx^2 + h'(v_m(x))$ is *singular* in the sense that $L\phi \notin C^0([0,l])$ if $\phi \in C^2([0,l])$ satisfies, e.g., $\phi(l) \ne 0$.

In order to prove these theorems we make use of the following

Lemma 4.3. *Assume that the function $g(u)$ is continuously differentiable in the closed interval $[0, U_0]$ and the derivative $g'(u)$ is Hölder continuous with exponent γ there. Put*

$$G(u) = \int_0^u g(t)dt.$$

Suppose that there exist two constants u_m, u_M such that $0 < u_m < u_M < U_0$ and the following (i)–(iv) *are satisfied:* (i) $g(0) = g(u_m) = 0$; (ii) $g'(0) < 0$, $g'(u_m) > 0$; (iii) $g(u) < 0$ *for* $0 < u < u_m$ *whereas* $g(u) > 0$ *if* $u_m < u \le U_0$; *and* (iv) $G(u_M) = 0$. *For* $0 < \alpha < u_m$, $u_m < \beta < u_M$ *we define*

$$I_0(\alpha) = \int_\alpha^{u_m} \frac{dv}{\sqrt{2(G(\alpha) - G(v))}}, \qquad I_1(\beta) = \int_{u_m}^{\beta} \frac{dv}{\sqrt{2(G(\beta) - G(v))}}.$$

Then, (a) $I_0(\alpha)$, $I_1(\beta)$ *are continuously differentiable in* $0 < \alpha < u_m$, $u_m < \beta < u_M$, *respectively;* (b) *as* $\alpha \uparrow u_m$, $I_0(\alpha) \to \pi/(2\sqrt{g'(u_m)})$, *and as* $\beta \downarrow u_m$, $I_1(\beta) \to \pi/(2\sqrt{g'(u_m)})$; (c) *for any* $\delta \in (0, u_m)$

$$\int_\alpha^{\delta} \frac{dv}{\sqrt{2(G(\alpha) - G(v))}} = \frac{1}{\sqrt{|g'(0)|}} \log \frac{1}{\alpha} + O(1),$$

$$\int_\delta^{u_m} \frac{dv}{\sqrt{2(G(\alpha) - G(v))}} = O(1) \qquad \text{as } \alpha \downarrow 0;$$

(d) *Assume furthermore* $g'(u_M) > 0$. *Then* $I_1(\beta)$ *remains bounded as* $\beta \uparrow u_M$.

Proof. The proof is elementary, see for instance [8].

Proof (of Theorem 4.1). (i) By the properties of $H(v)$, we know that

$$\min_{\underline{v} \le v \le v_M} H(v) = H(\bar{v}).$$

Recall that the monotone increasing solution satisfies

$$\frac{dv}{dx}(x) = \sqrt{\frac{2(H(a) - H(v(x)))}{D}}.$$

This solution is well-defined as long as $E(v) = H(a) - H(v)$ is nonnegative.

As $a \uparrow \bar{v}$, $b(a) \to \bar{v}$; therefore, $a \le v(x) \le b(a)$ implies assertion (i).

(ii) Define $I(a) = \int_a^b 1/\sqrt{2[H(a) - H(v)]}\, dv$, then we have $D(a) = l^2/I^2(a)$.

We define functions $I_0(a)$ and $I_1(b)$ by

$$I_0(a) = \int_a^{\bar{v}} \frac{dv}{\sqrt{2[H(a) - H(v)]}} \qquad \text{for } \underline{v} < a < \bar{v},$$

$$I_1(b) = \int_{\bar{v}}^{b} \frac{dv}{\sqrt{2[H(b) - H(v)]}} \qquad \text{for } \bar{v} < b < v_M.$$

Then we have $I(a) = I_0(a) + I_1(b(a))$.

Since $H(a) = H(b)$, we have $h(a) = h(b)b'(a)$. We see that $h(a) < 0$ for $a < \bar{v}$, and $h(b) > 0$ for $b > \bar{v}$. Therefore $db(a)/da < 0$. Hence, as $a \uparrow \bar{v}$, we obtain $b(a) \downarrow \bar{v}$.

Notice that (i) $h(\underline{v}) = h(\bar{v}) = 0$, (ii) $h'(\underline{v}) < 0$, $h'(\bar{v}) > 0$, (iii) $h(v) < 0$ for $\underline{v} < v < \bar{v}$, $h(v) > 0$ for $\bar{v} < v < v_M$ and (iv) $H(\bar{v}) = 0$. Hence, we see that $I_0(a)$ and $I_1(b)$ satisfy the conditions of Lemma 4.3. Therefore, by Lemma 4.3 (b), we have

$$I_0(a) \to \frac{\pi}{2\sqrt{h'(\bar{v})}} \quad \text{as } a \uparrow \bar{v} \qquad \text{and} \qquad I_1(b) \to \frac{\pi}{2\sqrt{h'(\bar{v})}} \quad \text{as } b \downarrow \bar{v}.$$

Consequently, $I(a) \to \pi/\sqrt{h'(\bar{v})}$, which means that $D(a) \to h'(\bar{v})/(\pi/l)^2$.

Proof (of Theorem 4.2). First we prove assertion 1). In the case $H(\underline{v}) < H(v_M)$, $h(v)$ is twice continuously differentiable in the closed interval $[0, b(\underline{v})]$; hence we can apply Lemma 4.3 to $I_0(a)$ and $I_1(b)$. By (c) of Lemma 4.3, $I(a) = I_0(a) + I_1(b(a)) \to +\infty$ as $a \downarrow \underline{v}$. Hence $D(a) \to 0$ as $a \downarrow \underline{v}$ because of (4.6).

Let κ be any positive number satisfying $\underline{v} < \underline{v} + \kappa < \bar{v}$. Let $x_\kappa(a) \in (0, l)$ be the unique point such that $v(x_\kappa(a), a) = \underline{v} + \kappa$. Then Lemma 4.3 (c) implies

$$\frac{x_\kappa(a)}{l} = \int_a^{\underline{v}+\kappa} \frac{dv}{\sqrt{2(H(a) - H(v))}} \Big/ I(a) \to 1$$

as $a \downarrow \underline{v}$. Note that $v(x; a)$ is monotone increasing in x, and hence we have $a \le v(x, a) \le \underline{v} + \kappa$ for $0 \le x \le x_\kappa(a)$. Since $x_\kappa(a) \to l$, we may conclude that $v(x, a) \to \underline{v}$ uniformly on $[0, l - \delta]$ as $a \downarrow \underline{v}$ for any $\delta > 0$. On the other hand, $v(l, a) = b(a) \uparrow b(\underline{v})$ as $a \downarrow \underline{v}$.

On the other hand, when $H(\underline{v}) = H(v_M)$, we have $b(\underline{v}) = v_M$ and $h(v)$ is not differentiable at $v = v_M$. However, we can prove that $I_1(b)$ remains bounded as $b \uparrow v_M$. Indeed, $H(v)$ is convex in $[\bar{v}, v_M]$ since $h'(v) > 0$ in the interval $[\bar{v}, v_M)$. Therefore $H(v) \ge H(b)(v - \bar{v})/(b - \bar{v})$ for $v \in [\bar{v}, b]$, where $\underline{v} < b \le v_M$. Hence, $H(b) - H(v) \ge H(b)(b - v)/(b - \bar{v})$ for $v \in [\bar{v}, b]$, from which it follows that

$$I_1(b) \le \frac{\sqrt{b - \bar{v}}}{\sqrt{2H(b)}} \int_{\bar{v}}^b \frac{dv}{\sqrt{b - v}} = \frac{\sqrt{2}(b - \bar{v})}{\sqrt{H(b)}}.$$

This verifies the assertion. Once we know the boundedness of $I_1(b(a))$ as $a \downarrow \underline{v}$, we can argue in exactly the same way as in the case $H(\underline{v}) < H(v_M)$ and obtain the conclusion also in this case.

Next we prove assertion 2). We have just proved that $I_1(b)$ remains bounded as $b \uparrow v_M$. Hence it is sufficient to verify that $I_0(a)$ remains bounded as $a \downarrow v_m$. If $h'(v_m) \geq 0$, then we can use the convexity of $H(v)$ in the interval $[v_m, \bar{v}]$ as in the arguments above.

If $h'(v_m) < 0$, we divide the integration interval $[v_m, \bar{v}]$ into $[v_m, v_I]$ and $[v_I, \bar{v}]$, where $h'(v_I) = 0$. We use the convexity of $H(v)$ in the interval $[v_I, \bar{v}]$ and obtain $H(v) \leq H(v_I)(v - \bar{v})/(v_I - \bar{v})$, and hence $H(a) - H(v) \geq H(v_I) - H(v) \geq H(v_I)(v - v_I)/(\bar{v} - v_I)$ on $[v_I, \bar{v}]$. On the interval $[v_m, v_I]$, the concavity of $H(v)$ implies $H(v) \leq h(a)(v - a) + H(a)$, so that $H(a) - H(v) \geq -h(a)(v - a)$. Using these estimates, we can easily derive a uniform bound on $I_0(a)$ as $a \downarrow v_m$. We omit the detail.

Therefore, in Case (A) (i.e., $H(\underline{v}) \leq H(v_M)$), the solution develops a boundary-layer. On the other hand, in Case (B), no layer appears in the monotone solutions, and $D(a)$ is bounded away from 0.

4.3. Global behavior of bifurcating branches. Let $\mathscr{S}$ be the set of all nonconstant solutions of the boundary value problem for the single equation (3.24) and $\mathscr{C}_j$ denote the connected component of $\overline{\mathscr{S}}$, the closure in $C^0([0, l]) \times (0, +\infty)$, which contains the bifurcation point $(\bar{v}, D_j)$. By the well-known result of Rabinowitz [10] (see also [9] and Appendix of [15]), we see that if $\mathscr{C}_j$ is compact then it contains another bifurcation point $(\bar{v}, D_k)$ for some $k \neq j$. The following lemma, however, rules out this possibility, and implies that $\mathscr{C}_j$ is *not* compact in $\mathfrak{D} \times (0, +\infty)$ where $\mathfrak{D} = \{v \in C^2([0, l]) \mid -1 < v(x) < v_M$ for all $x \in [0, l]\}$.

Lemma 4.4. *If $m \neq n$ then $\mathscr{C}_m \cap \mathscr{C}_n = \varnothing$.*

Proof. To prove this, we define the *mode* of a nonconstant solution $v(x)$ of (3.24). We say that a solution $v(x)$ is of mode n if $v'(x)$ has $n - 1$ zeros in the open interval $(0, l)$. Therefore, if $v(x)$ is monotone increasing (or decreasing), then the mode of $v(x)$ is one.

We claim that any nonconstant solution $(v(x), D)$ on $\mathscr{C}_m$ is of mode m. Clearly, near the bifurcation point $(\bar{v}, D_m)$, solutions on $\mathscr{C}_m \backslash \{(\bar{v}, D) \mid D > 0\}$ are of mode m. Assume for contradiction that $\mathscr{C}_m$ contains a solution $(w(x), D)$ of mode $n \neq m$. Then by continuity of $v'(x)$ with respect to D, the derivative $\tilde{v}'(x)$ of some solution $(\tilde{v}, \tilde{D}) \in \mathscr{C}_m$ must have a double zero. But, this implies that $\tilde{v}'(x) \equiv 0$ since

$$D \frac{d^2 \tilde{v}'}{dx^2} + h'(\tilde{v})\tilde{v}' = 0. \tag{4.7}$$

It is important to point out that the argument above works only in the case $-1 < \tilde{v} < v_M$ for all $x \in [0, l]$ since the uniqueness of the solution of the initial value problem is used to conclude $\tilde{v} \equiv 0$. The function $h(v)$ is not differentiable at $v = v_M$, though it is Hölder continuous there. However, if $\tilde{v}(x_M) = v_M$ for some $x_M \in [0, l]$, then x_M is a maximum point of $\tilde{v}(x)$, and hence $\tilde{v}'(x_M) = 0$. If in addition x_M is a double zero of $\tilde{v}'(x)$, then $\tilde{v}''(x_M) = 0$. Therefore, $h(\tilde{v}(x_M)) = 0$. On the other hand,

$$h(\tilde{v}(x_M)) = h(\tilde{v}_M) = g(p(v_M), v_M) > 0$$

since the point $(p(v_M), v_M)$ is in the region where $g(u, v) > 0$. This excludes the possibility that $\tilde{v}'$ has a double zero at $x = x_M$.

Therefore, $(\tilde{v}, \tilde{D})$ must be a bifurcation point. But this leads to a contradiction because there exist two sequences of solutions $\{(w_\nu(x), D_\nu)\}_{\nu=1}^{\infty}$ and $\{(v_\mu(x), D_\mu)\}_{\mu=1}^{\infty}$ such that $(w_\nu(x), D_\nu) \to (\bar{v}, \tilde{D})$ as $\nu \to \infty$, $(v_\mu(x), D_\mu) \to (\bar{v}, \tilde{D})$ as $\mu \to \infty$, $w_\nu(x)$ is of mode n whereas $v_\mu(x)$ is of mode m. Since $(\bar{v}, \tilde{D})$ is a simple bifurcation point, the solution set near $(\bar{v}, \tilde{D})$ consists of two curves: one is the trivial branch $\{(\bar{v}, D) \mid D > 0\}$ and the other is a branch of nonconstant solutions of some definite mode (see Theorem 3.6).

Now that $\mathscr{C}_j$ is not compact in $\mathfrak{D} \times (0, +\infty)$, we would like to know in what way the branch $\mathscr{C}_j$ approaches the boundary. First, we consider the case where D is sufficiently large.

Proposition 4.5. *Assume that condition* (*D*) *of Proposition* 2.2 *is satisfied and* k *is so small that* $\operatorname{tr} J < 0$ *is satisfied (see Lemma* 3.2*). Then there exists a positive constant* $D^\star$*, depending only on the function* $h(v)$ *and* l*, such that the boundary value problem* (3.24) *has only constant solutions if* $D > D^\star$.

Proof. It is convenient to put $\gamma = 1/D$, so that our equation becomes

$$v'' + \gamma h(v) = 0.$$

First, we prove that $v(x)$ needs to be close to a constant when γ is small. Since we know that v is *a priori* bounded, i.e.,

$$0 \le v(x) \le v_M \qquad \text{for } x \in [0, l],$$

$h(v)$ is also bounded by a positive constant M independent of γ:

$$|h(v(x))| \le M \qquad \text{for } x \in [0, l].$$

Put

$$c = \frac{1}{l}\int_0^l v(x)dx$$

and choose an $x_c \in [0, l]$ such that $v(x_c) = c$. Therefore, from the expression

$$v(x) = c - \gamma\left(\int_{x_c}^{x}(x-t)h(v(t))dt + \int_0^{x_c}(x_c - t)h(v(t))dt\right)$$

it follows that

$$|v(x) - c| \le \gamma\int_0^x (x-t)M\,dt + \gamma\int_0^{x_c}(x_c - t)M\,dt = \frac{\gamma M}{2}x^2 + \frac{\gamma M x_c^2}{2} \le \gamma M l^2.$$

This in particular yields that $\max v(x) - \min v(x) \le 2\gamma M l^2$. Using this fact we can verify that either $v(x) \equiv \underline{v}$ or it satisfies the inequality

$$\bar{v} - 2\gamma M l^2 \le v(x) \le \bar{v} + 2\gamma M l^2 \tag{4.8}$$

for $x \in [0, l]$. For the proof we observe that if $v(x_m) = \min v(x)$, then $v''(x_m) \ge 0$, so that $h(v(x_m)) \le 0$. Hence, $\underline{v} \le v(x_m) \le \bar{v}$. Now if $\max v(x) > \bar{v} + 2Ml^2$, then $v(x) > \bar{v}$ for all $x \in [0, l]$, which is impossible. Therefore, $\max v(x) \le \bar{v} + 2\gamma M l^2$. Next, assume that $\underline{v} < \min v < \bar{v} - 2\gamma M l^2$. Then $\max v(x) < \bar{v}$, so that $h(\max v(x)) < 0$. This contradicts the fact that $v''(x_M) \le 0$ for x_M such that $v(x_M) = \max v(x)$. Consequently we have either $v(x) \equiv \underline{v}$ or inequality (4.8).

Now we decompose v into

$$v(x) = c + \varphi(x), \qquad \text{where } c = \frac{1}{l}\int_0^l v(x)dx, \qquad \int_0^l \varphi(x)dx = 0.$$

Then

$$\varphi'' + \gamma h(c + \varphi) = 0,$$

which yields

$$\varphi'' + \gamma(h(c) + h'(c + \theta\varphi)\varphi) = 0,$$

where $\theta = \theta(x)$ satisfies $0 < \theta < 1$. Multiply both sides with φ and then integrate the resulting equation over the interval $[0, l]$. We obtain

$$-\int_0^l (\varphi')^2 + \gamma\int_0^l h'(c + \theta\varphi)\varphi^2\,dx = 0.$$

We fix a $\delta_0 > 0$ such that $\bar{v} + \delta_0 < v_M$, and assume that $2\gamma M l^2 \le \delta_0$. Then, thanks to (4.8), we have

$$|h'(c + \theta\varphi(x))| \le M_1 \qquad \text{for } x \in [0, l],$$

provided that $2\gamma M l^2 \le \delta_0$, where M_1 is a positive constant depending only on δ_0. Thus we obtain

$$\int_0^l (\varphi')^2 dx \le \gamma M_1 \int_0^l \varphi^2\, dx.$$

Recalling the Poincaré type inequality

$$\int_0^l w^2\, dx \le \frac{1}{\ell_1}\int_0^l (w')^2 dx + \frac{1}{l}\left(\int_0^l w(x)dx\right)^2 \qquad \text{with } \ell_1 = (\pi/l)^2,$$

we find that

$$\int_0^l (\varphi')^2 dx \le \gamma M_1 \cdot \frac{1}{\ell_1}\int_0^l (\varphi')^2 dx,$$

showing that $\varphi' \equiv 0$ if $\gamma M_1/\ell_1 < 1$. In other words, $v(x) \equiv \bar{v}$ if $D > M_1/\ell_1$.

Therefore, $\mathscr{C}_n$ cannot extend to the neighborhood of $D = +\infty$, so that the projection of the branch $\mathscr{C}_n$ on $\mathbb{R}$ forms either (i) an interval $\{D \mid 0 < D \le D_M\}$ or (ii) an interval $\{D \mid d_\star \le D \le D_M\}$, where D_M, $d_\star$ are positive constants. Combined with Theorem 4.2, this observation yields the following

Proposition 4.6. *Let* $\mathrm{Proj}_{\mathbb{R}}\, \mathscr{C}_n$ *denote the projection of* $\mathscr{C}_n$ *on* $\mathbb{R}$. (i) *If* $H(\underline{v}) \le H(v_M)$, *then* $\mathrm{Proj}_{\mathbb{R}}\, \mathscr{C}_n = (0, D_M]$ *for some* $D_M > 0$. (ii) *If* $H(\underline{v}) > H(v_M)$, *then* $\mathrm{Proj}_{\mathbb{R}}\, \mathscr{C}_n = [d_\star, D_M]$ *for some* $D_M > d_\star > 0$.

Remark. Let us consider the case $H(\underline{v}) > H(v_M)$ and the branch of monotone increasing solutions $\mathscr{C}_{1,+} = \{(v(\cdot, a), D(a)) \mid v_m < a \le \bar{v}\}$ where $v_m < \bar{v}$ satisfies $H(v_m) = H(v_M)$. The arguments at the beginning of Subsection 4.1 yield that the initial value problem for $Dv'' + h(v) = 0$ subject to $v(0) = a < v_m$ and $v'(0) = 0$ has a unique strictly increasing solution $v(x; a)$ which satisfies $v(\xi; a) = v_M$ for some $\xi > 0$, but $v'(\xi; a) > 0$. Thus, the boundary value problem (3.24) has no solution satisfying $v(0) < v_m$. The branch $\mathscr{C}_{1,+}$, therefore, cannot continue beyond the singular solution $(v(x, v_m), D(v_m))$ mentioned in Remark immediately after Theorem 4.2. Consequently, $\mathscr{C}_{1,+}$ is a curve connecting the bifurcation point $(\bar{v}, D_1)$ with $(v(\cdot, v_m), D(v_m)) \in (\partial\mathscr{D}) \times (0, \infty)$.

5. Appendix

Here we give some explicit formulas for $p_-(v)$, $h(v)$ and their derivatives.

$$p_-(v) = \frac{m_1 - \sqrt{E(v)}}{2k(1+v)}, \qquad \text{where } E(v) = m_1^2 - 4k(1+v)^2,$$

$$h(v) = -\mu_3 v + \left(\frac{m_2}{m_1} + \left(\frac{m_2}{m_1} - 1\right)v\right)p_-(v),$$

$$h'(v) = -\mu_3 + \left(\frac{m_2 - m_1}{m_1} + \frac{m_2 + (m_2 - m_1)v}{(1+v)\sqrt{E(v)}}\right)p_-(v),$$

$$h''(v) = 2p_-(v) \cdot \frac{(m_2 - m_1)E(v) + k(m_2 + (m_2 - m_1)v)(p_-(v)\sqrt{E(v)} + 2(1+v))}{(1+v)E(v)\sqrt{E(v)}}.$$

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