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Stable extendibility of some complex vector bundles over lens spaces and Schwarzenberger's theorem

Dedicated to the Memory of Professor Yusuke Kawamoto

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ABSTRACT. We obtain conditions for stable extendibility of some complex vector bundles over the $(2n+1)$-dimensional standard lens space $L^n(p)$ mod p, where p is a prime. Furthermore, we study stable extendibility of the bundle $\pi_n^*(\tau(\mathbf{C}P^n))$ induced by the natural projection $\pi_n : L^n(p) \to \mathbf{C}P^n$ from the complex tangent bundle $\tau(\mathbf{C}P^n)$ of the complex projective n-space $\mathbf{C}P^n$. As an application, we have a result on stable extendibility of $\tau(\mathbf{C}P^n)$ which gives another proof of Schwarzenberger's theorem.

1. Introduction

Let $\mathbf{F}$ denote either the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$. Let X be a space and A its subspace. A t-dimensional $\mathbf{F}$-vector bundle α over A is said to be extendible (respectively stably extendible) to X if and only if there exists a t-dimensional $\mathbf{F}$-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to α (cf. [8, p. 20] and [5, p. 273]). In this paper, we use the same letter for an $\mathbf{F}$-vector bundle and its equivalence class.

For a prime p, let $L^n(p) = S^{2n+1}/(\mathbf{Z}/p)$ denote the standard lens space mod p of dimension $2n+1$ and $\mathbf{C}P^n = S^{2n+1}/S^1$ the complex projective space of complex dimension n, where S^m is the standard sphere of dimension m and $\mathbf{Z}/q$ the group of integers mod q. Let μ_n be the canonical $\mathbf{C}$-line bundle over $\mathbf{C}P^n$. Then we define $\eta_n = \pi_n^*(\mu_n)$, the bundle induced by the natural projection $\pi_n : L^n(p) \to \mathbf{C}P^n$ from μ_n. We call η_n the canonical $\mathbf{C}$-line bundle over $L^n(p)$.

Throughout this paper, we denote by $[x]$ the largest integer q with $q \le x$. In [2], we have obtained the following result for the stable extendibility of $\mathbf{R}$-vector bundles over $L^n(p)$.

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THEOREM 1.1 ([2, Theorem 1]). *Let p be an odd prime and α a t-dimensional* **R**-*vector bundle over $L^n(p)$ which is stably equivalent to $sr(\eta_n)$, where s is an integer with $n/2 \leq s < p^{[n/(p-1)]}$, and $r(\eta_n)$ is the real restriction of η_n. Then the following three conditions are equivalent.*

(i) *α is stably extendible to $L^m(p)$ for every $m \geq n$.*

(ii) *α is stably extendible to $L^{2s}(p)$.*

(iii) *$s \leq [t/2]$.*

In this paper, we have

THEOREM 1. *Let p be a prime and α a t-dimensional* **C**-*vector bundle over $L^n(p)$ which is stably equivalent to $s\eta_n$, where s is an integer with $n \leq s < p^{[n/(p-1)]}$. Then the following three conditions are equivalent.*

(i) *α is stably extendible to $L^m(p)$ for every $m \geq n$.*

(ii) *α is stably extendible to $L^s(p)$.*

(iii) *$s \leq t$.*

It should be remarked that the implication (iii) $\Rightarrow$ (i) holds even in the cases where $s < n$ (see the proof of Theorem 1).

Let $\tau(\mathbf{C}P^n)$ denote the complex tangent bundle of $\mathbf{C}P^n$ and $\pi_n^*(\tau(\mathbf{C}P^n))$ the bundle induced by the natural projection $\pi_n : L^n(p) \to \mathbf{C}P^n$ from $\tau(\mathbf{C}P^n)$.

As an application of Theorem 1, we have

COROLLARY 2. *Let p be a prime and $\pi_n : L^n(p) \to \mathbf{C}P^n$ the natural projection. Then $\pi_n^*(\tau(\mathbf{C}P^n))$ is not stably extendible to $L^{n+1}(p)$ if $n \geq 2p-2$.*

Using Corollary 2, we have

THEOREM 3. *If $n \geq 2$, $\tau(\mathbf{C}P^n)$ is not stably extendible to $\mathbf{C}P^{n+1}$.*

In Appendix I of [3], R. L. E. Schwarzenberger proved the following.

THEOREM 1.2 ([3]). *If $n \geq 2$, $\tau(\mathbf{C}P^n)$ is not extendible to $\mathbf{C}P^{n+1}$.*

Clearly, extendibility implies stable extendibility. Hence Theorem 1.2 follows from Theorem 3. Conversely, we see

LEMMA 1.3. *If $\tau(\mathbf{C}P^n)$ is stably extendible to $\mathbf{C}P^{n+1}$, it is extendible to $\mathbf{C}P^{n+1}$.*

This shows that Theorem 3 follows also from Theorem 1.2.

For a sufficient condition of stable extendibility of **C**-vector bundles over lens spaces, the following holds.

THEOREM 4. *Let p be a prime and α a t-dimensional* **C**-*vector bundle over $L^n(p)$ which is stably equivalent to $s\eta_n$. Then α is stably extendible to $L^m(p)$*

for every $m \geq n$ *if there exists an integer* a *satisfying the inequalities:*

$$s - t \leq ap^{1+[(n-1)/(p-1)]} \leq s.$$

The converse does not hold in general. In fact, we have the following.

THEOREM 5. *The converse claim of Theorem* 4 *does not hold for* $n = 1$, $p \geq 3$ *and* $\alpha = \pi_1^*(\tau(\mathbf{C}P^1))$, *where* $\pi_1 : L^1(p) \to \mathbf{C}P^1 (= S^2)$ *is the natural projection.*

The following corollary will be used to prove Theorem 9 below.

COROLLARY 6. *Let* p *be a prime and* $\pi_{p-1} : L^{p-1}(p) \to \mathbf{C}P^{p-1}$ *the natural projection. Then* $\pi_{p-1}^*(\tau(\mathbf{C}P^{p-1}))$ *is stably extendible to* $L^m(p)$ *for every* $m \geq p - 1$.

It is shown in Theorem 5 that the converse of Theorem 4 does not hold in general, but for $p = 3$ and $n = 2k$ we can show that the converse holds as follows.

THEOREM 7. *Let* α *be a t-dimensional* **C***-vector bundle over* $L^{2k}(3)$ *which is stably equivalent to* $s\eta_{2k}$, *where* η_{2k} *is the canonical* **C***-line bundle over* $L^{2k}(3)$. *Then* α *is stably extendible to* $L^m(3)$ *for every* $m \geq 2k$ *if and only if there exists an integer* a *satisfying the inequalities:*

$$s - t \leq a3^k \leq s.$$

For $p = 3$, we have

THEOREM 8. *Let* $\pi_n : L^n(3) \to \mathbf{C}P^n$ *be the natural projection. Then* $\pi_n^*(\tau(\mathbf{C}P^n))$ *is not stably extendible to* $L^{n+1}(3)$ *if* $n \geq 3$.

THEOREM 9. $\pi_n^*(\tau(\mathbf{C}P^n))$ *is stably extendible to* $L^m(3)$ *for every* $m \geq n$ *if and only if* $n = 1, 2$.

This paper is organized as follows. In Section 2, we prove Theorem 1, Corollary 2, Theorem 3 and Lemma 1.3 by using results in [7] on the stable extendibility of some **C**-vector bundles over $L^n(p)$. In Section 3, we recall some known results on the structure of the K-ring of $L^n(p)$, and prove Theorems 4, 5 and Corollary 6. Detailed results for the case $p = 3$, that is, Theorems 7, 8 and 9, are proved in Sections 4 and 5.

2. Proofs of Theorem 1, Corollary 2, Theorem 3 and Lemma 1.3

The following result gives information about stable extendibility of some **C**-vector bundles over $L^n(p)$, and is useful for the proofs of Theorems 1 and 7.

THEOREM 2.1 ([7, Theorem 4.5]). *Let p be a prime and α a t-dimensional $\mathbf{C}$-vector bundle over $L^n(p)$. Assume that there exists a positive integer l such that α is stably equivalent to a sum of $t+l$ non-trivial $\mathbf{C}$-line bundles, where $t+l < p^{[n/(p-1)]}$. Then $n < t+l$ and α is not stably extendible to $L^{t+l}(p)$.*

We use the next lemma for the proof of Theorem 1.

LEMMA 2.2 ([1, Lemma 2.1]). *Let A be a subspace of a space X, and α and β be $\mathbf{F}$-vector bundles over A of respective dimensions a and b, where $b \leq a$. Suppose that α is stably equivalent to β. Then, if β is stably extendible to X, so is α.*

PROOF OF THEOREM 1. (i) $\Rightarrow$ (ii) is clear.

We prove (ii) $\Rightarrow$ (iii) by contraposition. Suppose $t < s$ and define $s - t = l$. Then $l > 0$ and $t + l = s < p^{[n/(p-1)]}$. Using Theorem 2.1, we have $n < s$ and α is not stably extendible to $L^s(p)$.

To prove (iii) $\Rightarrow$ (i), suppose $s \leq t$. Then, setting $A = L^n(p)$, $X = L^m(p)$ $(m \geq n)$, $\beta = s\eta_n$, $a = t$, $b = s$ in Lemma 2.2, we see that α is stably extendible to $L^m(p)$, since $i^*(s\eta_m) = si^*(\eta_m) = s\eta_n = \beta$, where $i^* : K(L^m(p)) \to K(L^n(p))$ is the homomorphism induced by the standard inclusion $i : L^n(p) \to L^m(p)$. □

PROOF OF COROLLARY 2. Recall that $\tau(\mathbf{C}P^n) \oplus 1 = (n+1)\mu_n$ (cf. [6, p. 145]), where $\oplus$ denotes the Whitney sum. Then $\pi_n^*(\tau(\mathbf{C}P^n)) \oplus 1 = (n+1)\eta_n$, where $\pi_n : L^n(p) \to \mathbf{C}P^n$ is the natural projection. Note that $n+1 < p^{[n/(p-1)]}$ if $n \geq 2p-2$. Thus the proof is completed by the implication (ii) $\Rightarrow$ (iii) of Theorem 1 by setting $\alpha = \pi_n^*(\tau(\mathbf{C}P^n))$, $t = n$ and $s = n+1$. □

PROOF OF THEOREM 3. Suppose that $\tau(\mathbf{C}P^n)$ is stably extendible to $\mathbf{C}P^{n+1}$. Then there exists an n-dimensional vector bundle β over $\mathbf{C}P^{n+1}$ such that $\tau(\mathbf{C}P^n)$ is stably equivalent to $j^*(\beta)$, where $j : \mathbf{C}P^n \to \mathbf{C}P^{n+1}$ is the standard inclusion. Consider the natural projection $\pi_m : L^m(2) \to \mathbf{C}P^m$, where $m = n$ and $n+1$. Then $\pi_n^*(\tau(\mathbf{C}P^n))$ is stably equivalent to $\pi_n^*(j^*(\beta))$ which is equal to $i^*(\pi_{n+1}^*(\beta))$ by naturality, where $i : L^n(2) \to L^{n+1}(2)$ is the standard inclusion. Hence $\pi_n^*(\tau(\mathbf{C}P^n))$ is stably extendible to $L^{n+1}(2)$. If $n \geq 2$, this contradicts to Corollary 2. □

To prove Lemma 1.3, we use the following result.

THEOREM 2.3 ([4, Theorem 1.5, p. 100]). *If α and β are two t-dimensional $\mathbf{F}$-vector bundles over an m-dimensional CW-complex X such that $\langle (m+2)/d - 1 \rangle \leq t$ and $\alpha \oplus k = \beta \oplus k$ for some k-dimensional trivial $\mathbf{F}$-vector* *bundle k over X, then $\alpha = \beta$, where $d = 1$ or 2 according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$ and $\langle x \rangle$ denotes the smallest integer q with $x \le q$.*

Proof of Lemma 1.3. Suppose that $\tau(\mathbf{C}P^n)$ is stably extendible to $\mathbf{C}P^{n+1}$. Then there exists an n-dimensional $\mathbf{C}$-vector bundle γ such that $i^*(\gamma) \oplus k = \tau(\mathbf{C}P^n) \oplus k$ for a k-dimensional trivial bundle k, where $i : \mathbf{C}P^n \to \mathbf{C}P^{n+1}$ is the standard inclusion. Putting $\mathbf{F} = \mathbf{C}$ (and thus $d = 2$), $\alpha = i^*(\gamma)$, $\beta = \tau(\mathbf{C}P^n)$, $X = \mathbf{C}P^n$, $m = 2n$ and $t = n$ in Theorem 2.3, we obtain $\alpha = \beta$, that is, $i^*(\gamma) = \tau(\mathbf{C}P^n)$. Hence $\tau(\mathbf{C}P^n)$ is extendible to $\mathbf{C}P^{n+1}$. $\square$

3. Proofs of Theorems 4, 5 and Corollary 6

For the canonical $\mathbf{C}$-line bundle η_n over $L^n(p)$, set $\sigma_n = \eta_n - 1 (\in \tilde{K}(L^n(p)))$. The structure of the ring $\tilde{K}(L^n(p))$ is determined in [6] as follows.

Theorem 3.1 ([6, Theorem 1]). *Let p be a prime and $n = s(p-1) + r$, where s and r are integers with $s \ge 0$ and $0 \le r < p - 1$. Then*

$$\tilde{K}(L^n(p)) \cong (\mathbf{Z}/p^{s+1})^r + (\mathbf{Z}/p^s)^{p-r-1}.$$

(Here, $(\mathbf{Z}/q)^k$ denotes the direct sum of k-copies of the additive group of integers mod *q.) The first r summands are generated by $\sigma_n^1, \sigma_n^2, \dots, \sigma_n^r$, and the last $(p - r - 1)$ summands by $\sigma_n^{r+1}, \sigma_n^{r+2}, \dots, \sigma_n^{p-1}$. Moreover, the ring structure is determined by the relations:*

$$(\sigma_n + 1)^p (= \eta_n^p) = 1 \qquad \textit{and} \qquad \sigma_n^{n+1} = 0.$$

Fact 3.2 ([6, (2.10)]). *Let p be a prime. Then, for $1 \le i \le p - 1$, σ_n^i is of order $p^{1+[(n-i)/(p-1)]}$.*

The cohomology groups of $L^n(p)$ are known as follows.

Fact 3.3 ([6, (2.1)]).

$$H^i(L^n(p); \mathbf{Z}) \cong \begin{cases} \mathbf{Z}/p & \text{if } i = 2k \text{ for some } 1 \le k \le n, \\ \mathbf{Z} & \text{if } i = 0 \text{ or } 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 4. In the setting of Theorem 4, let us assume that we can take an integer a satisfying the inequality below

$$s - t \le ap^{1+[(n-1)/(p-1)]} \le s.$$

Since α is stably equivalent to $s\eta_n$, we have $\alpha = s\eta_n + t - s$ in $K(L^n(p))$. By Fact 3.2, the equality $ap^{1+[(n-1)/(p-1)]}(\eta_n - 1) = 0$ holds in $\tilde{K}(L^n(p))$ for our integer a. Hence we obtain the equality

$$\alpha = (s - ap^{1+[(n-1)/(p-1)]})\eta_n + t - s + ap^{1+[(n-1)/(p-1)]}$$

in $K(L^n(p))$. Set $U = s - ap^{1+[(n-1)/(p-1)]}$ and $V = t - s + ap^{1+[(n-1)/(p-1)]}$. Then we have $\alpha = U\eta_n + V$, and $U \geq 0$ and $V \geq 0$ by the assumption for a. Since the Whitney sum $U\eta_n \oplus V$ is extendible to $L^m(p)$ for every $m \geq n$, α is stably extendible to $L^m(p)$ for every $m \geq n$. □

PROOF OF THEOREM 5. We show that the bundle $\alpha = \pi_1^*(\tau(\mathbf{C}P^1))$ is extendible to $L^m(p)$ for every $m \geq 1$, but there does not exist an integer a satisfying the inequalities of Theorem 4 for α and $p \geq 3$. Let $BU(1)$ be the classifying space for $U(1) = S^1$, and let $\zeta : L^1(p) \to BU(1)$ be the classifying map of the bundle α. The obstructions for extending ζ to $L^m(p)$ $(m \geq 1)$ consist in the groups

$$H^{r+1}(L^m(p), L^1(p); \pi_r(BU(1))) \ (\cong H^{r+1}(L^m(p), L^1(p); \pi_{r-1}(S^1)))$$

which are easily seen to be 0 for each r (cf. Fact 3.3). So α is extendible to $L^m(p)$ for every $m \geq 1$. Now, $t = \dim \alpha = 1$, and $s = 2$ since $\alpha \oplus 1 = 2\eta_1$. Then there does not exist an integer a satisfying the inequalities: $1 \leq ap \leq 2$ if $p \geq 3$. □

PROOF OF COROLLARY 6. Note that $\dim \pi_{p-1}^*(\tau(\mathbf{C}P^{p-1})) = p - 1$ and that $\pi_{p-1}^*(\tau(\mathbf{C}P^{p-1})) \oplus 1 = p\eta_{p-1}$. Then, for $n = p - 1$, $\alpha = \pi_{p-1}^*(\tau(\mathbf{C}P^{p-1}))$, $t = p - 1$ and $s = p$ in Theorem 4, we have the result, because $a = 1$ satisfies the inequalities: $1 \leq ap \leq p$. □

4. Proof of Theorem 7

PROOF OF THEOREM 7. The "if" part of the theorem follows immediately from Theorem 4 since $p^{1+[(n-1)/(p-1)]} = 3^k$ for $p = 3$ and $n = 2k$.

We prove the "only if" part of the theorem by contraposition. Assume that every integer a satisfies

$$a3^k < s - t \qquad \text{or} \qquad s < a3^k.$$

Let M be the minimum integer such that $s < M3^k$. Then, since $s \geq (M-1)3^k$, we have $(M-1)3^k < s - t$ by the above assumption. Put $l = s - t - (M-1)3^k$. Then $l > 0$,

$$t + l = s - (M-1)3^k < M3^k - (M-1)3^k = 3^k \qquad \text{and}$$

$$(t + l)\eta_{2k} = \{s - (M-1)3^k\}\eta_{2k} = s\eta_{2k} - (M-1)3^k$$

since $\{(M-1)3^k\}(\eta_{2k}-1)=0$ by Fact 3.2. Hence, by Theorem 2.1, $2k < s-(M-1)3^k$ and α is not stably extendible to $L^m(3)$ for $m=s-(M-1)3^k$. $\square$

5. Proofs of Theorems 8 and 9

We recall some known facts for the proof of Theorem 8.

Fact 5.1. *The total Chern class $C(\eta_n^i)$ of η_n^i is given by $C(\eta_n^i)=1+iz_n$, where $z_n=C_1(\eta_n)$ is the generator of $H^2(L^n(p);\mathbf{Z})(\cong \mathbf{Z}/p)$.*

Fact 5.2. *Let p be a prime and let $a=\sum_{0\le i\le m} a(i)p^i$ and $b=\sum_{0\le i\le m} b(i)p^i$, $(0\le a(i)<p,\ 0\le b(i)<p)$. Then*

$$\binom{b}{a}\equiv\prod_{0\le i\le m}\binom{b(i)}{a(i)} \bmod p.$$

Proof of Theorem 8. If $n\ge 4$, $\pi_n^*(\tau(\mathbf{C}P^n))$ is not stably extendible to $L^{n+1}(3)$ by Corollary 2.

We prove that $\pi_n^*(\tau(\mathbf{C}P^n))$ is not stably extendible to $L^{n+1}(p)$ for $p=3$ and $n=3$. Suppose that there exists a 3-dimensional $\mathbf{C}$-vector bundle β over $L^4(3)$ satisfying $i^*(\beta)=\pi_3^*(\tau(\mathbf{C}P^3))$, where $i: L^3(3)\to L^4(3)$ is the standard inclusion and $\pi_3: L^3(3)\to \mathbf{C}P^3$ is the natural projection. According to Theorem 3.1, there exist integers a and b such that

$$\beta-3=a\sigma_4+b\sigma_4^2\in \tilde{K}(L^4(3))(\cong \mathbf{Z}/3^2+\mathbf{Z}/3^2).$$

Applying the induced homomorphism $i^*:\tilde{K}(L^4(3))\to\tilde{K}(L^3(3))$ to the both sides of the above equality, we obtain

$$i^*(\beta-3)=a\sigma_3+b\sigma_3^2\in \tilde{K}(L^3(3))(\cong \mathbf{Z}/3^2+\mathbf{Z}/3).$$

On the other hand, we have

$$i^*(\beta-3)=\pi_3^*(\tau(\mathbf{C}P^3))-3=4\eta_3-4=4\sigma_3.$$

Hence $a=9x+4$ and $b=3y$ for some integers x and y. So

$$\begin{aligned}\beta-3&=(9x+4)\sigma_4+3y\sigma_4^2=(9x+4)(\eta_4-1)+3y(\eta_4-1)^2\\&=\{9(x-y)+3(y+1)+1\}\eta_4+3y\eta_4^2-9x+3y-4.\end{aligned}$$

Define $A=9(x-y)+3(y+1)+1$ and $B=3y$. Since we may take a and b with $a\ge 2b\ge 0$, we consider that x and y satisfy inequalities: $A\ge 0$ and $B\ge 0$.

Now, by Fact 5.1, the total Chern class of β is given by

$$C(\beta) = C(\eta_4)^A C(\eta_4^2)^B = (1+z_4)^A(1+2z_4)^B = (1+z_4)^A(1-z_4)^B,$$

where z_4 is the generator of $H^2(L^4(3);\mathbf{Z})(\cong \mathbf{Z}/3)$. Thus the 4-th Chern class of β is given by

$$C_4(\beta) = \sum_{i+j=4} \binom{A}{i}\binom{B}{j}(-1)^j z_4^4.$$

Here, by Fact 5.2, $\binom{A}{2} \equiv \binom{B}{j} \equiv 0 \pmod 3$ for $j = 1, 2, 4$, $\binom{A}{i} \equiv \binom{B}{0} \equiv 1 \pmod 3$ for $i = 0, 1$, $\binom{A}{i} \equiv y+1 \pmod 3$ for $i = 3, 4$, and $\binom{B}{3} \equiv y \pmod 3$. Hence we have $C_4(\beta) = (-y+y+1)z_4^4 = z_4^4 \neq 0$.

On the other hand, $C_4(\beta) = 0$ since β is 3-dimensional. This is a contradiction. □

PROOF OF THEOREM 9. In the proof of Theorem 5, it is proved that $\pi_1^*(\tau(\mathbf{C}P^1))$ is extendible to $L^m(3)$ for every $m \geq 1$. Putting $p = 3$ in Corollary 6, we see that $\pi_2^*(\tau(\mathbf{C}P^2))$ is stably extendible to $L^m(3)$ for every $m \geq 2$.

The "only if" part follows immediately from Theorem 7. □

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