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On the average of some arithmetical functions under a constraint on the sum of digits of squares

Dedicated to my parents Hédi and Wiem for their endless support, with love

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ABSTRACT. Let $q \geq 2$ be an integer and $S_q(n)$ denote the sum of the digits in base q of the positive integer n. We look for an estimate of the average of some multiplicative arithmetical functions defined by sums over divisors d of n satisfying $S_q(d^2) \equiv r \bmod m$ for some integers r and m.

1. Introduction

Throughout this paper, we denote by $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the sets of positive integers, non negative integers, integers, real and complex numbers respectively. Given a real number x, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$ and $e(x) = e^{2i\pi x}$. The greatest common divisor of two integers a and b will be denoted by (a, b) and if $a \leq b$ we denote by $[\![a, b]\!]$ the set $\{a, a+1, \dots, b\}$. The number of distinct prime factors of a positive integer n will be denoted $\omega(n)$.

First, we shall introduce the following definition: let $n \in \mathbf{N}_0$ and q be an integer ≥ 2. The sequence $(a_j(n))_{j \in \mathbf{N}_0} \in \{0, 1, \dots, q-1\}^{\mathbf{N}_0}$ is defined to be the unique sequence satisfying

$$n = \sum_{k=0}^{\infty} a_k(n) q^k. \tag{1.1}$$

The right hand side of the expression (1.1) shall be called the *expansion* of n to the *base* q. We shall set

$$S(n) = S_q(n) = \sum_{k=0}^{\infty} a_k(n).$$

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A function $f : \mathbf{N}_0 \to \mathbf{C}$ is called *completely q-additive* if $f(0) = 0$ and $f(aq^k + b) = f(a) + f(b)$ for any integers $a \geq 1$, $k \geq 1$ and $0 \leq b < q^k$. Such functions were introduced by Gelfond [7] and further studied by Delange [5], Bésineau [3], Coquet [4], Kàtai [9] and others. Using the base q expansion (1.1), we find that the function f is completely q-additive if and only if $f(0) = 0$ and

$$f(n) = \sum_{k=0}^{\infty} f(a_k(n)).$$

It follows that a completely q-additive function is completely determined by its values on the set $\{0, 1, \ldots, q-1\}$. A typical example of a completely q-additive function is the sum of digits function S.

Another kind of arithmetic functions is the *multiplicative* ones, i.e. that satisfy $f(1) = 1$ and whenever a and b are coprime integers, then $f(ab) = f(a)f(b)$ (see [2, chapter 2] for further informations).

In this paper, we shall focus on the following functions depending on a positive integer n:

- the number of positive divisors function, $\tau : n \mapsto \sum_{d|n} 1$.
- The sum of the s-th powers of all the positive divisors function (for $s \in \mathbf{R}$), $\sigma_s : n \mapsto \sum_{d|n} \left(\frac{n}{d}\right)^s$. In particular, $\sigma_0 = \tau$.
- The number of positive integers $\leq n$ and coprime to n, $\varphi : n \mapsto \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} 1$.
- The Möbius function,

$$\mu : n \mapsto \begin{cases} 1, & \text{if } n = 1, \\ (-1)^r, & \text{if } n = p_1 \ldots p_r, \text{ a product of distinct primes}, \\ 0, & \text{otherwise}. \end{cases}$$

- The non principal Dirichlet character modulo 4,

$$\chi : n \mapsto \begin{cases} 0, & \text{if } 2|n, \\ (-1)^{(n-1)/2}, & \text{otherwise}. \end{cases}$$

- The number of representations of n as the sum of two integral squares denoted by $r(n)$.

Except the last one, all these functions are multiplicative and it can be shown (see [8, chapter 16] for instance) that

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d},$$

$$r(n) = 4 \sum_{d|n} \chi(d).$$

In fact, this implies that $\frac{r(n)}{4}$ is multiplicative and thus $r(n)$ is "almost multiplicative".

For every $r \in \mathbf{Z}$, q and $m \geq 2$ such that $(m, q-1) = 1$, we define the following functions that depend on n, q, r and m (but we will omit the latter ones for briefness)

$$\bar{\tau}(n) = \sum_{\substack{d|n \\ S(d^2) \equiv r \bmod m}} 1,$$

$$\overline{\sigma_s}(n) = \sum_{\substack{d|n \\ S(d^2) \equiv r \bmod m}} \left(\frac{n}{d}\right)^s, \qquad \text{for } s \in \mathbf{R},$$

$$\bar{\varphi}(n) = \sum_{\substack{d|n \\ S(d^2) \equiv r \bmod m}} \mu(d)\frac{n}{d},$$

$$\bar{r}(n) = 4 \sum_{\substack{d|n \\ S(d^2) \equiv r \bmod m}} \chi(d).$$

Note that the assumption $(m, q-1) = 1$ is crucial since it implies that $(q-1)\frac{j}{m} \in \mathbf{R}\backslash\mathbf{Z}$ for all $j \in [\![1, m-1]\!]$, a result that allows to use the Theorem **A** which will be stated later.

Our target, in this paper, is to estimate $\sum_{n \leq x} \bar{\tau}(n)$, $\sum_{n \leq x} \overline{\sigma_s}(n)$, $\sum_{n \leq x} \bar{\varphi}(n)$ and $\sum_{n \leq x} \bar{r}(n)$ (representing the averages of the functions $\bar{\tau}$, $\overline{\sigma_s}$, $\bar{\varphi}$ and $\bar{r}$ respectively, in accordance to the study made in [8, chapter 18]).

In order to detect the congruences, we shall use the classic orthogonality relation

$$\frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{j(a-b)}{m}\right) = \begin{cases} 1, & \text{if } a \equiv b \bmod m, \\ 0, & \text{else.} \end{cases} \qquad (m \in \mathbf{N},\ a, b \in \mathbf{Z}) \tag{1.2}$$

Finally, Gelfond [7] alluded the problem of giving an estimate for the number of values of a polynomial P (P takes only integer values on the set $\mathbf{N}$) satisfying the condition $S(P(n)) \equiv r \bmod m$.

Basically, we shall need the following result proved by Mauduit and Rivat [10], answering the question of Gelfond in the case $P(n) = n^2$.

Theorem A [Mauduit-Rivat, 2007]. *Let $q \geq 2$ be an integer and $\alpha \in \mathbf{R}$ such that $(q-1)\alpha \in \mathbf{R}\backslash\mathbf{Z}$ then there exists $\sigma_q(\alpha) > 0$ and $x_0 := x_0(q, \alpha) \geq 2$*

such that for every real number $x \geq x_0$, we have

$$\left|\sum_{n \leq x} e(\alpha S(n^2))\right| \leq 4q^{7/2}(\log q)^{5/2}\tau(q)^{1/2}\left(1 + \frac{\log x}{\log q}\right)^{(1/2)\omega(q)+4} x^{1-\sigma_q(\alpha)}.$$

In particular, this means that for x sufficiently large

$$\sum_{n \leq x} e(\alpha S(n^2)) \ll (\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_q(\alpha)}.$$

Recently, Mkaouar and Wannès [11] used an improved result of Drmota, Mauduit and Rivat [6], in addition to the classic Abel's summation formula, to prove interesting results about the average of some additive functions (namely, the number of distinct prime factors ω and the total number of prime factors Ω of a positive integer n) under digital constraints.

Using the same ideas, we will follow a similar path, in this article, in order to study the multiplicative functions under constraints on the sum of the digits of squares. Indeed, this is our second approach concerning this topic after the first study done in [1].

2. A lemma on power sums

We need a classical lemma estimating some expressions that will be needed later. In part a), the constant γ is Euler-Mascheroni's constant defined by the equation

$$\gamma = \lim_{x \to +\infty}\left(\sum_{n \leq x} \frac{1}{n} - \log x\right).$$

In part b), $\zeta(s)$ denotes the Riemann zeta function defined by the equations

$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^s}, & \text{if } s > 1, \\ \displaystyle\lim_{x \to +\infty}\left(\sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right), & \text{if } 0 < s < 1, \end{cases}$$

a proof can be found in [2, chapter 3].

Lemma 2.1. *As $x \to +\infty$, we have*

a) $\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$.

b) $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$, *if* $s > 0$, $s \neq 1$.

c) $\sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$, *if* $s > 1$.

d) $\sum_{n \le x} n^s = \frac{x^{s+1}}{s+1} + O(x^s)$, *if* $s \ge 0$.

3. Average of $\bar{\tau}$

Theorem 3.1. *Let* $q \ge 2$ *and* $m \ge 2$ *be integers such that* $(q-1, m) = 1$, *let* $r \in \mathbf{Z}$. *Then we have*

$$\sum_{n \le x} \bar{\tau}(n) = \frac{1}{m} x \log x + \frac{\alpha + 2\gamma - 1}{m} x + O((\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}/2}),$$

as $x \to +\infty$, *where* γ *stands for Euler-Mascheroni's constant,* $\sigma_{q,m} = \min_{j \in [\![1, m-1]\!]} \sigma_q\left(\frac{j}{m}\right)$ *(*σ_q *being the constant stated in Theorem* **A***) and*

$$\alpha = \sum_{j=1}^{m-1} e\left(-\frac{rj}{m}\right)\left(\int_1^{+\infty}\left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right)\frac{du}{u^2}\right).$$

Proof. Given x large enough, we have

$$\sum_{n \le x} \bar{\tau}(n) = \sum_{\substack{d,l \\ dl \le x \\ S(d^2) \equiv r \bmod m}} 1$$

$$= A_1 + A_2 - A_3,$$

where

$$A_1 = \sum_{\substack{d \le \sqrt{x} \\ S(d^2) \equiv r \bmod m}} \sum_{l \le x/d} 1,$$

$$A_2 = \sum_{l \le \sqrt{x}} \sum_{\substack{d \le x/l \\ S(d^2) \equiv r \bmod m}} 1$$

and

$$A_3 = \sum_{\substack{d \le \sqrt{x} \\ S(d^2) \equiv r \bmod m}} \sum_{l \le \sqrt{x}} 1.$$

First, we may write using the orthogonality relation (1.2)

$$A_1 = \sum_{\substack{d \le \sqrt{x} \\ S(d^2) \equiv r \bmod m}} \left\lfloor \frac{x}{d} \right\rfloor$$

$$= \frac{1}{m} \sum_{d \le \sqrt{x}} \left\lfloor \frac{x}{d} \right\rfloor \sum_{j=0}^{m-1} e\left(\frac{j}{m}(S(d^2) - r)\right)$$

$$= \frac{1}{m} x \sum_{d \le \sqrt{x}} \frac{1}{d} + \frac{1}{m} x \sum_{j=1}^{m-1} e\left(-\frac{rj}{m}\right) S_j(x) + O(\sqrt{x}), \tag{3.1}$$

where

$$S_j(x) = \sum_{d \le \sqrt{x}} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d}, \qquad \text{for each } j \in [\![1, m-1]\!].$$

The first sum can be estimated as a consequence of Lemma 2.1 giving

$$\frac{1}{m} x \sum_{d \le \sqrt{x}} \frac{1}{d} = \frac{1}{2m} x \log x + \frac{\gamma}{m} x + O(\sqrt{x}). \tag{3.2}$$

Next, using Abel's summation formula, we get

$$S_j(x) = \sum_{d \le \sqrt{x}} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d}$$

$$= \frac{1}{\sqrt{x}} \sum_{d \le \sqrt{x}} e\left(\frac{j}{m} S(d^2)\right) + \int_1^{\sqrt{x}} \left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^2}.$$

Thanks to Theorem **A**, we write for $j \in [\![1, m-1]\!]$

$$\left|\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right| \frac{1}{u^2} \ll \frac{(\log u)^{(1/2)\omega(q)+4}}{u^{1+\sigma_q(j/m)}}.$$

Hence, we obtain

$$\int_1^{\sqrt{x}} \left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^2} = \alpha_j + O\left(\frac{(\log x)^{(1/2)\omega(q)+4}}{x^{\sigma_q(j/m)/2}}\right), \tag{3.3}$$

where

$$\alpha_j = \int_1^{+\infty} \left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^2}, \qquad \text{for every } j \in [\![1, m-1]\!].$$

Using Theorem **A** again, we have

$$\frac{1}{\sqrt{x}}\sum_{d\le\sqrt{x}} e\left(\frac{j}{m}S(d^2)\right) = O\left(\frac{(\log x)^{(1/2)\omega(q)+4}}{x^{\sigma_q(j/m)/2}}\right). \tag{3.4}$$

Taking the identities (3.3) and (3.4) jointly and setting

$$\sigma_{q,m} = \min_{j\in[\![1,m-1]\!]} \sigma_q\left(\frac{j}{m}\right),$$

we find

$$S_j(x) = \alpha_j + O\left(\frac{(\log x)^{(1/2)\omega(q)+4}}{x^{\sigma_{q,m}/2}}\right).$$

Considering (3.2) and choosing $\sigma_{q,m}$ small enough, we go back to (3.1) in order to find

$$A_1 = \frac{1}{2m}x\log x + \frac{\alpha+\gamma}{m}x + O((\log x)^{(1/2)\omega(q)+4}x^{1-\sigma_{q,m}/2}), \tag{3.5}$$

where $\alpha = \sum_{j=1}^{m-1} e\left(-\frac{rj}{m}\right)\alpha_j$.

Next, we write

$$\begin{aligned} A_2 &= \frac{1}{m}\sum_{l\le\sqrt{x}}\sum_{d\le x/l}\sum_{j=0}^{m-1} e\left(\frac{j}{m}(S(d^2)-r)\right) \\ &= \frac{1}{m}\sum_{l\le\sqrt{x}}\sum_{d\le x/l} 1 + \frac{1}{m}\sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right)\sum_{l\le\sqrt{x}}\sum_{d\le x/l} e\left(\frac{j}{m}S(d^2)\right). \end{aligned}$$

It is easy to check, using Lemma 2.1, that

$$\begin{aligned} \frac{1}{m}\sum_{l\le\sqrt{x}}\sum_{d\le x/l} 1 &= \frac{1}{m}\sum_{l\le\sqrt{x}}\left(\frac{x}{l}+O(1)\right) \\ &= \frac{1}{2m}x\log x + \frac{\gamma}{m}x + O(\sqrt{x}). \end{aligned} \tag{3.6}$$

As a consequence of Theorem **A**, we get for $j\in[\![1,m-1]\!]$,

$$\begin{aligned} \sum_{l\le\sqrt{x}}\sum_{d\le x/l} e\left(\frac{j}{m}S(d^2)\right) &\ll \sum_{l\le\sqrt{x}} (\log x)^{(1/2)\omega(q)+4}\left(\frac{x}{l}\right)^{1-\sigma_q(j/m)} \\ &\ll (\log x)^{(1/2)\omega(q)+4}x^{1-\sigma_q(j/m)/2}. \end{aligned} \tag{3.7}$$

The last bound follows from Lemma 2.1.

Combining (3.6) and (3.7) and choosing $\sigma_{q,m}$ small enough gives

$$A_2 = \frac{1}{2m} x \log x + \frac{\gamma}{m} x + O((\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}/2}), \tag{3.8}$$

Finally, thanks to Theorem **A**, we obtain

$$\begin{aligned} A_3 &= \frac{1}{m} \sum_{d \le \sqrt{x}} (\sqrt{x} + O(1)) + \frac{1}{m} \sum_{j=1}^{m-1} \sum_{d \le \sqrt{x}} e\left(\frac{j}{m}(S(d^2) - r)\right)(\sqrt{x} + O(1)) \\ &= \frac{1}{m} x + O((\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}/2}), \end{aligned} \tag{3.9}$$

Gathering (3.5), (3.8) and (3.9) together, we get the desired conclusion. ■

4. Average of $\overline{\sigma_s}$

The case $s = 0$ was considered in the previous paragraph. We shall first consider $s > 0$ and deal carefully with the subcase $s = 1$.

THEOREM 4.1. *Let $q \ge 2$ and $m \ge 2$ be integers such that $(m, q-1) = 1$, let $r \in \mathbf{Z}$ and $s > 0$ be a real number. We assert that*

$$\sum_{n \le x} \overline{\sigma_s}(n) = \frac{\frac{\zeta(s+1)}{s+1} + \beta}{m} x^{s+1} + \begin{cases} O(x \log x), & \text{if } s = 1, \\ O(x^t), & \text{if } s \ne 1, \end{cases} \qquad \text{as } x \to +\infty,$$

where ζ stands for the Riemann zeta function defined in Lemma 2.1, $t = \max(1, s)$ and $\beta = \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right)\left(\int_1^{+\infty}\left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right)\frac{du}{u^{s+2}}\right)$.

PROOF. Let x be large enough, we may write

$$\begin{aligned} \sum_{n \le x} \overline{\sigma_s}(n) &= \sum_{n \le x} \sum_{\substack{d \mid n \\ S(d^2) \equiv r \bmod m}} \left(\frac{n}{d}\right)^s \\ &= \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \sum_{\substack{n \le x \\ n = hd}} \left(\frac{n}{d}\right)^s \\ &= \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \sum_{h \le x/d} h^s \\ &= \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \left\{ \frac{\left(\frac{x}{d}\right)^{s+1}}{s+1} + O\left(\frac{x^s}{d^s}\right) \right\}. \end{aligned} \tag{4.1}$$

Here, we should treat the subcases $s = 1$ and $s \neq 1$ separately, using in both cases Lemma 2.1 and applying (1.2).

- If $s = 1$, then

$$\begin{aligned}
\sum_{n \le x} \overline{\sigma_1}(n) &= \frac{1}{2} x^2 \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \frac{1}{d^2} + O\left(x \sum_{d \le x} \frac{1}{d} \right) \\
&= \frac{1}{2m} x^2 \sum_{j=0}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^2} + O(x \log x) \\
&= \frac{1}{2m} x^2 \left\{ -\frac{1}{x} + \zeta(2) + O\left(x^{-2}\right) \right\} \\
&\quad + \frac{1}{2m} x^2 \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^2} + O(x \log x).
\end{aligned}$$

- If $s \neq 1$, we set $t = \max(1, s)$ so that

$$\begin{aligned}
\sum_{n \le x} \overline{\sigma_s}(n) &= \frac{1}{s+1} x^{s+1} \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \frac{1}{d^{s+1}} + O\left(x^s \sum_{d \le x} \frac{1}{d^s} \right) \\
&= \frac{1}{(s+1)m} x^{s+1} \sum_{j=0}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^{s+1}} \\
&\quad + O\left(x^s \left\{ \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \right\} \right) \\
&= \frac{1}{(s+1)m} x^{s+1} \left\{ -\frac{x^{-s}}{s} + \zeta(s+1) + O(x^{-s-1}) \right\} \\
&\quad + \frac{1}{(s+1)m} x^{s+1} \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^{s+1}} + O(x^t).
\end{aligned}$$

Going back to (4.1), we can summarize by the following formula

$$\sum_{n \le x} \overline{\sigma_s}(n) = \frac{\zeta(s+1)}{(s+1)m} x^{s+1} + B + \begin{cases} O(x \log x), & \text{if } s = 1, \\ O(x^t), & \text{if } s \neq 1, \end{cases} \tag{4.2}$$

where

$$B = \frac{1}{(s+1)m} x^{s+1} \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^{s+1}}. \tag{4.3}$$

We apply Abel's formula to B in order to get, for $j \in [\![1, m-1]\!]$,

$$T_j(x) = \sum_{d \le x} \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^{s+1}}$$
$$= \frac{1}{x^{s+1}} \sum_{d \le x} e\left(\frac{j}{m} S(d^2)\right) + (s+1) \int_1^x \left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^{s+2}}.$$

As a consequence of Theorem **A**, the latter integral is again absolutely convergent. Therefore, we get

$$\int_1^x \left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^{s+2}} = \beta_j + O((\log x)^{(1/2)\omega(q)+4} x^{-s-\sigma_q(j/m)}), \quad (4.4)$$

with

$$\beta_j = \int_1^{+\infty} \left(\sum_{d \le u} e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^{s+2}}, \qquad \text{for every } j \in [\![1, m-1]\!].$$

Using Theorem **A** once again, we have

$$\frac{1}{x^{s+1}} \sum_{d \le x} e\left(\frac{j}{m} S(d^2)\right) = O((\log x)^{(1/2)\omega(q)+4} x^{-s-\sigma_q(j/m)}). \quad (4.5)$$

Considering the identities (4.4) and (4.5) jointly, we find

$$T_j(x) = (s+1)\beta_j + O((\log x)^{(1/2)\omega(q)+4} x^{-s-\sigma_q(j/m)}).$$

Going back to (4.3), we write

$$B = \frac{\beta}{m} x^{s+1} + O((\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}}), \quad (4.6)$$

with $\sigma_{q,m} = \min_{j \in [\![1, m-1]\!]} \sigma_q\left(\frac{j}{m}\right)$ and $\beta = \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right)\beta_j$.

Hence, taking (4.2) and (4.6) jointly enables to reach our result. ■

In order to find the average order of $\overline{\sigma_s}(n)$ for negative s, we shall set $u = -s$ where $u > 0$.

Theorem 4.2. *Let $q \ge 2$ and $m \ge 2$ be integers such that $(m, q-1) = 1$, let $r \in \mathbf{Z}$ and $s < 0$ be a real number. Thus*

$$\sum_{n\le x}\overline{\sigma_s}(n)=\frac{\zeta(1-s)}{m}x+\begin{cases}O(1), & \text{if } v<0,\\ O((\log x)^{(1/2)\omega(q)+4}x^{v}), & \text{if } v\ge 0 \text{ and } s\ne-\sigma_{q,m},\\ O((\log x)^{(1/2)\omega(q)+5}x^{1+s}), & \text{if } \sigma_q\left(\frac{j}{m}\right)=\sigma_{q,m},\\ \forall j\in[\![1,m-1]\!] \text{ and } s=-\sigma_{q,m}\ge -1,\end{cases}$$

as $x\to+\infty$, *where* ζ *is the Riemann zeta function*, $\sigma_{q,m}=\min_{j\in[\![1,m-1]\!]}\sigma_q\left(\frac{j}{m}\right)$ *and* $v=\max(1+s,1-\sigma_{q,m})$.

Proof. In fact, if we set $u=-s>0$, it follows from the orthogonality relation (1.2) that

$$\begin{aligned}\sum_{n\le x}\overline{\sigma_s}&=\sum_{n\le x}\sum_{\substack{d\mid n\\ S(d^2)\equiv r \bmod m}}\left(\frac{d}{n}\right)^u\\ &=\sum_{h\le x}\frac{1}{h^u}\sum_{\substack{d\le x/h\\ S(d^2)\equiv r \bmod m}}1\\ &=\Gamma_1+\Gamma_2, \end{aligned}\tag{4.7}$$

where

$$\Gamma_1=\frac{1}{m}\sum_{h\le x}\frac{1}{h^u}\left\lfloor\frac{x}{h}\right\rfloor$$

and

$$\Gamma_2=\frac{1}{m}\sum_{j=1}^{m-1}e\left(-\frac{rj}{m}\right)\sum_{h\le x}\frac{1}{h^u}\sum_{d\le x/h}e\left(\frac{j}{m}S(d^2)\right).$$

Therefore,

$$\begin{aligned}\Gamma_1&=\frac{1}{m}x\sum_{h\le x}\frac{1}{h^{1+u}}+O\left(\sum_{h\le x}\frac{1}{h^u}\right)\\ &=\frac{1}{m}x\{\zeta(1+u)+O(x^{-u})\}+\begin{cases}O(\log x), & \text{if } u=1,\\ O\left(\left\{\frac{x^{1-u}}{1-u}+\zeta(u)+O(x^{-u})\right\}\right), & \text{else},\end{cases}\\ &=\frac{\zeta(1+u)}{m}x+\begin{cases}O(\log x), & \text{if } u=1,\\ O(1), & \text{if } u>1,\\ O(x^{1-u}), & \text{if } u<1.\end{cases}\end{aligned}$$

Besides, Theorem **A** enables us to get for $j \in [\![1, m-1]\!]$,

$$\sum_{h \le x} \frac{1}{h^u} \sum_{d \le x/h} e\left(\frac{j}{m} S(d^2)\right) \ll \sum_{h \le x} \frac{1}{h^u} \left(\log \frac{x}{h}\right)^{(1/2)\omega(q)+4} \left(\frac{x}{h}\right)^{1-\sigma_q(j/m)}$$

$$\ll (\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_q(j/m)} \sum_{h \le x} \frac{1}{h^{1+u-\sigma_q(j/m)}}.$$

Thanks to Lemma 2.1,

$$\Gamma_2 = \begin{cases} O((\log x)^{(1/2)\omega(q)+5} x^{1-u}), & \text{if } \sigma_q\left(\frac{j}{m}\right) = \sigma_{q,m},\ \forall j \in [\![1, m-1]\!] \\ & \text{and } u = \sigma_{q,m}, \\ O((\log x)^{(1/2)\omega(q)+4} x^{1-u}), & \text{if } u < \sigma_{q,m}, \\ O((\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}}), & \text{if } u > \sigma_{q,m}. \end{cases}$$

A discussion on the possible orders of the real numbers 0, $1-u$ and $1-\sigma_{q,m}$ completes the proof. ■

5. Average of $\bar{\varphi}$

Theorem 5.1. *Let $q \ge 2$ and $m \ge 2$ be integers such that $(m, q-1) = 1$, let $r \in \mathbf{Z}$. We assert that*

$$\sum_{n \le x} \bar{\varphi}(n) = \frac{\frac{3}{\pi^2} + \rho}{m} x^2 + O(x \log x), \qquad \text{as } x \to +\infty,$$

where $\rho = \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right)\left(\int_1^{+\infty}\left(\sum_{d \le u} \mu(d) e\left(\frac{j}{m} S(d^2)\right)\right)\frac{du}{u^3}\right)$.

Proof. Let x be large enough, due to (1.2) and Lemma 2.1 we have

$$\sum_{n \le x} \bar{\varphi}(n) = \sum_{n \le x} \sum_{\substack{d \mid n \\ S(d^2) \equiv r \bmod m}} \mu(d) \frac{n}{d}$$

$$= \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \mu(d) \sum_{h \le x/d} h$$

$$= \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \mu(d) \left(\frac{x^2}{2d^2} + O\left(\frac{x}{d}\right)\right)$$

$$= \frac{1}{2m} x^2 \sum_{j=0}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \mu(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^2} + O(x \log x)$$

$$= \frac{1}{2m} x^2 \sum_{d=1}^{+\infty} \frac{\mu(d)}{d^2} + O\left(x^2 \sum_{d>x} \frac{1}{d^2}\right)$$

$$+ \frac{1}{2m} x^2 \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \mu(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^2} + O(x \log x)$$

$$= \frac{1}{2m} x^2 \sum_{d=1}^{+\infty} \frac{\mu(d)}{d^2} + C + O(x \log x), \tag{5.1}$$

where

$$C = \frac{1}{2m} x^2 \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le x} \mu(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^2}. \tag{5.2}$$

First, using the Möbius inversion formula (see [2, p. 32]), we get

$$\left(\sum_{d=1}^{+\infty} \frac{1}{d^2}\right)\left(\sum_{m=1}^{+\infty} \frac{\mu(m)}{m^2}\right) = \sum_{k=1}^{+\infty} \frac{1}{k^2} \left(\sum_{\substack{d,m \\ dm=k}} \mu(d)\right) = \sum_{k=1}^{+\infty} \frac{1}{k^2} \left(\sum_{m|k} \mu(m)\right) = 1,$$

so that

$$\sum_{d=1}^{+\infty} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2}. \tag{5.3}$$

Next, we may apply Abel's summation formula again to C in order to obtain

$$U_j(x) = \sum_{d \le x} \mu(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d^2}$$

$$= \frac{1}{x^2} \sum_{d \le x} \mu(d) e\left(\frac{j}{m} S(d^2)\right) + 2 \int_1^x \left(\sum_{d \le u} \mu(d) e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^3}.$$

The integral is trivially convergent. Subsequently, we get

$$\int_1^x \left(\sum_{d \le u} \mu(d) e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^3} = \rho_j + O\left(x^{-1}\right), \tag{5.4}$$

with

$$\rho_j = \int_1^{+\infty} \left(\sum_{d \le u} \mu(d) e\left(\frac{j}{m} S(d^2)\right)\right) \frac{du}{u^3}, \qquad \text{for each } j \in [\![1, m-1]\!].$$

Clearly, we have

$$\frac{1}{x^2}\sum_{d\le x}\mu(d)e\left(\frac{j}{m}S(d^2)\right) = O\left(x^{-1}\right). \tag{5.5}$$

Considering the identities (5.4) and (5.5) jointly, we find

$$U_j(x) = 2\rho_j + O\left(x^{-1}\right).$$

Going back to (5.2), we write

$$C = \frac{\rho}{m}x^2 + O(x), \tag{5.6}$$

with $\sigma_{q,m} = \min_{j\in[\![1,m-1]\!]}\sigma_q\left(\frac{j}{m}\right)$ and $\rho = \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right)\rho_j$.

Hence, considering (5.1), (5.3) and (5.6) together gives the desired result. ■

6. Average of $\bar{r}$

Theorem 6.1. *Let $q \ge 2$ and $m \ge 2$ be integers such that $(m, q-1) = 1$, let $r \in \mathbf{Z}$. We state that*

$$\sum_{n\le x}\bar{r}(n) = \frac{\pi + 4v}{m}x + O((\log x)^{(1/2)\omega(q)+4}x^{1-\sigma_{q,m}/2}), \qquad \text{as } x \to +\infty,$$

where

$$v = \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right)\left(\int_1^{+\infty}\left(\sum_{d\le u}\chi(d)e\left(\frac{j}{m}S(d^2)\right)\right)\frac{du}{u^2}\right)$$

and $\sigma_{q,m} = \min_{j\in[\![1,m-1]\!]}\sigma_q\left(\frac{j}{m}\right)$ (σ_q being the constant stated in Theorem **A***).*

Proof. Given x large enough, we have

$$\begin{aligned}\sum_{n\le x}\bar{r}(n) &= 4\sum_{n\le x}\sum_{\substack{d\mid n\\ S(d^2)\equiv r \bmod m}}\chi(d)\\ &= 4\sum_{\substack{dh\le x\\ S(d^2)\equiv r \bmod m}}\chi(d)\\ &= 4\sum_{\substack{d\le x\\ S(d^2)\equiv r \bmod m}}\chi(d)\sum_{h\le x/d}1\end{aligned}$$

$$= 4 \sum_{\substack{d \le x \\ S(d^2) \equiv r \bmod m}} \chi(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$= 4 \sum_{\substack{d \le \sqrt{x} \\ S(d^2) \equiv r \bmod m}} \chi(d) \left\lfloor \frac{x}{d} \right\rfloor + 4 \sum_{h \le \sqrt{x}} \sum_{\substack{\sqrt{x} < d \le x/h \\ S(d^2) \equiv r \bmod m}} \chi(d). \tag{6.1}$$

Since

$$\sum_{d=1}^{+\infty} \frac{\chi(d)}{d} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4},$$

the identity (1.2) gives

$$\sum_{\substack{d \le \sqrt{x} \\ S(d^2) \equiv r \bmod m}} \chi(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{\substack{d \le \sqrt{x} \\ S(d^2) \equiv r \bmod m}} \chi(d) \frac{x}{d} + O(\sqrt{x})$$

$$= \frac{1}{m} x \sum_{j=0}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le \sqrt{x}} \chi(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d} + O(\sqrt{x})$$

$$= \frac{1}{m} x \left(\sum_{d=1}^{+\infty} \frac{\chi(d)}{d} + O\left(\frac{1}{\sqrt{x}}\right) \right)$$

$$+ \frac{1}{m} x \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le \sqrt{x}} \chi(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d} + O(\sqrt{x})$$

$$= \frac{\pi}{4m} x + D + O(\sqrt{x}), \tag{6.2}$$

with

$$D = \frac{1}{m} x \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le \sqrt{x}} \chi(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d}. \tag{6.3}$$

Applying Abel's formula to D gives, for $j \in [\![1, m-1]\!]$,

$$V_j(x) = \sum_{d \le \sqrt{x}} \chi(d) \frac{e\left(\frac{j}{m} S(d^2)\right)}{d}$$

$$= \frac{1}{\sqrt{x}} \sum_{d \le \sqrt{x}} \chi(d) e\left(\frac{j}{m} S(d^2)\right) + \int_1^{\sqrt{x}} \left(\sum_{d \le u} \chi(d) e\left(\frac{j}{m} S(d^2)\right) \right) \frac{du}{u^2}.$$

The integral converges absolutely and following Theorem **A**

$$\int_1^{\sqrt{x}}\left(\sum_{d\le u}\chi(d)e\left(\frac{j}{m}S(d^2)\right)\right)\frac{du}{u^2}=v_j+O((\log x)^{(1/2)\omega(q)+4}x^{-\sigma_q(j/m)/2}),\tag{6.4}$$

where

$$v_j=\int_1^{+\infty}\left(\sum_{d\le u}\chi(d)e\left(\frac{j}{m}S(d^2)\right)\right)\frac{du}{u^2},\qquad \text{for every } j\in[\![1,m-1]\!].$$

Indeed, applying the identity

$$\chi(d)=\frac{e\left(\frac{d-1}{4}\right)+e\left(-\frac{d-1}{4}\right)}{2},\tag{6.5}$$

we are left with two sums of the type $\sum_{d\le x} e(\alpha d+\beta S(d^2))$ (with $\alpha=\frac{1}{4}$ and $\alpha=-\frac{1}{4}$) which can be handled exactly as in Theorem **A** since Mauduit and Rivat start by applying a Van der Corput inequality in [10, Lemme 15] that will cancel the term αd written above.

Obviously, according to the previous argument, we have

$$\frac{1}{\sqrt{x}}\sum_{d\le\sqrt{x}}\chi(d)e\left(\frac{j}{m}S(d^2)\right)=O((\log x)^{(1/2)\omega(q)+4}x^{-\sigma_q(j/m)/2}).\tag{6.6}$$

Considering the identities (6.4) and (6.6) together, we find

$$V_j(x)=v_j+O((\log x)^{(1/2)\omega(q)+4}x^{-\sigma_q(j/m)/2}).$$

Going back to (6.3), we write

$$D=\frac{v}{m}x+O((\log x)^{(1/2)\omega(q)+4}x^{1-\sigma_{q,m}/2}),\tag{6.7}$$

with $\sigma_{q,m}=\min_{j\in[\![1,m-1]\!]}\sigma_q\left(\frac{j}{m}\right)$ and $v=\sum_{j=1}^{m-1}e\left(-\frac{jr}{m}\right)v_j$.

Finally, it remains to bound the second sum in (6.1) as follows

$$\begin{aligned}\sum_{h\le\sqrt{x}}\ \sum_{\substack{\sqrt{x}<d\le x/h\\ S(d^2)\equiv r \bmod m}}\chi(d)&=\frac{1}{m}\sum_{h\le\sqrt{x}}\sum_{j=0}^{m-1}e\left(-\frac{jr}{m}\right)\sum_{\sqrt{x}<d\le x/h}\chi(d)e\left(\frac{j}{m}S(d^2)\right)\\&=\Lambda_1+\Lambda_2-\Lambda_3.\end{aligned}\tag{6.8}$$

with

$$\Lambda_1 = \frac{1}{m} \sum_{h \le \sqrt{x}} \sum_{\sqrt{x} < d \le x/h} \chi(d),$$

$$\Lambda_2 = \frac{1}{m} \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{h \le \sqrt{x}} \sum_{d \le x/h} \chi(d) e\left(\frac{j}{m} S(d^2)\right)$$

and

$$\Lambda_3 = \frac{\sqrt{x}}{m} \sum_{j=1}^{m-1} e\left(-\frac{jr}{m}\right) \sum_{d \le \sqrt{x}} \chi(d) e\left(\frac{j}{m} S(d^2)\right).$$

But, $\sum \chi(d)$, between any limits, is 0 or ± 1, hence

$$\Lambda_1 \ll \sqrt{x}. \tag{6.9}$$

Furthermore, the identity (6.5) and its following remark imply

$$\Lambda_3 \ll (\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}/2} \tag{6.10}$$

and

$$\sum_{h \le \sqrt{x}} \sum_{d \le x/h} \chi(d) e\left(\frac{j}{m} S(d^2)\right) \ll (\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}} \sum_{h \le \sqrt{x}} \frac{1}{h^{1-\sigma_{q,m}}}$$

$$\ll (\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_{q,m}/2}. \tag{6.11}$$

The last bound follows from Lemma 2.1.

Thus, putting (6.9), (6.10) and (6.11) in (6.8), then combining (6.2), (6.7) and (6.8) jointly in (6.1) allows to reach our result. ■

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References

[1] K. Aloui, On the order of magnitude of some arithmetical functions under digital constraint I, to appear in Proceedings of the Indian Academy of Sciences—Mathematical Sciences.

[2] T. M. Apostol, Introduction to analytic number theory, Undergraduate Texts in Mathematics, New York-Heidelberg, Springer-Verlag (1976).

[3] J. Bésineau, Indépendance statistique d'ensembles liés à la fonction "somme des chiffres", Acta Arithmetica **20** (1972), 401–416.

[4] J. Coquet, Sur les fonctions Q-multiplicatives et Q-additives, Thèse $3^{ème}$ cycle, Orsay (1975).

[5] H. Delange, Sur les fonctions q-additives ou q-multiplicatives, Acta Arithmetica **21** (1972), 285–298.

[6] M. Drmota, C. Mauduit, J. Rivat, Primes with an average sum of digits, Compositio Mathematica **145** (2009), 271–292.

[7] A. O. Gelfond, Sur les nombres qui ont des propriétés additives et multiplicatives données, Acta Arithmetica **13** (1968), 259–265.

[8] G. H. Hardy, E. M. Wright, An Introduction to the Theory of Numbers, Fifth edition, Oxford Univ. Press, New York (1979).

[9] I. Kàtai, Distribution of q-additive functions, in "Probability Theory and Applications", Kluwer Academic, Dordrecht (1992), 309–318.

[10] C. Mauduit, J. Rivat, La somme des chiffres des carrés, Acta Mathematica **203** (2009), 107–148.

[11] M. Mkaouar, W. Wannès, On the number of restricted prime factors of an integer, Acta Mathematica Hungarica **143 (1)** (2014), 88–95.

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