HIROSHIMA MATH. J.
46 (2016), 1–36

The space of geometric limits of abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$

Hyungryul BAIK and Lucien CLAVIER

(Received October 3, 2013)
(Revised February 12, 2015)

ABSTRACT. We give a local description of the topology of the space of all geometric limits of closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. More precisely, we give geometric descriptions for all possible neighborhoods of a point of this space. Intuition from hyperbolic geometry plays an important role by identifying $\mathrm{PSL}_2(\mathbb{C})$ with the group of isometries of $\mathbb{H}^3$. The tools and ideas developed in the authors' previous paper on one-generator closed subgroups of $\mathrm{PSL}_2(\mathbb{R})$ allow one to reduce this problem to a problem about the geometric limits of certain closed subgroups of $\mathbb{C}$ and $\mathbb{C}^*$.

1. Introduction

The present article was announced in [3], in which the authors, motivated by the desire to understand the closure of the faithful discrete type-preserving $\mathrm{PSL}_2(\mathbb{C})$-representations of the fundamental group of the once-punctured torus, gave a complete description of the closure of the space of one-generator closed subgroups of $\mathrm{PSL}_2(\mathbb{R})$ for the Chabauty topology. See [5] for a general exposition of Chabauty topology; we also included in [3] a mini-History of Chabauty topology and related topics.

In the world of geometric limits of Kleinian groups, a sequence of infinite cyclic groups each of which is generated by one hyperbolic isometry can converge to a subgroup isomorphic to $\mathbb{Z}^2$, whose generators are both parabolic isometries. This fact can be equivalently stated using Chabauty topology, a topology one puts on the space of the closed subgroups of $\mathrm{PSL}_2(\mathbb{C})$. The existence of such a behaviour was first observed by Jorgensen. See [9], Section 5 and [11], Example 9.14 for more detail.

A natural question arising is how to find conditions on a given sequence of groups for the limit group to exist, and to describe this limit group.

In this paper we answer this question for an arbitrary sequence of abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ by using an "exhaustion of cases" approach (see Subsection 2.4: Strategy).

The first author was partially supported by Bill Thurston's NSF grant DMS-0910516 and the ERC Grant Nb. 10160104.

2010 *Mathematics Subject Classification*. 30F40, 20H10.
Key words and phrases. Chabauty topology, Kleinian Group, Geometric Limit.

Here is now a summary of the paper.

Section 2. We recall some properties of the Chabauty topology, provide a reminder of results from [3], and basic properties of subgroups of $\mathrm{PSL}_2(\mathbb{C})$. These have been included mainly for notational purposes.

Section 3. The carefully chosen matrix representation of non-parabolic isometries of $\mathbb{H}^3$ (Subsections 3.1 and 3.2) leads to Propositions 1 and 2: the space of all non-trivial, non-parabolic, closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is homeomorphic to $\Theta \times (\mathscr{C}(\mathbb{C}^*)\backslash\{1\})$; the space of all non-trivial discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ generated by one elliptic (resp. hyperbolic) generator is homeomorphic to $\Theta \times \mathbb{N}_{\geq 2}$ (resp. $\Theta \times (\mathbb{C}\backslash\overline{\mathbb{D}})$), where Θ is the space of pairs of distinct points on $\hat{\mathbb{C}}$ (Subsection 3.3). The matrix representation of parabolic isometries of $\mathbb{H}^3$ yields Propositions 4 and 5: $\mathbf{P}_1'$ (resp. $\mathbf{P}_2'$), the space of all non-trivial discrete cyclic (resp. abelian) parabolic subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is a 4-twist fiber bundle of $\mathbb{C}^*$ over S^2 (resp. is homeomorphic to $S^2 \times (\mathbb{C}^2\backslash\{0\})$).

Section 4. We use the Reduction Lemma introduced in [3] to reduce the problem of convergence of closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ to some problem of convergence of closed subgroups of $\mathbb{C}$ (Subsection 4.1). We give geometrical interpretations for the parameters introduced (Subsection 4.2) and for the closed subgroups of $\mathbb{C}$ we are left to study in the enrichment case (Subsection 4.3).

Section 5. We describe the whole exhaustion of cases for sequences of closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. This exhaustion is radically simplified by the reducing results in Subsection 4.1, and involves continued fraction in a rather unexpected way.

Section 6. We provide local models for neighborhoods of parabolic subgroups inside the space of all closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. This section is independent from Section 5.

Section 7. This short section consists only of a Summary Statement, collection of the different results in this paper, and of a short conclusion.

2. Preliminaries

2.1. Chabauty topology. Recall that the Chabauty topology of a locally compact group G is the topology on the space $\overline{F}(G)$ of all its closed subgroups induced by the Hausdorff distance on the one-point compactification $\overline{G}$ of G (see [3] for instance, or [4]). Equipped with this topology, $\overline{F}(G)$ becomes a compact metric space; $\overline{F}(G)$, together with the Hausdorff distance d_H, will be usually referred to as the *Chabauty space* of G. We write it $\mathscr{C}(G)$. In the context of Kleinian groups, the limit of a convergent sequence in the Chabauty topology is called the *geometric limit* of the sequence.

In the previous paper [3] of the authors, we obtained the following theorem, where $\mathbf{C}$ (resp. $\mathbf{E}$, $\mathbf{H}$, $\mathbf{P}$) is the closure of the space of discrete cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$ (resp. discrete subgroups generated by an elliptic, hyperbolic, parabolic element of $\mathrm{PSL}_2(\mathbb{R})$).

Theorem 1. *The space of geometric limits of closed subgroups of* $\mathrm{PSL}_2(\mathbb{R})$ *with one generator is* $\mathbf{C} = \mathbf{E} \cup \mathbf{H}/\sim$, *where*

(1) $\mathbf{E}$ *is a wedge sum of countably many* 2*-spheres* $D_n/\partial D_n$, *which accumulate to a disk* D_∞ *and to the cone* $\mathbf{P}$ *on the circle* ∂D_∞. *(see* 5 *in* [3]*).*

(2) $\mathbf{H}$ *is the cone on a closed Möbius band, the inside of which is foliated by "bent" open Möbius bands, which accumulate to an open Möbius band* M_0 *and the cone* $\mathbf{P}$ *on the circle* ∂M_0 *(see Figure* 6 *in* [3]*).*

(3) $\sim$ *represents the gluing of* $\mathbf{E}$ *and* $\mathbf{H}$ *along* $\mathbf{P}$.

The geometric convergence of Kleinian groups is not easy to handle directly in general; we developed a tool to reduce the problem of the convergence for the Hausdorff topology in some complicated space (e.g. $\mathbf{C} \subset \mathscr{C}(\mathrm{PSL}_2(\mathbb{R}))$) to the convergence for the Hausdorff topology in a better-known space (e.g. some particular families of closed subsets of $\mathbb{C}$). See Proposition 6 in Section 4.

2.2. Transformations of $\mathrm{PSL}_2(\mathbb{C})$**.** Note, after identification of $\mathrm{PSL}_2(\mathbb{C})$ with the group $\mathrm{Aut}(\mathbb{H}^3)$ of conformal automorphisms of $\mathbb{H}^3$, that each element of $\mathrm{PSL}_2(\mathbb{C})$ acts on the sphere at infinity $\hat{\mathbb{C}} = \partial\mathbb{H}^3$. Let us recall that the conformal automorphisms (i.e. isometries) of $\mathbb{H}^3$ are of three types:

- *parabolic* if they have one fixed point in $\hat{\mathbb{C}} = \partial\mathbb{H}^3$.
- *elliptic* if they have two fixed points in $\hat{\mathbb{C}}$, and act on $\mathbb{H}^3$ as a rotation along the axis defined by these fixed points.
- *hyperbolic* if they have two fixed points in $\hat{\mathbb{C}}$, and act on $\mathbb{H}^3$ as a translation with skew along the axis defined by these fixed points.

More precisely, elliptic (resp. hyperbolic) elements of $\mathrm{PSL}_2(\mathbb{C})$ are conjugated to a map $[z \mapsto az]$, by sending one of the fixed point to 0, and the other to ∞; we have $a \in S^1$ (resp. $a \in \mathbb{C}^* \setminus S^1$), and we call a the *multiplier* of the element. Note that a is only defined up to the inverse mapping $[z \mapsto z^{-1}]$, except when there is a way to decide which of the fixed points is sent to 0 and which is sent to ∞.

2.3. Closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$**.** For each isometry $g \in \mathrm{Aut}(\mathbb{H}^3)$ let $\mathrm{Fix}(g)$ be the set of fixed points of g on the sphere at infinity $\hat{\mathbb{C}}$. Recall that abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ are exactly the subgroups $G \subset \mathrm{PSL}_2(\mathbb{C})$ such that all elements have the same fixed points at infinity, i.e.

$$\forall g_1, g_2 \in G \backslash \{1\}, \qquad \mathrm{Fix}(g_1) = \mathrm{Fix}(g_2)$$

Note that a non-trivial abelian subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ can be of two rather different kinds, namely:

- G is a parabolic subgroup, i.e. each of its non-trivial element is parabolic and fixes the same point $z \in \hat{\mathbb{C}}$. Then G is conjugated to a subgroup Γ of translations of $\hat{\mathbb{C}}$, under a map sending z to ∞; for z fixed, if this map is chosen once and for all, G is entirely and uniquely described by Γ (see Section 3.4).
- G is a non-parabolic subgroup, i.e. each of its non-trivial element is non-parabolic and fixes the same points $z_1, z_2 \in \hat{\mathbb{C}}$. Then the space of all multipliers of its elements is a subgroup Ξ of $\mathbb{C}^*$, and G is entirely described by the unordered pair $\{z_1, z_2\}$ and by Ξ.

As a generalization of [3], the authors were originally interested solely in the closure of the space of cyclic closed subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Since, as we will see, this closure already contains the space of abelian closed parabolic subgroups of $\mathrm{PSL}_2(\mathbb{C})$, it is very natural to consider also the closure of the space of all abelian closed subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Thus, let us define $\mathbf{C}_1$ (resp. $\mathbf{C}_2$) to be the closure of the space of cyclic (resp. abelian) closed subgroups of $\mathrm{PSL}_2(\mathbb{C})$, for the Chabauty topology on the space $\mathscr{C}(\mathrm{PSL}_2(\mathbb{C}))$ of all closed subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Also, define $\mathbf{P}_1$ (resp. $\mathbf{P}_2$) to be the closure of the space of cyclic (resp. abelian) closed parabolic subgroups of $\mathrm{PSL}_2(\mathbb{C})$. There will be no special need for analogs for the spaces $\mathbf{E}$ and $\mathbf{H}$ introduced in [3].

2.4. Strategy. One goal of this article is to understand all limits of sequences G_n in $\mathbf{C}_2$. Let us explain in more detail what we mean by this. In the present paper, we associate to each $G \in \mathbf{C}_2$ a finite list of parameters p_j (e.g. fixed points, multiplicity, etc.) that lie in some compact sets K_j where convergence is well understood (e.g. $[0, \infty]$, S^1, SO_3). Then, given a sequence G_n such that all parameters $p_{j,n}$ converge in K_j to some $p_{j,\infty}$, G_n converges to a group G which we describe explicitely from the limit parameters $p_{j,\infty}$.

If $G_n \to G$, there is a subsequence ψ such that all parameters $p_{j,\psi(n)}$ converge (this is because all $p_{j,n}$ lie in compact sets). Call these limit parameters $p_{j,\psi}$. We describe $G = G(p_{\cdot,\psi})$ explicitly from them. Moreover, if we want to know whether a sequence G_n converges, it is enough to look at all possible subsequences ψ of G_n such that all $p_{j,\psi(n)}$ converge. Then $G_n \to G$

if, and only if all $G(p_{\cdot,\psi})$ are equal. In this very specific sense, we regard this as describing all possible limits of sequences G_n in $\mathbf{C}_2$. We used this strategy in [3] to show Theorem 1.

Here is now a list of the parameters we introduce and a brief description of them.

- If G is non-parabolic: $z_1, z_2 \in \hat{\mathbb{C}}$ (alternatively $(\zeta_1, \xi_1), (\zeta_2, \xi_2) \in \mathbb{CP}^1$): the fixed points common to all elements of G; $\Xi \in \mathscr{C}(\mathbb{C}^*)$ the set of multipliers of elements of G (see Subsection 3.2).
- If G is parabolic: $z \in \hat{\mathbb{C}}$ (alternatively $(\zeta, \xi) \in \mathbb{CP}^1$): the fixed point common to elements of G; $\Gamma \in \mathscr{C}(\mathbb{C})$, the set of multipliers of elements of G (the multiplier here is not well-defined, so we need some normalization. See Subsection 3.4 for a precise definition).

Case 0: If G_n are all parabolic, the parameters $z_\infty = \lim z_n$ and $\Gamma_\infty = \lim \Gamma_n$ completely describe G_∞ (see Subsection 3.4 and in particular Corollary 2).

Case 1: If G_n are all non-parabolic, the parameters $z_{1,\infty}$, $z_{2,\infty}$ and Ξ_∞ completely describe G_∞ provided that $z_{1,\infty} \neq z_{2,\infty}$, by Theorem 3.

Case 2: If G_n are all non-parabolic and $z_{1,\infty} = z_{2,\infty}$, we introduce the following parameters (see Section 4):

- $R \geq 1$: the inverse of the spherical distance between z_1 and z_2 (in the case considered, $R_n \to \infty$).
- $\omega \in [0, 2\pi]$: the opposite of the angle between the horizontal line and the line passing through z_1 and z_2.
- $\Gamma = R\,\mathrm{Log}(\Xi)$: multiplying by R is a way of zooming around $0 \in \mathrm{Log}\,\Xi$ (i.e. around $1 \in \Xi \subset \mathbb{C}^*$).

In Case 2, the limit parameters z_∞, ω_∞ and Γ_∞ now completely determine G_∞. Compared to Ξ in Case 1, however, Γ_∞ is rather esoteric in that it is obtained by zooming further and further (as $R_n \to \infty$) around $0 \in \mathrm{Log}\,\Xi_n$, with Ξ_n (possibly) becoming denser and denser.

As a notational reminder, the letter H (resp. P) will always be reserved for non-parabolic (resp. parabolic) elements of $\mathrm{PSL}_2(\mathbb{C})$, *$G$ for subgroups of* $\mathrm{PSL}_2(\mathbb{C})$, *$\Xi$ for multiplicative subgroups of $\mathbb{C}^*$ and Γ for additive subgroups of* $\mathbb{C}$.

Before further study, it will be helpful to have the following example in mind. Consider a sequence of elements α_n of $\mathrm{PSL}_2(\mathbb{C})$ of the form

$$\alpha_n : z \mapsto \rho_n^2 e^{2\pi i/n}(z - a_n) + a_n.$$

Each α_n is a hyperbolic isometry which fixes both a_n and ∞. We will choose both ρ_n and a_n so that (α_n) converges to $[z \mapsto z + 1]$ and (α_n^n) converges to the parabolic element $[z \mapsto z + \gamma]$ with $\gamma \notin \mathbb{R}$.

For that purpose, set

$$a_n = \frac{1}{1 - \rho_n^2 e^{2\pi i/n}}$$

so that $\alpha_n(0) = 1$. In particular, this forces $\rho_n \to 1$.

When $\rho_n = 1$, $\alpha_n^n = \mathrm{id}$ for all n, so (α_n^n) obviously converges to the identity map. When $\rho_n = 1 + \frac{\alpha}{n}$ for some constant α, (α_n^n) is not convergent. When $\rho_n = 1 + o\left(\frac{1}{n^2}\right)$ then α_n^n converges to a parabolic element, and when ρ_n is defined by

$$\rho_n = 1 + \frac{\alpha}{n^2}$$

then (α_n^n) converges to the map $\left[z \mapsto z - i\frac{\alpha}{\pi}\right]$. In this example, the sequence of infinite cyclic groups $\langle \alpha_n \rangle$ converges to $\mathbb{Z}^2$ generated by $[z \mapsto z + 1]$ and $\left[z \mapsto z - i\frac{\alpha}{\pi}\right]$.

This example is a part of Case 2 with $R_n = \sqrt{a_n^2 + 1}$, $\omega_n = 0$ and $\operatorname{Log} \Xi_n$ the lattice generated by $2i\pi/n$ and $2 \log \rho_n$. We saw that it is not easy to predict the limit subgroup as $a_n \to \infty$ (i.e. $R_n \to \infty$) and $\operatorname{Log} \Xi_n$ gets denser: this limit subgroup depends crucially on how fast $a_n \to \infty$ (so that $\alpha_n(0) = 1$) and how fast $\rho_n \to 1$.

We solve this issue in Subsection 5.3 by introducing a parameter θ whose expansion in continuous fraction contains most of the information we need to understand *explicitely* the limit group G_∞. See Section 5 for the explicit algorithm.

3. Matrix representations

3.1. $\mathbb{CP}^1$ as a quotient. Of crucial importance in [3] was the particular matrix representations of elliptic and hyperbolic isometries.

To mimic this (the upshot being Propositions 1 and 2), let us first start by finding a subspace of $\mathbb{C}^2 \backslash \{0\}$ mapped homeomorphically to $\mathbb{C}$ via the projectivizing map $[(\zeta, \xi) \mapsto \zeta/\xi]$, and which stays away from both 0 and ∞ (i.e. has a compact closure that does not contains 0). The classical choice of such a subspace as the plane $\mathbb{C} \times \{1\}$ does not answer this condition, since it is not compact; the choice of the sphere in $\mathbb{C} \times \mathbb{R} \subset \mathbb{C}^2$ of radius 1/2 and centered at (0,1/2), with the south pole removed:

$$S^2 \backslash S = \{(\zeta, \xi) \in \mathbb{C}^2 \backslash \{0\}; \xi \in (0, 1], |\zeta|^2 + (\xi - 1/2)^2 = 1/4\}$$

does not either, since its closure S^2 contains the south pole $S = (0, 0)$.

Let us therefore define $\mathbb{D}^+$ to be the unit upper hemisphere in $\mathbb{C} \times \mathbb{R} \subset \mathbb{C}^2$:

$$\mathbb{D}^+ = \{(\zeta, \xi) \in \mathbb{C}^2 \backslash \{0\}; \xi \in (0, 1], |\zeta|^2 + \xi^2 = 1\}.$$

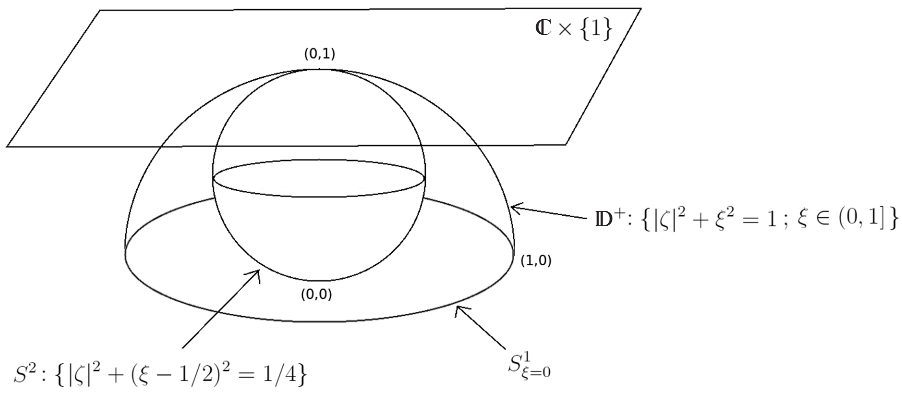

Fig. 1. Three models for $\mathbb{CP}^1$ with a point removed: the plane $\mathbb{C} \times \{1\}$, the sphere S^2 minus its south pole $(0,0)$, and the upper unit hemisphere $\mathbb{D}^+$.

Note that $(\zeta, \xi) \mapsto [\zeta : \xi]$ induces an homeomorphism

$$\overline{\mathbb{D}^+}/S^1_{\xi=0} \cong \mathbb{CP}^1.$$

Remark 1. *It is straightforward to see that the stereographic projection from* $\overline{\mathbb{D}^+}$ *to* S^2 *is given by*

$$(\zeta, \xi) \mapsto (\zeta\xi, \xi^2).$$

Let us define

$$\mathbb{D}^- = (\overline{\mathbb{D}^+}/S^1_{\xi=0}) \backslash \{(0,1)\}$$

and

$$S^1_{\mathrm{eq}} = \left\{ (\zeta, \xi) \in \mathbb{D}^+ ; \xi = \frac{\sqrt{2}}{2} \right\}.$$

Under the stereographic projection, $\mathbb{D}^+$, $\mathbb{D}^-$ and S^1_{eq} are respectively mapped to $S^2 \backslash S$, $S^2 \backslash N$ (where S and N are the south and north poles $(0,0)$ and $(0,1)$) and to the equator of S^2 for which ξ is constantly 1/2.

3.2. Matrix representations of elliptic and hyperbolic isometries. In this section, we show how to represent every elliptic and hyperbolic element of $\mathrm{PSL}_2(\mathbb{C})$ as a 2×2 matrix.

Recall that an elliptic (resp. hyperbolic) element of $\mathrm{PSL}_2(\mathbb{C})$ fixes a unique axis joining two distinct points of $\hat{\mathbb{C}}$, and that it then acts like a rotation (resp. a screw motion) around this axis. Moreover, an elliptic (resp. hyperbolic) element is entirely determined by its two fixed points and its rotational multiplier (resp. screw motion multiplier).

Let H be either elliptic or hyperbolic, with multiplier $a \in \mathbb{C}^*$; suppose first that the two distinct points z_1 and z_2 of H are in $\mathbb{C}$. Then it is straightforward to check that the matrix of $\mathrm{PSL}_2(\mathbb{C})$ representing H is

$$\frac{1}{\sqrt{a}(z_2 - z_1)} \begin{pmatrix} az_2 - z_1 & z_1 z_2 (a-1) \\ 1-a & z_2 - az_1 \end{pmatrix}.$$

Indeed, $\phi = \left[z \mapsto \frac{z - z_1}{z - z_2} \right]$ is an automorphism of $\hat{\mathbb{C}}$ mapping z_1 to 0 and z_2 to ∞; moreover, the element with multiplier a and fixed points 0 and ∞ is simply $[z \mapsto az]$. Note that exchanging z_1 with z_2 while replacing a by a^{-1} does not change this matrix, thus the ordering of the two fixed point does not matter, as long as we study the subgroups, not particular elements, of $\mathrm{PSL}_2(\mathbb{C})$.

This description using $z_i \in \mathbb{C}$ has the drawback of blowing off when one of the fixed points approaches $\infty \in \hat{\mathbb{C}}$. To circumvent this, let us replace z_1 and z_2 by projectivized quantities ζ_1/ξ_1 and ζ_2/ξ_2, with $(\zeta_i, \xi_i) \in \mathbb{D}^+$. Then the matrix becomes

$$\frac{1}{\sqrt{a}} \begin{pmatrix} 1 + \mu\zeta_2\xi_1 & \mu\zeta_1\zeta_2 \\ -\mu\xi_1\xi_2 & 1 - \mu\zeta_1\xi_2 \end{pmatrix}$$

with $\mu = \dfrac{a-1}{\zeta_2\xi_1 - \zeta_1\xi_2}$.

DEFINITION 1. For every pair of distinct points $(\zeta_1, \xi_1), (\zeta_2, \xi_2) \in \mathbb{D}^+$, and for every $a \in \mathbb{C}^*$ satisfying $|a| = 1$ (resp. $|a| \neq 1$), the elliptic (resp. hyperbolic) element of $\mathrm{PSL}_2(\mathbb{C})$ fixing both $[\zeta_i : \xi_i] \in \mathbb{CP}^1$ and with multiplier a is written

$$H_{(\zeta_1, \xi_1), (\zeta_2, \xi_2), a} = \frac{1}{\sqrt{a}} \begin{pmatrix} 1 + \mu\zeta_2\xi_1 & \mu\zeta_1\zeta_2 \\ -\mu\xi_1\xi_2 & 1 - \mu\zeta_1\xi_2 \end{pmatrix}$$

with $\mu = \frac{a-1}{\zeta_2\xi_1 - \zeta_1\xi_2}$.

With the choice of $\mathbb{D}^+$ for representing $\mathbb{CP}^1 \backslash \{[1:0]\}$, $H_{(\zeta_1, \xi_1), (\zeta_2, \xi_2), a}$ can be extended continuously when either one of the (ζ_i, ξ_i) approaches the boundary of $\mathbb{D}^+$. We write this continuation in the obvious way, e.g. if $\xi_1 = 0$ (thus $|\zeta_1| = 1$):

$$H_{(\zeta_1, 0), (\zeta_2, \xi_2), a} = \frac{1}{\sqrt{a}} \begin{pmatrix} 1 & \mu\zeta_1\zeta_2 \\ 0 & 1 - \mu\zeta_1\xi_2 \end{pmatrix} = \frac{1}{\sqrt{a}} \begin{pmatrix} 1 & 1-a \\ 0 & a \end{pmatrix},$$

which does not depend on ζ_1; thus we can think of the map $[(x, y, a) \mapsto H_{x,y,a}]$ as taking its first two components inside $\mathbb{CP}^1 = \overline{\mathbb{D}^+}/S^1_{q=0}$.

Denote by Θ the space of pairs of distinct points in S^2:

$$\Theta = ((\mathbb{CP}^1 \times \mathbb{CP}^1) \backslash \Delta)/((x, y) \sim (y, x)),$$

where Δ is the diagonal $\{(x, x); x \in \mathbb{CP}^1\}$; we definitely think of $\mathbb{CP}^1$ here as being $\overline{\mathbb{D}^+}/S^1_{q=0}$.

The following two propositions hold.

Proposition 1. *The space of all non-trivial non-parabolic closed abelian subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *is homeomorphic to*

$$\Theta \times (\mathscr{C}(\mathbb{C}^*)\backslash\{1\}).$$

Proof. It is immediate to see that the desired homeomorphism is induced by

$$((\mathbb{CP}^1 \times \mathbb{CP}^1)\backslash\Delta) \times (\mathscr{C}(\mathbb{C}^*)\backslash\{1\}) \to \mathbf{C}_2$$

$$(x, y, \Xi) \mapsto \{H_{x,y,a}; a \in \Xi\},$$

where by convention $H_{x,y,1}$ is always the identity of $\mathrm{PSL}_2(\mathbb{C})$. Indeed, the map above descends to a homeomorphism from $\Theta \times (\mathscr{C}(\mathbb{C}^*)\backslash\{1\})$ onto its image, which is the space of all non-trivial non-parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. $\square$

Proposition 2. *The space of all non-trivial discrete subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *generated by one elliptic generator is homeomorphic to*

$$\Theta \times \mathbb{N}_{\geq 2}.$$

The space of non-trivial discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ *generated by one hyperbolic generator is homeomorphic to*

$$\Theta \times (\mathbb{C}\backslash\overline{\mathbb{D}}),$$

where $\mathbb{D}$ *is the unit disk in* $\mathbb{C}$.

Proof. This is similar to the proof of Proposition 1. $\square$

Before studying parabolic subgroups, let us give a more recognizable form to Θ.

3.3. Θ as a subspace of $\mathbb{CP}^2$. This subsection is due to John H. Hubbard, and we thank him for explaining it to us.

For each pair $([\zeta_1 : \xi_1], [\zeta_2 : \xi_2]) \in (\mathbb{CP}^1)^2$, consider the polynomial

$$P_{[\zeta_1:\xi_1],[\zeta_2:\xi_2]}(x) = (\xi_1 x - \zeta_1)(\xi_2 x - \zeta_2)$$

defined up to a multiplicative constant.

Note that $P_{[\zeta_1:\xi_1],[\zeta_2:\xi_2]}$ and $P_{[\zeta_3:\xi_3],[\zeta_4:\xi_4]}$ differ by a multiplicative constant if and only if $\{[\zeta_1 : \xi_1], [\zeta_2 : \xi_2]\}$ coincides with $\{[\zeta_3 : \xi_3], [\zeta_4 : \xi_4]\}$ as a set. Also, we have $[\zeta_1 : \xi_1] = [\zeta_2 : \xi_2]$ if and only if the square-rooted discriminant of $P_{[\zeta_1:\xi_1],[\zeta_2:\xi_2]}$

$$\sqrt{\Delta_P} = \pm(\zeta_2\xi_1 - \zeta_1\xi_2)$$

is zero (remark that this number is, up to sign, the denominator of the quantity μ defined in the matrix representations above, Definition 1).

Therefore, the map

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$$

$$[\zeta_1 : \xi_1], [\zeta_2 : \xi_2] \mapsto [\xi_1\xi_2 : -\zeta_2\xi_1 - \zeta_1\xi_2 : \zeta_1\zeta_2]$$

descends to a homeomorphism between Θ and $\mathbb{CP}^2$ minus the curve (actually a sphere) of homogeneous equation

$$Y^2 - 4XZ = 0.$$

3.4. Matrix representations of parabolic isometries. In this section, we show how to represent every parabolic element of $\mathrm{PSL}_2(\mathbb{C})$ as a 2×2 matrix. This will lead to a complete description of $\mathbf{P}_1$ and $\mathbf{P}_2$.

It is well known that any two parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$ are conjugate. We saw in [3] that we could find a way of normalizing parabolic elements, by asking them to be conjugated to a particular parabolic element by a chosen matrix. This led to a homeomorphism between the space of non-trivial parabolic elements of $\mathrm{PSL}_2(\mathbb{R})$ and $S^1 \times \mathbb{R}^*$, the upshot being a description of the space of parabolic cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$. We cannot immediately extend this method to the case of $\mathrm{PSL}_2(\mathbb{C})$; we will actually see in a while that the space of non-trivial parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$ is a non-trivial bundle (see Proposition 3).

Write $\mathscr{P}$ for the space of all non-trivial parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$. Since for each point $z \in S^2 = \hat{\mathbb{C}}$ the space of parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$ fixing z is homeomorphic to $\mathbb{C}^*$, we see that $\mathscr{P}$ is a SO_2-bundle, with base space S^2 and fiber $\mathbb{C}^*$. Recall that a G-bundle with fiber F is a fiber bundle such that the topological group G acts on F as a group of symmetries, and such that transition functions between charts are continuous.

Since $\mathscr{P}$ is a SO_2-bundle over a sphere S^2, one can understand its structure via its clutching map. We briefly recall the basic description here. Consider S^2 as the union of two disks $\mathbb{D}^+$ and $\mathbb{D}^-$ glued along the equator S^1_{eq}. In general, if trivialized fiber bundles over $\mathbb{D}^\pm$ with fiber F and structure group G, and a map $f : S^1_{\mathrm{eq}} \to G$ (called the clutching map) are given, then one can glue the two trivial bundles together via f to get a bundle over S^2 with fiber F. Two homotopy-equivalent clutching maps produce equivalent bundles. If a

G-bundle over S^2 is given, f is the transition function between the two charts $\mathbb{D}^{\pm}$ of S^2.

In our case, $F = \mathbb{C}^*$ and G is the circle SO_2. The clutching map $f : S^1_{\mathrm{eq}} \to \mathrm{SO}_2$ is determined, up to homotopy, by its twisting number. Let us compute the twisting number in our case.

We have two local trivializations

$$\begin{aligned} &\mathbb{D}^+ \times \mathbb{C}^* \to \mathscr{P} \subset \mathrm{PSL}_2(\mathbb{C}) \\ &((\zeta,\xi),\rho) \mapsto \begin{pmatrix} 1-\rho\zeta\xi & \rho\zeta^2 \\ -\rho\xi^2 & 1+\rho\zeta\xi \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{D}^- \times \mathbb{C}^* \to \mathscr{P} \subset \mathrm{PSL}_2(\mathbb{C}) \\ &((\zeta,\xi),\rho) \mapsto \begin{pmatrix} 1-\rho\bar{\zeta}\xi & \rho|\zeta|^2 \\ -\rho\dfrac{\bar{\zeta}}{\zeta}\xi^2 & 1+\rho\bar{\zeta}\xi \end{pmatrix}. \end{aligned}$$

The clutching map associated with these two trivializations is

$$\begin{aligned} &S^1_{\mathrm{eq}} \to \mathrm{SO}_2 \\ &e^{i\phi} \mapsto [\rho \mapsto e^{2i\phi}\rho], \end{aligned}$$

which represents the number 2 in $H_1(\mathrm{SO}_2) \cong \mathbb{Z}$. Thus we have reproven the following known fact:

Proposition 3. *The space $\mathscr{P}$ of non-trivial parabolic elements of* $\mathrm{PSL}_2(\mathbb{C})$ *is a* 2*-twist $\mathbb{C}^*$-bundle over S^2.*

3.5. The spaces $\mathbf{P}_1$ and $\mathbf{P}_2$. Define $\mathbf{P}_1'$ (resp. $\mathbf{P}_2'$) to be the space of all non-trivial discrete cyclic (resp. abelian) parabolic subgroups of $\mathrm{PSL}_2(\mathbb{C})$; of course $\mathbf{P}_i' \subset \mathbf{P}_i$. As above, we see that $\mathbf{P}_i'$ is a fiber-bundle over S^2 with fiber the space of all non-trivial discrete cyclic subgroups (resp. non-trivial discrete subgroups) of $\mathbb{C}$. The former fiber is easily seen to be simply $\mathbb{C}^*/(z \sim -z) \cong \mathbb{C}^*$, hence SO_2 is the structure group of the bundle P_1'; the latter is known from [7] to be homeomorphic to $(\mathbb{C}^2)^* = \mathbb{R}^4 \setminus \{0\}$, hence SO_4 is the structure group of P_2', where SO_4 acts on $(\mathbb{C}^2)^*$ in the usual way (namely, as a 4×4 real-matrix group acts on $(\mathbb{C}^2)^*$).

Proposition 4. *$\mathbf{P}_1'$ is a* 4*-twist $\mathbb{C}^*$-bundle over S^2.*

PROOF. As above we have two trivializations

$$\mathbb{D}^+ \times (\mathbb{C}^*/(z \sim -z)) \to \mathbf{P}_1'$$
$$((\zeta, \xi), u) \mapsto \left\{ \begin{pmatrix} 1-\rho\zeta\xi & \rho\zeta^2 \\ -\rho\xi^2 & 1+\rho\zeta\xi \end{pmatrix}; \rho \in \langle u \rangle \right\}$$

and

$$\mathbb{D}^- \times (\mathbb{C}^*/(z \sim -z)) \to \mathbf{P}_1'$$
$$((\zeta, \xi), u) \mapsto \left\{ \begin{pmatrix} 1-\rho\bar{\zeta}\xi & \rho|\zeta|^2 \\ -\rho\dfrac{\bar{\zeta}}{\zeta}\xi^2 & 1+\rho\bar{\zeta}\xi \end{pmatrix}; \rho \in \langle u \rangle \right\}$$

where $\langle u \rangle$ is here the additive subgroup of $\mathbb{C}$ generated by $\pm u \in \mathbb{C}^*$.

The clutching map associated with these two trivializations is now

$$e^{i\phi} \mapsto [\langle u \rangle \mapsto e^{2i\phi}\langle u \rangle],$$

which becomes, after identifying $\mathbb{C}^*/(z \sim -z)$ with $\mathbb{C}^*$:

$$S^1_{\mathrm{eq}} \to \mathrm{SO}_2 \cong S^1$$
$$e^{i\phi} \mapsto e^{4i\phi}. \qquad \square$$

COROLLARY 1. *$\mathbf{P}_1$ is the one-point compactification of a* 4*-twist* SO_2*-bundle of $\overline{\mathbb{D}}\backslash\{0\}$ over S^2.*

PROOF. The closure of the space of discrete cyclic subgroups of $\mathbb{C}$ for the Chabauty topology of $\mathbb{C}$ is just a closed disc $\overline{\mathbb{D}}$ (see for instance [7]). Thus it follows from Proposition 4 that $\mathbf{P}_1\backslash\{\mathrm{Id}\}$ is a 4-twist bundle of $\overline{\mathbb{D}}\backslash\{0\}$ over S^2. One recovers the compact set $\mathbf{P}_1$ from $\mathbf{P}_1\backslash\{\mathrm{Id}\}$ by taking the one-point compactification. $\square$

PROPOSITION 5. *$\mathbf{P}_2'$ is homeomorphic to $S^2 \times (\mathbb{C}^2)^*$.*

PROOF. The space $\mathscr{C}_d(\mathbb{C})$ of all discrete subgroups of $\mathbb{C}$ is homeomorphic to $\mathbb{C}^2$ via a map F explicited in Section 3 of [7]. This map F is the inverse of

$$\Gamma \mapsto \left(\frac{1}{60} \sum_{z \in \Gamma\backslash 0} \frac{1}{z^4}, \frac{1}{140} \sum_{z \in \Gamma\backslash 0} \frac{1}{z^6} \right).$$

One recognizes at once the modular invariants g_2 and g_3 of an elliptic curve.

Now as above we have two trivializations

$$\mathbb{D}^+ \times (\mathscr{C}_d(\mathbb{C})\backslash\{0\}) \to \mathbf{P}_2'$$
$$((\zeta,\xi),\Gamma) \mapsto \left\{ \begin{pmatrix} 1-\rho\zeta\xi & \rho\zeta^2 \\ -\rho\xi^2 & 1+\rho\zeta\xi \end{pmatrix}; \rho \in \Gamma \right\}$$

and

$$\mathbb{D}^- \times (\mathscr{C}_d(\mathbb{C})\backslash\{0\}) \to \mathbf{P}_2'$$
$$((\zeta,\xi),\Gamma) \mapsto \left\{ \begin{pmatrix} 1-\rho\bar{\zeta}\xi & \rho|\zeta|^2 \\ -\rho\dfrac{\bar{\zeta}}{\zeta}\xi^2 & 1+\rho\bar{\zeta}\xi \end{pmatrix}; \rho \in \Gamma \right\}.$$

The clutching map associated with these two trivializations is now

$$e^{i\phi} \mapsto [\Gamma \mapsto e^{2i\phi}\Gamma]$$

which becomes, using F:

$$S^1_{\mathrm{eq}} \to \mathrm{SO}_4$$
$$e^{i\phi} \mapsto [(a,b) \mapsto (e^{8i\phi}a, e^{12i\phi}b)]$$

(see Lemma 2 in [7]). Since SO_4 admits $S^3 \times S^3$ (which is simply connected) as double cover, and since the clutching map above is the double of some loop in SO_4, we see that it can be homotoped to the trivial loop

$$e^{i\phi} \mapsto [(a,b) \mapsto (a,b)] = \mathrm{Id}_{\mathrm{SO}_4},$$

therefore $\mathbf{P}_2'$ is homeomorphic to $S^2 \times \mathscr{C}_d(\mathbb{C})\backslash\{0\}$. □

Corollary 2. *$\mathbf{P}_2$ is homeomorphic to the one-point compactification of $S^2 \times \mathbb{R}^4$.*

Proof. We can conclude, simply by considering the possible limits of elements of $\mathbf{P}_2\backslash\{\mathrm{Id}\}$, that the homeomorphism described at the end of the proof of Proposition 5 extends to a homeomorphism between $\mathbf{P}_2\backslash\{\mathrm{Id}\}$ and $S^2 \times (\mathscr{C}(\mathbb{C})\backslash\{0\})$; $\mathscr{C}(\mathbb{C})$ is homeomorphic to S^4 (see [7]), so $\mathscr{C}(\mathbb{C})\backslash\{0\}$ is homeomorphic to $\mathbb{R}^4$. One recovers the compact set $\mathbf{P}_2$ from $\mathbf{P}_2\backslash\{\mathrm{Id}\}$ by taking the one-point compactification. □

4. Reduction lemma

4.1. The two reducing arguments. Any non-trivial non-parabolic closed abelian subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is well defined by two fixed points and a closed subgroup of $\mathbb{C}^*$ (see Proposition 1). Let us consider a sequence (G_n) of non-trivial non-parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. For all n, let us define $((\zeta_1)_n, (\xi_1)_n), ((\zeta_2)_n, (\xi_2)_n) \in \overline{\mathbb{D}^+}$ so that $[(\zeta_1)_n : (\xi_1)_n], [(\zeta_2)_n : (\xi_2)_n] \in \mathbb{CP}^1$ are the distinct fixed points of one (hence all) non-trivial element of G_n (the order does not matter); let us also define Ξ_n to be the subgroup of $\mathbb{C}^*$ consisting of the multipliers of the elements of G_n (see Proposition 1). For notational purposes, let us finally define $R_n \geq 1$ and $\omega_n \in \mathbb{R}/2\pi\mathbb{Z}$ by

$$R_n e^{i\omega_n} = \frac{1}{(\zeta_2)_n(\xi_1)_n - (\zeta_1)_n(\xi_2)_n}.$$

Taking extractions if necessary, we can always assume that $((\zeta_1)_n, (\xi_1)_n)$ and $((\zeta_2)_n, (\xi_2)_n)$ converge in $\overline{\mathbb{D}^+}$, that (R_n) converges in $[1, \infty]$ and $(e^{i\omega_n})$ converges in S^1. We denote the limits of these quantities by the subscript ∞.

Theorems 2 and 3 below are the arguments needed to reduce the problem of the convergence of non-parabolic groups in $\mathbf{C}_2$ to problems about convergence in $\mathscr{C}(\mathbb{C})$. The former deals with the case where the geometric limit is parabolic, the latter deals with the easier case where the geometric limit is non-parabolic.

Let us now recall the Reduction Lemma, a key tool in the reducing arguments below.

PROPOSITION 6 (Reduction Lemma). *Let (X, d_X), (Y, d_Y) be two second countable, locally compact metric spaces. Let (φ_n) be a sequence of maps from X to Y, converging to a continuous proper map φ, uniformly on every compact subset. Assume that for every compact subset $K \subset Y$, the closed subset*

$$\overline{\bigcup_{n \geq N} \varphi_n^{-1}(K)}$$

is compact for N large enough.

Then whenever a sequence of closed subsets $F_n \subset X$ converges to a closed subset F in the Hausdorff topology of X, the subsets $\overline{\varphi_n(F_n)}$ converge to $\overline{\varphi(F)}$ in the Hausdorff topology of Y.

PROOF. See Section 4 in [3]. □

REMARK 2. *If the maps φ_n are only defined on some domains $\Omega_n \subset X$ satisfying that for any compact subset $K \subset X$, we can find an integer N such that for all $n \geq N$, $K \subset \Omega_n$ (or, equivalently, if for every neighborhood $\mathscr{N}$ of the infinity-point $\infty \in \overline{X}$ and for all n large enough, $\Omega_n^c \subset \mathscr{N}$), then the conclusion of Proposition 6 still holds if $F_n \subset \Omega_n$ for every n, simply by declaring that $\widetilde{\varphi_n}$ sends every point of Ω_n^c to $\infty \in \tilde{Y}$ (see Section 5 in* [3]*).*

THEOREM 2. *Let (G_n) be a sequence of non-trivial, non-parabolic, closed abelian subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *such that (R_n) converges to $R_\infty = \infty$ (equivalently*

$((\zeta_i)_n, (\xi_i)_n)$ converges to the same point for $i = 1, 2$), and (ω_n) converges to some ω_∞.

For all n, let Γ_n be the closed subgroup of $\mathbb{C}$ defined by $\Gamma_n = R_n \operatorname{Log}(\Xi_n)$. Then (G_n) converges if and only if (Γ_n) converges to some closed subgroup Γ_∞ of $\mathbb{C}$ (for the Chabauty topology of $\mathbb{C}$), and in that case the geometric limit of G_n is the subgroup

$$G_\infty = \left\{ \begin{pmatrix} 1 + \rho e^{i\omega_\infty} \zeta_\infty \xi_\infty & \rho e^{i\omega_\infty} \zeta_\infty^2 \\ -\rho e^{i\omega_\infty} \xi_\infty^2 & 1 - \rho e^{i\omega_\infty} \zeta_\infty \xi_\infty \end{pmatrix}; \rho \in \Gamma_\infty \right\},$$

where we defined ζ_∞ to be $(\zeta_1)_\infty = (\zeta_2)_\infty$ and ξ_∞ to be $(\xi_1)_\infty = (\xi_2)_\infty$. Note that non-trivial elements of G_∞ (if any) are parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$ fixing $[\zeta_\infty : \xi_\infty] \in \mathbb{CP}^1$.

Proof. We will only prove here the indirect implication. The direct implication follows from Remark 3 which is slightly more general. This is because convergence of both (ω_n) and $(\widehat{\Gamma_n})$ implies convergence of (Γ_n) (see Remark 3 for notations).

The indirect direction will be proved by applying the Reduction Lemma twice.

First, let us define for all n a map $\psi_n : \mathbb{C} \to \mathrm{PSL}_2(\mathbb{C})$ by

$$z \mapsto (1 + z/R_n)^{-1/2} \begin{pmatrix} 1 + z e^{i\omega_n} (\zeta_2)_n (\xi_1)_n & z e^{i\omega_n} (\zeta_1)_n (\zeta_2)_n \\ -z e^{i\omega_n} (\xi_1)_n (\xi_2)_n & 1 - z e^{i\omega_n} (\zeta_1)_n (\xi_2)_n \end{pmatrix}.$$

Let us also define $\psi : \mathbb{C} \to \mathrm{PSL}_2(\mathbb{C})$ by

$$z \mapsto \begin{pmatrix} 1 + z e^{i\omega_\infty} \zeta_\infty \xi_\infty & z e^{i\omega_\infty} \zeta_\infty^2 \\ -z e^{i\omega_\infty} \xi_\infty^2 & 1 - z e^{i\omega_\infty} \zeta_\infty \xi_\infty \end{pmatrix}.$$

We need to check that this family satisfies the condition of the Reduction Lemma. This will be done through the following lemmas.

Lemma 1. *ψ is proper and continuous.*

Proof. This is clear, since whenever $z \to \infty$, at least one of the four entries in the matrix $\psi(z)$ blows off to infinity. □

Lemma 2. *(ψ_n) converges uniformly to ψ on every compact set.*

Proof. It is sufficient to prove it for every compact $K_M = \{z \in \mathbb{C}; |z| \leq M\}$. Fix some $\varepsilon > 0$. Since $R_n \to \infty$, we can find for every $M > 0$ some integer N such that, for all $n \geq N$ and all $z \in K_M$,

$$1 - \varepsilon \leq (1 + z/R_n)^{-1/2} \leq 1 + \varepsilon.$$

Therefore, we can also find an integer such that for every n larger than this N,

$$\|\psi_n(z) - \psi(z)\| \leq \varepsilon$$

holds for every $z \in K_M$, thus the proof is completed. $\square$

Lemma 3. *For any compact subset K of* $\mathrm{PSL}_2(\mathbf{C})$*, the closed subset of* $\mathbf{C}$

$$\overline{\bigcup_{n \geq N} \psi_n^{-1}(K)}$$

is compact for N large enough.

Proof. It is sufficient to prove that for every $M > 0$ and for every z with $|z| > M$, one of the entries of $(\psi_n(z))$ has a modulus greater that some quantity $A(M)$ depending only on M, with $A(M) \to \infty$ as $M \to \infty$. Recall that $(\zeta_2)_\infty(\xi_1)_\infty$, $(\zeta_1)_\infty(\zeta_2)_\infty$, $(\xi_1)_\infty(\xi_2)_\infty$ and $(\zeta_1)_\infty(\xi_2)_\infty$ cannot vanish at the same time. Suppose for instance that the first entry does not vanish (other cases are similar). Then there is a constant $c > 0$ for which we have $|(\zeta_2)_n(\xi_1)_n| \geq c$ for every n larger than some integer N. Also by taking a larger N if necessary, we may assume that $R_n \geq M$. Thus we have

$$\begin{aligned}
&|(1 + z/R_n)|^{-1/2}|1 + ze^{i\omega_n}(\zeta_2)_n(\xi_1)_n| \\
&\qquad \geq \frac{|z|c - 1}{\sqrt{|z|}\sqrt{1/|z| + 1/R_n}} \geq \frac{|z|c - 1}{\sqrt{2|z|/M}} \geq \sqrt{M}\frac{|z|c - 1}{\sqrt{2|z|}} \geq (c/\sqrt{2})M - \frac{1}{\sqrt{2}}
\end{aligned}$$

and the proof is completed. $\square$

Define for all n the subset $\mathscr{F}_n \subset \mathbf{C}$ by

$$\begin{aligned}
\mathscr{F}_n &= \left\{ \frac{a - 1}{|(\zeta_2)_n(\xi_1)_n - (\zeta_1)_n(\xi_2)_n|} ; a \in \Xi_n \right\} \\
&= \{R_n(a - 1); a \in \Xi_n\}.
\end{aligned}$$

Putting together Lemmas 1, 2, 3 and Proposition 6, we now see that if $(\mathscr{F}_n)$ converges to some closed subset $\mathscr{F}_\infty \subset \mathbf{C}$ for the Hausdorff topology, then $(G_n) = (\psi_n(\mathscr{F}_n))$ converges to $\psi(\mathscr{F}_\infty)$ for the Chabauty topology, hence $G_\infty = \psi(\mathscr{F}_\infty)$. The first step of the proof of Theorem 2 is completed.

The second step consists of applying the Reduction Lemma again. For all n, define $\varphi_n : \mathbf{C} \to \mathbf{C}$ by

$$\varphi_n(z) = R_n(e^{z/R_n} - 1).$$

By the series expansion of exp, we see that the sequence (φ_n) converges to the identity map of $\mathbf{C}$, uniformly on every compact subset, which is obviously continuous and proper. Since φ_n is periodic of period $2i\pi R_n$, there is no chance for

$$\overline{\bigcup_{n\geq N} \varphi_n^{-1}(K)}$$

to be ever compact. Thus, let us use Remark 2, by defining for all n Ω_n to be the band

$$\Omega_n = \{z \in \mathbb{C}; |\mathrm{Im}(z)| \leq \pi R_n\},$$

which satisfies the required conditions of Remark 2, because $R_n \to \infty$.

Then, we have the following, where $\|z\|_\infty$ stands for $\mathrm{Max}\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\}$:

Lemma 4. *For every $M > 0$, for every $z \in \Omega_n$, $\|z\|_\infty \geq M$ implies $|\varphi_n(z)| \geq M/2$ provided n is large enough.*

Proof. Let us suppose that n is large enough so that $R_n \geq M$. If $|\mathrm{Re}(z)| \geq M$, then $|\varphi_n(z)| \geq R_n(|e^{z/R_n}| - 1) \geq R_n(e^{M/R_n} - 1) \geq M$. Thus, let us suppose that z is in the closed subset

$$U_n = \{z \in \mathbb{C}; |\mathrm{Re}(z)| \leq M, M \leq |\mathrm{Im}(z)| \leq \pi R_n\}.$$

Now U_n is mapped homeomorphically by $[z \mapsto e^{z/R_n}]$ onto a horseshoe

$$\{z \in \mathbb{C}; e^{-M/R_n} \leq |z| \leq e^{M/R_n}, M/R_n \leq \mathrm{Arg}\, z \leq 2\pi - M/R_n\}$$

that avoids an open ball of radius $\sin(M/R_n)$ around 1. Thus $\varphi_n(U_n)$ avoids a ball of radius $R_n \sin(M/R_n)$ around 0. In view of the series expansion of sin, and since $R_n \to \infty$, we must have $R_n \sin(M/R_n) \geq M/2$ for n large enough, and the proof is completed. □

Thus, applying Proposition 6 again, we see that if $\Gamma_n \to \Gamma_\infty$, then $\mathscr{F}_n = \varphi_n(\Gamma_n) \to \Gamma_\infty$, hence $\mathscr{F}_\infty = \Gamma_\infty$.

All in all, we have $G_\infty = \psi(\Gamma_\infty)$, and this completes the proof of Theorem 2 for the indirect direction. □

Remark 3. *The equivalence of the convergence of (G_n) and the convergence of (Γ_n) in Theorem 2 does not hold in general if (ω_n) does not converge. In that case, the sequence (G_n) could converge while (Γ_n) does not converge. To prevent this, we could modify (Γ_n) by multiplying it by the adjustment factor $e^{i\omega_n}$. Namely, Theorem 2 actually says that the convergence of (G_n) is equivalent to the convergence of $(\widehat{\Gamma_n}) = (R_n e^{i\omega_n} \mathrm{Log}(\Xi_n))$, and of the fixed points $(f_{i,n})$ to the same limit f_∞.*

Proof. The indirect implication immediately follows from the proof of Theorem 2. This is because we can always assume that (ω_n) converges by extracting a subsequence. Checking the formula for G_∞, we see that it does not depend on the extraction.

To prove the direct implication, assume (G_n) converges. Then it is necessary that the fixed points $f_{i,n}$ converge to ζ_∞/ξ_∞. Now since the modified $\widehat{\Gamma_n}$ lie in the compact set $\mathscr{C}(\mathbb{C})$, it is enough to show that every converging subsequence of $(\widehat{\Gamma_n})$ converges to the same subgroup $\widehat{\Gamma_\infty}$. But applying Theorem 2 to all converging subsequences for (Γ_n) and (ω_n) produces the same G_∞. Direct inspection shows that this is only possible if all converging subsequences for $(\widehat{\Gamma_n})$ converge to the same subgroup. $\square$

THEOREM 3. *Let (G_n) be a sequence of non-trivial non-parabolic closed abelian subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *such that (R_n) converges to $R_\infty < \infty$ (equivalently $((\zeta_i)_n, (\xi_i)_n)$ converge to the distinct points for $i = 1, 2$). (G_n) converges if and only if (Ξ_n) converges to some Ξ_∞ (for the Chabauty topology of $\mathbb{C}$), and then the geometric limit of (G_n) is the subgroup*

$$G_\infty = \{H_{((\zeta_1)_\infty, (\xi_1)_\infty), ((\zeta_2)_\infty, (\xi_2)_\infty), a}; a \in \Xi_\infty\}.$$

PROOF. Applying the Reduction Lemma with $\psi_n : \mathbb{C}^* \to \mathrm{PSL}_2(\mathbb{C})$ defined by

$$z \mapsto z^{-1/2}\begin{pmatrix} 1 + (z-1)R_n e^{i\omega_n}(\zeta_2)_n(\xi_1)_n & (z-1)R_n e^{i\omega_n}(\zeta_1)_n(\zeta_2)_n \\ -(z-1)R_n e^{i\omega_n}(\xi_1)_n(\xi_2)_n & 1 - (z-1)R_n e^{i\omega_n}(\zeta_1)_n(\xi_2)_n \end{pmatrix},$$

and $\psi_\infty : \mathbb{C}^* \to \mathrm{PSL}_2(\mathbb{C})$ by

$$z \mapsto z^{-1/2}\begin{pmatrix} 1 + (z-1)R_n e^{i\omega_\infty}(\zeta_2)_\infty(\xi_1)_\infty & (z-1)R_n e^{i\omega_\infty}(\zeta_1)_\infty(\zeta_2)_\infty \\ -(z-1)R_n e^{i\omega_\infty}(\xi_1)_\infty(\xi_2)_\infty & 1 - (z-1)R_n e^{i\omega_\infty}(\zeta_1)_\infty(\xi_2)_\infty \end{pmatrix},$$

we conclude that if (Ξ_n) converges to Ξ_∞, $(G_n) = (\psi_n(\Xi_n))$ converges to $\psi(\Xi_\infty)$; thus $G_\infty = \psi(\Xi_\infty)$.

For the other direction, assume (G_n) converges. For any extraction ϕ such that $(\Xi_{\phi(n)})$ converges to some Ξ_ϕ, we see that Ξ_ϕ is entirely determined by G. Since $\Xi_n \in \mathscr{C}(\mathbb{C})$ which is compact, this proves that (Ξ_n) converges, and the proof is completed. $\square$

4.2. Geometric view of R_n and ω_n. Let us give geometric interpretations for R and ω, defined as above by

$$Re^{i\omega} = \frac{1}{\zeta_2\xi_1 - \zeta_1\xi_2},$$

where $(\zeta_1, \xi_1), (\zeta_2, \xi_2) \in \mathbb{D}^+$. We already saw in Subsection 3.3 that $\zeta_2\xi_1 - \zeta_1\xi_2$ can be interpreted as a square rooted discriminant in a model of $\mathbb{CP}^2$.

Lemma 5. *R is the inverse of the "spherical distance" between $[\zeta_1 : \xi_1]$ and $[\zeta_2 : \xi_2] \in \mathbb{CP}^1$. Namely, $1/R = |\zeta_2\xi_1 - \zeta_1\xi_2|$ is equal to the distance, in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, between the respective stereographic projections of (ζ_1, ξ_1) and $(\zeta_2, \xi_2) \in \mathbb{D}^+$ on the sphere S^2 of center $(0, 1/2)$ and radius $1/2$ (see Figure 1).*

Proof. Recall from Remark 1 that the stereographic projection of $\mathbb{D}^+$ onto $S^2 \backslash S$ is given by

$$(\zeta, \xi) \mapsto (\zeta\xi, \xi^2).$$

Now, using that $|\zeta_i|^2 = 1 - \xi_i^2$, it is a straightforward computation to show that

$$|\zeta_2\xi_1 - \zeta_1\xi_2|^2 = |\zeta_2\xi_2 - \zeta_1\xi_1|^2 + (\xi_1^2 - \xi_2^2)^2. \qquad \square$$

Lemma 6. *ω is the opposite of the angle between $[\zeta_1 : \xi_1]$ and $[\zeta_2 : \xi_2] \in \mathbb{CP}^1$. Namely, $-\omega = \mathrm{Arg}(\zeta_2\xi_1 - \zeta_1\xi_2)$ is equal to the argument, in $\mathbb{C} \times \{1\} \equiv \mathbb{C}$, of the vector $z_2 - z_1$ for the respective stereographic projections z_i of $(\zeta_i, \xi_i) \in \mathbb{D}^+$ on the horizontal plane passing through $N = (0, 1)$.*

Proof. Since multiplying a number by a positive real does not change the argument, we have

$$-\omega = \mathrm{Arg}(\zeta_2\xi_1 - \zeta_1\xi_2) = \mathrm{Arg}(\zeta_2/\xi_2 - \zeta_1/\xi_1). \qquad \square$$

4.3. Geometric limits seen with cylinders. We would like now to give a geometric interpretation of Theorem 2. Let (G_n) be a sequence of non-trivial, non-parabolic, closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ as above. Recall notations for $((\zeta_1)_n, (\xi_1)_n), ((\zeta_2)_n, (\xi_2)_n) \in \overline{\mathbb{D}^+}$, $R_n \geq 1$, $\omega_n \in [0, 2\pi]$. Assuming here that $R_n \to \infty$, i.e. that the distance between the fixed points $[(\zeta_1)_n : (\xi_1)_n]$ and $[(\zeta_2)_n : (\xi_2)_n] \in \mathbb{CP}^1$ of G_n tends to 0, recall the notation $\Gamma_n = R_n \,\mathrm{Log}(\Xi_n)$ with Ξ_n the multiplicative group of multipliers of elements of G_n (see the beginning of Section 4).

For all n, Γ_n is a subgroup of $\mathbb{C}$ containing $2i\pi R_n$. Equivalently, $\Gamma_n/2i\pi R_n\mathbb{Z}$ is a subgroup of $\mathbb{C}/2i\pi R_n\mathbb{Z}$, which is a cylinder. Let us view this cylinder in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ as being the cylinder with circumference $2i\pi R_n$ (i.e. radius R_n) and with center line:

$$\{(z, t) \in \mathbb{C} \times \mathbb{R}; \mathrm{Im}(z) = 0 \text{ and } t = R_n\}.$$

This cyclinder intersects the plane $\mathbb{C} \times \{0\}$ in the line $\{\mathrm{Im}(z) = 0 \text{ and } t = 0\}$.

For notational convenience, let us denote this cylinder by C_n. Otherwise put, C_n is the image of $\mathbb{C}$ under the map

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \times \mathbb{R} \\ x + iy &\mapsto (x + iR_n \sin y, R_n(1 - \cos y)). \end{aligned}$$

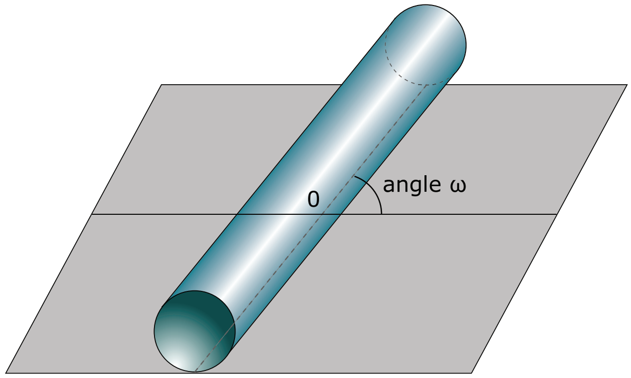

Fig. 2. Rotated cylinder C_n associated to G_n sitting on the plane $\mathbb{C} \times \{0\}$.

Better yet, imagine C_n as being rotated by an angle ω_n, as in the following drawing, Figure 2.

Now when $n \to \infty$, $R_n \to \infty$ also, i.e. the cylinders C_n become wider and wider; therefore (C_n) converges, for the Hausdorff topology of $\mathbb{R}^3$, to the plane $\mathbb{C} \times \{0\}$.

The last step of the description is to draw for all n the subgroup $\Gamma_n/2i\pi R_n\mathbb{Z}$ on C_n, simply as the image of Γ_n under the map

$$\mathbb{C} \to \mathbb{C} \times \mathbb{R}$$
$$x + iy \mapsto (e^{i\omega_n}(x + iR_n \sin y), R_n(1 - \cos y)).$$

As $n \to \infty$ and the cylinders C_n become wider and wider, these images look more and more like a closed subgroup of $\mathbb{C} \times \{0\}$, that we recognize to be $e^{i\omega_\infty}\Gamma_\infty$.

Finally, plug the values of this subgroup $e^{i\omega_\infty}\Gamma_\infty \subset \mathbb{C}$ in the matrix representation so that we obtain

$$G_\infty = \left\{ \begin{pmatrix} 1 + \rho\zeta_\infty\xi_\infty & \rho\zeta_\infty^2 \\ -\rho\xi_\infty^2 & 1 - \rho\zeta_\infty\xi_\infty \end{pmatrix}; \rho \in e^{i\omega_\infty}\Gamma_\infty \right\},$$

which is the geometric limit of G_n.

5. Exhaustion of cases

In this section, we consider a converging sequence (G_n) of non-trivial, non-parabolic, closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$, as in Section 4. Our goal is to provide a way of prescribing the limit subgroup G_∞, provided that we allow extracting subsequences to make the introduced parameters (R_n, ω_n, Ξ_n, etc.) converge. See notations introduced in Section 4. This will lead to a complete exhaustion of cases for converging sequences (G_n). See also Subsection 2.4 for further details about this strategy.

5.1. Case 1: $|\mathrm{Fix}(G_\infty)| = 2$. Let us consider the case where G_∞ is non-trivial, non-parabolic. By Theorem 3 the exhaustion of cases for sequences (G_n) reduces (in an explicit way) to the exhaustion for sequences $(n \mapsto \Xi_n) \in \mathscr{C}(\mathbb{C}^*)$, i.e. to the description of the Chabauty space of $\mathbb{C}^*$, which is well-known. See for instance [1] (or [2] with pictures).

It is interesting to note that the Reduction Lemma (Proposition 6) implies that we could assume without loss of generality that all G_n have the same fixed points in $\hat{\mathbb{C}}$:

$$\mathrm{Fix}\, G_n = \{0, \infty\}.$$

This follows from minor changes in the proof of Lemma 7.

5.2. Case 2: $|\mathrm{Fix}(G_\infty)| = 1$. Let us consider the case where G_∞ is a non-trivial parabolic subgroup. By Theorem 2, the exhaustion of cases for sequences (G_n) reduces (in an explicit way) to the exhaustion for sequences of subgroups

$$(\Gamma_n) = (R_n \,\mathrm{Log}\, \Xi_n \in \mathscr{C}(\mathbb{C})).$$

In order to unify the notation between Section 4 and the Example at the end of Section 2.4, we write $R_n = \sqrt{1 + a_n^2}$, so that R_n is the inverse of the spherical distance between the stereographic projections of $a_n \in \mathbb{R}^{\geq 0} \subset \hat{\mathbb{C}}$ and $\infty \in \hat{\mathbb{C}}$. The following Lemma allows us to assume further that $\mathrm{Fix}(G_n) = \{a_n, \infty\}$ with $a_n \in \mathbb{R}^{\geq 0}$.

Lemma 7. *We can assume without loss of generality that the fixed points of G_n are*

$$\mathrm{Fix}(G_n) = \{a_n, \infty\}$$

with $a_n \in \mathbb{R}^{\geq 0}$, $a_n \to \infty$.

More precisely, given $R_n = \sqrt{1 + a_n^2}$ and Γ_n, there is a unique subgroup $\widetilde{G_n}$ with $\mathrm{Fix}(\widetilde{G_n}) = \{a_n, \infty\}$ and Γ_n as group of multipliers. Then (G_n) converges precisely when $(\widetilde{G_n})$ converges and then

$$\widetilde{G_\infty} = \left\{ \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}; \rho \in \Gamma_\infty \right\}$$

with $\Gamma_\infty = \lim \Gamma_n$. Moreover, G_∞ is obtained from $\widetilde{G_\infty}$ by conjugating with an appropriate rotation.

Proof. We use the Reduction Lemma (Proposition 6).

For any two points $z_1 \neq z_2 \in \hat{\mathbb{C}}$, there is a unique rotation $\varphi \in SO_3$ that sends z_2 to ∞ and z_1 inside $\mathbb{R}^{\geq 0}$. Here SO_3 acts on $\hat{\mathbb{C}}$ after identifying it with

S^2 via the stereographic projection. Since SO_3 acts on $PSL_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{H}^3)$ by conjugation, we think of φ as a homeomorphism from $PSL_2(\mathbb{C})$ to itself. We wish to apply the Reduction Lemma with $X = Y = PSL_2(\mathbb{C})$ and the rotations φ_n. Given a sequence of fixed points $z_{1,n}$, $z_{2,n}$ converging to the same limit z_∞, if (ω_n) (defined in Section 4 as the angle $\mathrm{Arg}(z_{2,n} - z_{1,n})$) converges to some $\omega_\infty \in \mathbb{R}/2\pi\mathbb{Z}$, then (φ_n) converges uniformly on compact subsets of $PSL_2(\mathbb{C})$ to φ_∞, the only rotation that sends the vector centered at z_∞ and pointing in direction ω_∞ to the vector centered at ∞ and pointing toward the positive reals.

Now let K be a compact set of $PSL_2(\mathbb{C})$. Since compact sets of $PSL_2(\mathbb{C})$ are exactly the bounded closed subsets and since SO_3 is compact, we see that for any N,

$$\overline{\bigcup_{n \geq N} \varphi_n^{-1}(K)} \subset SO_3^{-1} \cdot K$$

is bounded thus compact. Therefore we can apply the Reduction Lemma and we conclude that if $(\varphi_n(G_n))$ converges, then (G_n) converges. To show the converse, apply the same argument replacing G_n by $\varphi_n(G_n)$ and φ_n by φ_n^{-1}.

We conclude that (G_n) converges if and only if $(\widetilde{G_n}) = (\varphi_n(G_n))$ converges. $\square$

With this in mind, we proceed to the exhaustion of cases. It reduces completely to the exhaustion of cases for sequences (Γ_n) of subgroups of $\mathbb{C}$ satisfying $2i\pi R_n \in \Gamma_n$ with $R_n \to \infty$, $R_n \in \mathbb{R}$.

We can reduce it further, by defining

$$l_n = \mathrm{Inf}\{x > 0 \mid ix \in \Gamma_n\}.$$

When $l_n \neq 0$, the integer $2\pi R_n / l_n$ is the maximal order of an elliptic element in G_n.

LEMMA 8. *Let $l > 0$. The followings are all the closed subgroups Γ of $\mathbb{C}$ such that $[0, il] \cap \Gamma = \{0, il\}$ or $[0, il]$:*

- $A^l := il\mathbb{Z}$,
- $B_z^l := z\mathbb{Z} + il\mathbb{Z}$ *for* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > 0$ *and* $\mathrm{Im}(z) \in [0, l]$,
- $C_x := x\mathbb{Z} + i\mathbb{R}$ *for* $x > 0$,
- $D_t^l := il\mathbb{Z} + (1 + it)\mathbb{R}$ *with* $t \in \mathbb{R}$,
- $A^0 = C_\infty := i\mathbb{R}$,
- $C_0 := \mathbb{C}$.

In Remark 4 below, we recall how letters for subgroups Γ correspond to different kinds of non-parabolic subgroups G of $PSL_2(\mathbb{C})$.

Remark 4. *Let G be a non-trivial, non-parabolic, closed abelian subgroup of $\mathrm{PSL}_2(\mathbb{C})$; let $\Gamma = R\,\mathrm{Log}\,\Xi$ (see the beginning of Section 4 for notations). Then exactly one of the following holds:*

- *$\Gamma = A^{2\pi R/m}$ if G is generated by an elliptic element of order m,*
- *Γ is some $B_z^{2\pi R/m}$ if G is generated by an elliptic element of order m and a non-trivial hyperbolic element; these two generators need to have the same fixed points in $\mathbb{CP}^1$ in order for G to be abelian,*
- *Γ is some C_x if $G \neq \mathbb{C}$, G contains every elliptic element fixing a pair of points in $\mathbb{CP}^1$, and G contains a non-trivial hyperbolic element fixing the pair of points,*
- *Γ is some $D_t^{2\pi R/m}$ if G contains exactly m elliptic elements, and has exactly m connected components homeomorphic to $\mathbb{R}$. Otherwise put, $\Xi = e^{\Gamma}$ is a m-branched logarithmic spiral,*
- *$\Gamma = i\mathbb{R}$ if G consists of the elliptic elements sharing the same fixed point set,*
- *$\Gamma = \mathbb{C}$ if G consists of the elliptic and hyperbolic elements sharing the same fixed point set.*

Let us continue the exhaustion of cases by mentioning easy cases first. Note that by extracting a subsequence if necessary, we may assume that all Γ_n are of the same type (A, B, C or D) described in Lemma 8, and that (l_n) converges in $[0, \infty]$.

If $l_n \to 0$ (resp. $l_n \to l$ with $l \in (0, \infty)$), explicit limits Γ_∞ for (Γ_n) can be computed, with $\Gamma_\infty = C_x$ for $x \in [0, \infty]$ (resp. a subgroup of letter A, B or D). This is performed by separating cases as in [2]. The whole description is somewhat tedious so we refer to [2] for the general picture.

Now we see that we are left with the problem of exhaustion of cases for convergent sequences (Γ_n) such that there is an $l_n > 0$ with $il_n \in \Gamma_n$, $[0, il_n] \cap \Gamma_n = \{0, il_n\}$ and $l_n \to \infty$. The following lemma deals with the easier cases.

Lemma 9. *In the Chabauty topology, we have the following convergence results, for $l_n \to \infty$ throughout.*

- $A^{l_n} \to \{0\}$,
- $$B_{z_n}^{l_n} \to \begin{cases} \{0\} \text{ if } \mathrm{Re}(z_n) \to \infty \\ (x+iy)\mathbb{Z} \text{ if } \mathrm{Re}(z_n) \to x \text{ with } x > 0, \text{ and } \widetilde{y_n} \to y \\ \quad \text{for } \widetilde{y_n} \text{ the unique } (\mathrm{Im}(z_n) \bmod il_n) \text{ in } (-l_n/2, l_n/2] \\ \{0\} \text{ if } \mathrm{Re}(z_n) \to x \text{ with } x > 0 \text{ and } \widetilde{y_n} \to \pm\infty, \end{cases}$$
- $$D_{t_n}^{l_n} \to \begin{cases} (1+it)\mathbb{R} \text{ if } t_n \to t \in \mathbb{R} \\ C_x \text{ if } t_n \to \pm\infty \text{ and } \frac{|l_n|}{|t_n|} \to x \in [0, \infty]. \end{cases}$$

Proof. These assertions result from elementary manipulation of Hausdorff limits. $\square$

Here again when all the Γ_n are of type B, we can further assume that $(\mathrm{Re}(z_n))$ converges in $[0,\infty]$ and we see that the case $\Gamma_n = B_{z_n}^{l_n}$ with $l_n \to \infty$ and $\mathrm{Re}(z_n) \to 0$ is the only remaining case. It will be studied separately in Subsection 5.3 below. Note that this case exactly describes the exhaustion of cases for sequences of abelian subgroups generated by an elliptic element of order m_n (this condition being vacuous for $m_n = 1$) and by a non-trivial hyperbolic element, and such that $R_n/m_n \to \infty$. In particular, this case englobes the exhaustion of cases for sequences of cyclic subgroups of $\mathrm{PSL}_2(\mathbb{C})$ generated by one hyperbolic element, converging to a parabolic group (example in Section 1, with $l_n = 2\pi\sqrt{1+a_n^2}$, $x_n = 2\sqrt{1+a_n^2}\ln(\rho_n)$, $\theta_n = 1/n$).

5.3. Remaining case. We will now study the convergence of lattices $\Gamma_n \subset \mathbb{C}$ of the form

$$\Gamma_n = \langle il_n, x_n + i\theta_n l_n\rangle$$

with $l_n > 0$, $x_n > 0$ and $\theta_n \in [0,1]$, in the case where $l_n \to \infty$ and $x_n \to 0$.

Taking extractions if necessary, we can assume that $\theta_n \to \theta_\infty \in [0,1]$.

The strategy to describe the explicit limit of the sequence (Γ_n) is to replace the generators $(il_n, x_n + i\theta l_n)$ by a "better" pair of generators, "better" here meaning roughly "closer to the origin". The intuition is that while (il_n) and $(x_n + i\theta l_n)$ converge/diverge possibly in very different speeds, linear combinations of these generators may very well end up close to the origin. Better generators will prevent this enrichment behavior to happen.

This is where the continued fractions enter to the picture. See for instance [6] for a geometry-flavoured exposition to continued fraction.

We write $\theta = [\alpha_0; \alpha_1, \alpha_2, \alpha_3, \ldots]$ for the expansion in continued fraction of θ:

$$\theta = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cdots}}}$$

with $\alpha_0 \in \mathbb{Z}$ and $\alpha_i \geq 1$ for $i \geq 1$.

Recall that for j less than or equal to the length of the expansion in continued fraction of θ (this condition being vacuous for θ irrational), the jth convergent

$$\frac{p_j}{q_j} = [\alpha_0; \alpha_1, \alpha_2, \ldots, \alpha_j]$$

with p_j, q_j coprime, satisfies the following properties.

Lemma 10. *By convention,* $p_{-1} = 1$, $p_0 = \alpha_0$, $q_{-1} = 0$, $q_0 = 1$. *For all* $j \geq 1$,

(1) $p_j = \alpha_j p_{j-1} + p_{j-2}$, $q_j = \alpha_j q_{j-1} + q_{j-2}$,
(2) $q_{j+1} p_j - q_j p_{j+1} = (-1)^j$,
(3) p_j/q_j *alternates around* θ. *More precisely,* $\mathrm{sign}(\theta - p_n/q_n) = (-1)^n$,
(4) $|\theta - p_j/q_j| < 1/q_j q_{j+1} < 1/q_j^2$.

Proof. These are standard facts. For instance, see [10]. □

Recall the previous notations $\Gamma_n = \langle i l_n, x_n + i\theta_n l_n \rangle$ with $l_n > 0$, $x_n > 0$, $\theta_n \in [0,1]$, $l_n \to \infty$, $x_n \to 0$ and $\theta_n \to \theta_\infty \in [0,1]$. Let as before $\theta_n = [0; \alpha_{n,1}, \ldots, \alpha_{n,j}, \ldots]$ be the continued fraction expansion of θ_n.

Define for all integers n and j

$$u_{n,j} = q_{n,j} x_n + i l_n (q_{n,j}\theta_n - p_{n,j}).$$

Note that we always have $\Gamma_n = \langle u_{n,j}, u_{n,j+1} \rangle$. Indeed, $\{u_{n,j}, u_{n,j+1}\}$ is linearly independent by Lemma 10-(3) and generates Γ_n since (2) and (4) imply that there is no point of Γ_n in the interior of the triangle with vertices 0, $u_{n,j}$ and $u_{n,j+1}$.

At this point, it does not seem that the way we expressed Γ_n using the continued fraction expansion of θ_n is by any mean more concrete that the Weierstrass elliptic function used in [7]. Contrary to this appearance, the following two lemmas show that a lot of the properties of the pair $(u_{n,j}, u_{n,j+1})$ can be "read" in the continued fraction expansion of θ_n.

Lemma 11. $\rho_{n,k} = \dfrac{\mathrm{Re}(u_{n,k+1})}{\mathrm{Re}(u_{n,k})}$ *has continued fraction expansion*

$$\rho_{n,k} = [\alpha_{n,k+1}; \alpha_{n,k}, \alpha_{n,k-1}, \ldots, \alpha_{n,1}],$$

and $\eta_{n,k} = \left|\dfrac{\mathrm{Im}(u_{n,k+1})}{\mathrm{Im}(u_{n,k})}\right|$ *has continued fraction expansion*

$$\eta_{n,k} = [0; \alpha_{n,k+2}; \alpha_{n,k+3}, \ldots].$$

In other words, $\rho_{n,k}$ *is obtained by reading the continued fraction of* θ_n *backwards, starting at the index* $k+1$, *and* $\eta_{n,k}$ *is obtained by reading the continued fraction of* θ_n *forwards, starting at the index* $k+2$.

Proof. We prove the first assertion by a simple induction on k for $\rho_{n,k+1} = q_{n,k+1}/q_{n,k}$, using Lemma 10-(1). Indeed, $\rho_{n,1} = q_{n,1}/q_{n,0} = \alpha_{n,1}$ and

$$\rho_{n,k+1} = \frac{q_{n,k+1}}{q_{n,k}} = \frac{\alpha_{n,k+1} q_{n,k} + q_{n,k-1}}{q_{n,k}} = \alpha_{n,k+1} + \frac{1}{\rho_{n,k}}.$$

Similarly for the second assertion, define for all k

$$\eta_{n,k+1} = \left|\frac{q_{n,k+1}\theta_n - p_{n,k+1}}{q_{n,k}\theta_n - p_{n,k}}\right|.$$

Then $\eta_{n,0} = \theta_n$ and by (1)

$$\eta_{n,k+1} = -\frac{q_{n,k+1}\theta_n - p_{n,k+1}}{q_{n,k}\theta_n - p_{n,k}} = -\alpha_{n,k+1} + \frac{1}{\eta_{n,k}}. \qquad \square$$

Lemma 12. *The set of lattices of* $\mathbb{C}$ *generated by a pair of vectors* $u, v \in \mathbb{C}$ *satisfying*

$$\begin{cases} 0 < \mathrm{Re}(u) < \mathrm{Re}(v), \\ 0 < |\mathrm{Im}(v)| < |\mathrm{Im}(u)|, \\ \mathrm{Im}(u) \cdot \mathrm{Im}(v) < 0, \\ \mathrm{Re}(v)/\mathrm{Re}(u) \in \mathbb{R}\backslash\mathbb{Q}, \end{cases}$$

is dense in the space of closed subgroups of $\mathbb{C}$, *for the Chabauty topology.*

Proof. This follows from a standard density argument. $\square$

The following theorem translates in the world of $\mathrm{PSL}_2(\mathbb{C})$ as saying that we can obtain any parabolic group P as the limit of a sequence of cyclic groups H_n with hyperbolic generators. Additionally, this remains true even if we ask the fixed points of H_n to converge radially. More precisely, suppose for instance that the fixed point of P is $0 \in \hat{\mathbb{C}} \cong \mathbb{CP}^1$ and choose some preferred angle ω. Then we can find a sequence (H_n) converging to P with $\mathrm{Fix}(H_n) = \{0, f_n\}$ and $\mathrm{Arg}\, f_n = -\omega$ for all n.

Theorem 4. *Let* Γ *be any closed subgroup of* $\mathbb{C}$, *and* $\theta \in [0,1)$. *Then there exist sequences* $l_n \to \infty$, $x_n \to 0$ *and* $\theta_n \to \theta$ *such that the sequence of lattices*

$$(\Gamma_n) = (\langle il_n, x_n + i\theta_n l_n\rangle)$$

converges to Γ *in the Chabauty topology.*

Proof. Assume that $\theta \in [0,1]\backslash\mathbb{Q}$ (the case $\theta \in \mathbb{Q}$ is only different in the fact that the expansion in continued fractions of θ is finite; it can be dealt with by minor changes to the present proof). Assume that Γ is generated by a pair of vectors $u, v \in \mathbb{C}$ satisfying the condition of Lemma 12, and consider $\rho = \mathrm{Re}(v)/\mathrm{Re}(u)$ and $\eta = |\mathrm{Im}(v)/\mathrm{Im}(u)|$.

Suppose for instance that $\mathrm{Im}(u) > 0$, the case $\mathrm{Im}(u) < 0$ being similar. Also, define the following continued fraction expansions:

$$\begin{cases} \theta = [0; \alpha_1, \alpha_2, \dots], \\ \rho = [\beta_0; \beta_1, \beta_2, \dots], \\ \eta = [0; \gamma_1, \gamma_2, \dots]. \end{cases}$$

Here by assumption, the first two expansions are infinite, $\beta_0 > 0$, and the last expansion is either finite or infinite.

Let us define the sequence θ_n by:

$$\theta_n = [0; \alpha_1, \alpha_2, \dots, \alpha_n, \beta_n, \beta_{n-1}, \dots, \beta_0, \gamma_1, \gamma_2, \dots].$$

Recall that $p_{n,2n}$ and $q_{n,2n}$ are defined by

$$p_{n,2n}/q_{n,2n} = [0; \alpha_1, \alpha_2, \dots, \alpha_n, \beta_n, \beta_{n-1}, \dots, \beta_1].$$

Set x_n and l_n so that $x_n q_{n,2n} = \mathrm{Re}(u)$ and $l_n(q_{n,2n}\theta_n - p_{n,2n}) = \mathrm{Im}(u)$, and let $\Gamma_n = \langle il_n, x_n + i\theta_n l_n \rangle$. By definition of x_n and l_n, we have that $u_{n,2n} = u$, and we want to prove that $u_{n,2n+1} \to v$. But by Lemma 11, $\rho_{n,2n}$ and $\eta_{n,2n}$ verify

$$\begin{cases} \rho_{n,2n} = [\beta_0; \beta_1, \dots, \beta_n, \alpha_n, \dots, \alpha_1], \\ \eta_{n,2n} = [0; \gamma_1, \gamma_2, \dots], \end{cases}$$

therefore $u_{n,2n+1} = \rho_{n,2n}\,\mathrm{Re}(u_{n,2n}) - \eta_{n,2n}\,\mathrm{Im}(u_{n,2n}) \to \rho\,\mathrm{Re}(u) - \eta\,\mathrm{Im}(u) = v$. Because u and v are linearly independent, $\Gamma_n = \langle u_{n,2n}, u_{n,2n+1} \rangle \to \Gamma = \langle u, v \rangle$ and the proof is completed for this case.

Now by Lemma 12 and by a standard density and diagonal argument, the proof is completed in all the remaining cases. □

Finally, let us describe explicitely the limit of a converging sequence (Γ_n) as above, using only the sequences (l_n), (x_n) and the coefficients α_i of the expansion in continued fraction of θ. Let us start with an easy lemma.

Lemma 13. *For any n, two minimal values of the sequence $(j \mapsto \|u_{n,j}\|_\infty)$ for the max norm $\|x + iy\|_\infty = \mathrm{Max}(|x|, |y|)$ are obtained for two consecutive integers, that we write $(j_n^{\min}, j_n^{\min} + 1)$. Then $u_{n,j_n^{\min}}$ and $u_{n,j_n^{\min}+1}$ are also minimal for the max norm amongst all non-zero elements of Γ_n.*

Proof. Since $(j \mapsto q_{n,j}x_n)$ is increasingly converging to ∞ and $(j \mapsto l_n|q_{n,j}\theta_n - p_{n,j}|)$ is decreasingly converging to 0,

$$j \mapsto \|q_{n,j}x_n + il_n(q_{n,j}\theta_n - p_{n,j})\|_\infty$$

is first decreasing and then increasing; this proves the first assertion. The second follows from Lemma 10-(2) and (4). □

Definition 2. For all n, define u_n, v_n by

$$\begin{cases} u_n = u_{n,j_n^{\min}}, \\ v_n = u_{n,j_n^{\min}+1}. \end{cases}$$

Theorem 5. *Suppose that for instance* $\mathrm{Im}(u_n) > 0$ *for all* n *(the case* $\mathrm{Im}(u_n) < 0$ *for all* n *is similar, and we can assume either one of the two by extracting subsequences if necessary). Define* $t_u = \lim \mathrm{Arg}\, u_n \in [0, \pi/2]$, $t_v = \lim \mathrm{Arg}\, v_n \in [-\pi/2, 0]$, *assuming these limits exist by taking an extraction if necessary. If* t_u *and* t_v *are neither both* 0 *nor both* $\pm\pi/2$, *then the limit subgroup* $\Gamma_\infty = \lim\langle u_n, v_n\rangle$ *is the one we expect, namely*

$$\Gamma_\infty = \Gamma_u + \Gamma_v$$

with

$$\Gamma_u = \lim\langle u_n\rangle = \begin{cases} (1 + it_u)\mathbb{R} & \text{if } u_n \to 0, \\ u_\infty \mathbb{Z} & \text{if } u_n \to u_\infty \in \mathbb{C}\backslash\{0\}, \\ \{0\} & \text{if } u_n \to \infty. \end{cases}$$

and similarly for v.

If t_u *and* t_v *are either both* 0 *or both* $\pm\pi/2$, *then:*

- $\Gamma_\infty = iy\mathbb{Z} + \mathbb{R}$ *if* $t_u = t_v = 0$, $u_n \to 0$ *and* $\frac{\mathrm{Im}(u_n)}{\mathrm{Re}(u_n)} \mathrm{Re}(v_n) + |\mathrm{Im}(v_n)| \to y$, *with by convention* $y\mathbb{Z} = \mathbb{R}$ *if* $y = 0$, $y\mathbb{Z} = \{0\}$ *if* $y = \infty$,
- $\Gamma_\infty = x\mathbb{Z}$ *if* $t_u = t_v = 0$ *and* $u_n \to x \in \mathbb{R}$, $x > 0$,
- $\Gamma_\infty = x\mathbb{Z} + i\mathbb{R}$ *if* $t_u = +\pi/2$, $t_v = -\pi/2$, $v_n \to 0$ *and* $\mathrm{Im}(u_n)\frac{\mathrm{Re}(v_n)}{|\mathrm{Im}(v_n)|} + \mathrm{Re}(u_n) \to x$, *with by convention* $x\mathbb{Z} = \mathbb{R}$ *if* $x = 0$, $x\mathbb{Z} = \{0\}$ *if* $x = \infty$,
- $\Gamma_\infty = iy\mathbb{Z}$ *if* $t_u = +\pi/2$, $t_v = -\pi/2$ *and* $v_n \to -iy$, $y > 0$.

Proof. The first part follows easily from the minimality of the generators (u_n, v_n), see Lemma 13. The two cases $t_u = t_v = 0$ and $t_u = +\pi/2$, $t_v = -\pi/2$ are similar; let us prove only the result for the former case.

If $t_u = t_v = 0$ and $u_n \to 0$, draw the line passing through v_n and parallel to the line through 0 and u_n; consider its intersection $-iy_n$ with the vertical axis. It is easy to see that $y_n = \frac{\mathrm{Im}(u_n)}{\mathrm{Re}(u_n)} \mathrm{Re}(v_n) + |\mathrm{Im}(v_n)|$, and since $u_n \to 0$ and $t_u = 0$, we conclude that $\Gamma_\infty = iy\mathbb{Z} + \mathbb{R}$.

If $t_u = t_v = 0$ and $u_n \to x$ with $x > 0$, then consider Figure 3. By the minimality of the generators (u_n, v_n) for the max norm, there can not be any element of $\Gamma_n = \langle u_n, v_n\rangle$ in the two left shaded squares. As a consequence, there can not be any element of Γ_n in any of the shaded region. Now $|\mathrm{Im}(v_n)| < \mathrm{Im}(u_n)$, so $\mathrm{Re}(v_n)$ must be bigger than the real part of the point represented by z on Figure 3, which is easily seen to be $\mathrm{Re}(z) = \frac{\mathrm{Re}(u_n)}{\mathrm{Im}(u_n)}(\mathrm{Re}(u_n) - \mathrm{Im}(u_n))$. Since $u_n \to x > 0$ we conclude that $\Gamma_\infty = x\mathbb{Z}$, and the proof is completed. □

6. Local models for $\mathbf{C}_2$

We would like now to provide local models for neighborhoods of elements in $\mathbf{C}_2$. Recall that the space of non-trivial, non-parabolic elements of $\mathbf{C}_2$ is

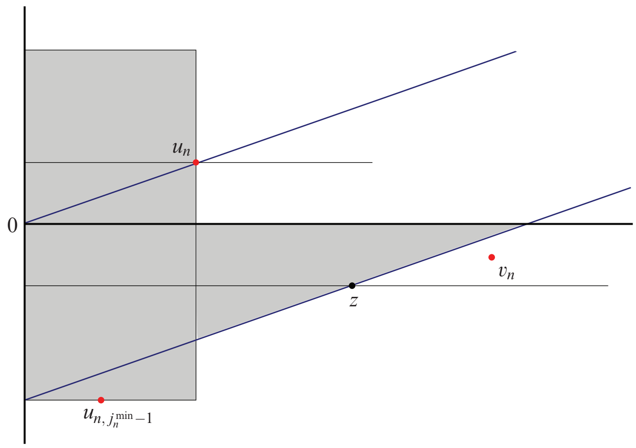

Fig. 3. If $u_n \to x$ with $x \in \mathbb{R}$, $x > 0$, then $\mathrm{Re}(v_n) \to \infty$.

homeomorphic to $\Theta \times (\mathscr{C}(\mathbb{C}^*)\backslash 1)$ (see Proposition 1). Since we know how to describe geometrically $\mathscr{C}(\mathbb{C}^*)$ (see [2]), we have a clear enough picture of what a neighborhood of a non-trivial, non-parabolic element of $\mathbf{C}_2$ looks like.

6.1. Local models in terms of marked subgroups of $\mathbb{C}$. Let G be a parabolic group in $\mathbf{C}_2$. For clarity, assume that its fixed point is 0. In the following, we will look at groups H in $\mathbf{C}_2$ close enough to G that:

- if H is parabolic, its fixed point is not ∞. In this case, we get "the" subgroup Γ associated to H, via the specific local trivialization $\mathbb{D}^- \times \mathbb{C}^* \to \mathscr{P}$ from Section 3.4.
- if H is non-parabolic, its fixed point set does not include ∞. Therefore it makes sense to talk about the argument ω of $z_2 - z_1$, for $z_i \in \mathbb{C}$ the stereographic projections for the two fixed points f_i on $\hat{\mathbb{C}}$.

Call Γ_1 the subgroup of $\mathbb{C}$ associated to G.

Non-parabolic groups in $\mathbf{C}_2$ are specified by two fixed points f_1 and f_2, and by a closed subgroup $\Gamma = Re^{i\omega} \operatorname{Log} \Xi \in \mathscr{C}(\mathbb{C})$. Γ contains the element $\delta = 2i\pi Re^{i\omega}$ with $R \geq 1$ (see Subsection 4.3). Note that for notational purposes, Γ in this section refers to $\hat{\Gamma}$ from Remark 3 (the hat symbol is dropped). This should not introduce any confusion.

Alternatively, non-parabolic groups can be specified by the giving of *one* fixed point f_1, and by a marked closed subgroup (Γ, δ), the marking δ being of absolute value $\geq 2\pi$. This is because we can recover f_2 from f_1, R and ω. Thus, let us define:

$$\mathscr{M} = \{(\Gamma, \delta); \Gamma \in \mathscr{C}(\mathbb{C}), \delta \in \Gamma \cup \{\infty\}, |\delta| \geq 2\pi\}.$$

We put on $\mathscr{M}$ the topology inherited by inclusion into $\mathscr{C}(\mathbb{C}) \times (\mathbb{C} \cup \{\infty\})$.

Consider the fixed point exchange map $(f_1, \delta = 2i\pi Re^{i\omega}) \mapsto f_2$, where f_2 are defined by $d(f_1, f_2) = 1/R$ and ω is the angle between the horizontal line and the line through f_1 and f_2.

Theorem 6. *Let B be an open ball around 0 inside S^2, small enough that $\infty \notin B$ (recall the choice of S^2 as a subset of $\mathbb{C}^2$ in Subsection 3.1). Let $\mathscr{N}$ be a neighborhood of $(\Gamma_1, \infty) \in \mathscr{M}$, small enough that: $\forall f_1 \in B$, $\forall (\Gamma, \delta) \in \mathscr{N}$, $f_2 \neq \infty$. Then the following map is a homeomorphism onto its image:*

$$(B \times \mathscr{N})/((f_1, \Gamma, \delta) \sim (f_2, \Gamma, -\delta)) \to \mathbf{C}_2$$
$$(f_1, \Gamma, \delta) \mapsto H(f_1, \Gamma, \delta),$$

where $H(f_1, \Gamma, \delta)$ is the non-parabolic group with fixed points f_1, f_2 and the associated group Γ if $\delta \neq \infty$, or the parabolic group with a fixed point f_1 and the associated group Γ if $\delta = \infty$.

Proof. Injectivity comes from the classification of abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Remark 3 implies that a sequence in the source space converges if and only if its image converges in the target space. $\square$

Theorem 6 reduces the problem of describing neighborhoods of parabolic groups in $\mathbf{C}_2$ to the one of describing the neighborhoods of marked subgroups $(\Gamma_1, \infty) \in \mathscr{M}$. The authors produced several pictures of these neighborhoods depending on the type of Γ as a subgroup of $\mathbb{C}$. Describing these pictures in detail would require lengthy explanations that we decided not to include in the present paper. We will discuss accumulation behavior depending on the type of Γ_1.

6.2. Dichotomy of accumulation behavior. As the space $\mathbf{C}_2$ accumulates to $\mathbf{P}_2$, we face the situation of a 6-dimensional space accumulating on another 6-dimensional space. We expect spiraling behaviors of some sort; this subsection is an attempt to make this precise.

In general, there is one simple nice dichotomy for the case when an n-dimensional space X accumulates to another n-dimensional space Y (say X, Y metric spaces). Let $p \in Y$ be a limit point of X. Then either there is a continuous path $\gamma : [0,1] \to X \cup Y$ such that $\gamma([0,1)) \subset X$ and $\gamma(1) = p \in Y$, or there is no such a path. Otherwise put, either for every neighborhood U of p in $X \cup Y$ the arcwise-connected component of U containing p contains an element of X, or for every neighborhood U of p in $X \cup Y$ the arcwise-connected component of U containing p contains no element of X. We would like to reserve the term "spiraling of X toward p in Y" for the latter behavior,

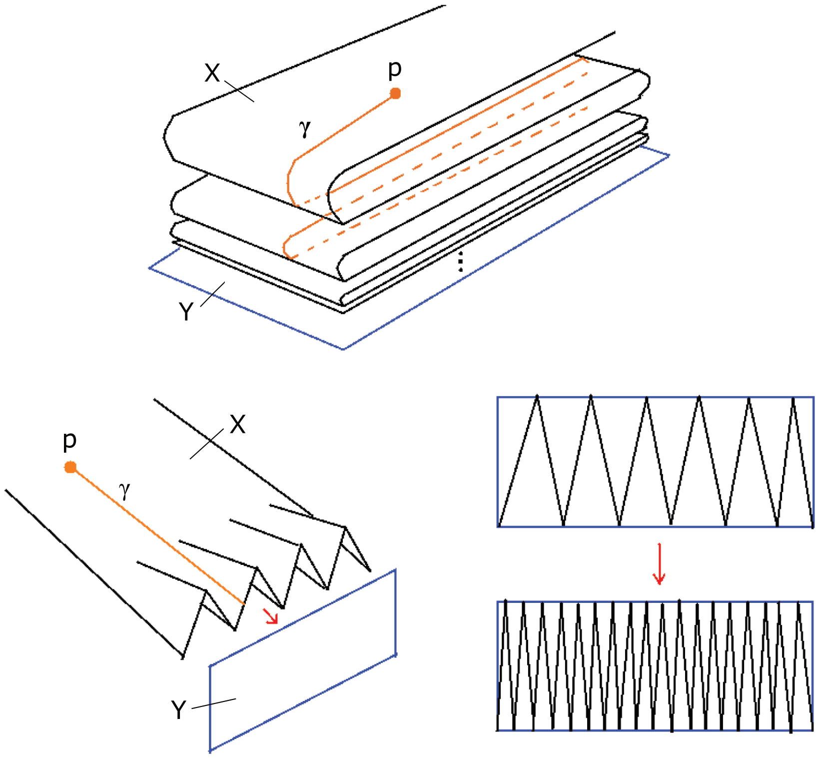

Fig. 4. In both pictures, a 2-dimensional space X is accumulating onto a 2-dimensional space Y. In the picture, we only show the approximation of a pleating of one end of X. It looks like an accordion with more and more pleating. As a limit of this process, X finally accumulates onto a the square Y. The fundamental difference between these two cases is captured by the paths γ. In the first case, if you pick a point p in X, then there is no finite path starting at this point and reaching Y. But such a path exists in the second case.

since it is similar to $[1, \infty) \subset \mathbb{R}$ accumulating onto S^1 via $x \mapsto \left(1 - \frac{1}{x}\right)e^{ix} \in \mathbb{C}$. We do not think that this terminology is standard.

Let us see an example in dimension 2 showing the two different situations. See Figure 4.

For notational convenience, define now $\mathbb{X} = \mathbf{C}_2 \backslash \mathbf{P}_2$ to be the space of all non-trivial non-parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$. In the following subsections, we will prove the following theorem.

THEOREM 7. *Let G be a group in $\mathbf{P}_2$. Then*

1. *if G is isomorphic to $\mathbb{Z}^2$, then $\mathbb{X}$ accumulates towards G in a spiraling way.*

2. *if G is not isomorphic to $\mathbb{Z}^2$, $\mathbb{X}$ accumulates toward G in a non-spiraling way.*

6.3. The spiraling case. In this subsection, we prove Theorem 7, part 1.

Let G be a parabolic group in $\mathbf{C}_2$ isomorphic to $\mathbb{Z}^2$. We can assume without loss of generality that its fixed point is 0 and its associated subgroup Γ_1 is $\mathbb{Z} + i\mathbb{Z} \subset \mathbb{C}$.

Assume there is a path $t \in [0,1] \mapsto G_t \in \mathbf{C}_2$ such that $G_t \in \mathbb{X}$ for every $t \in [0,1]$, and with $G_1 = G$. We would like to find a contradiction. It is sufficient to look at the induced path in the space $\mathscr{M}$ introduced in Section 6.1.

By shortening G_t if necessary, we can assume that for all t, the fixed points of G_t, $f_1(t)$ and $f_2(t)$ are not ∞. Let $\Gamma_t = R(t)e^{i\omega(t)}\,\mathrm{Log}\,\Xi(t) \in \mathscr{C}(\mathbb{C})$, and $\delta(t) = 2i\pi R(t)e^{i\omega(t)} \in \Gamma_t$. Then since $G_1 = G$ is parabolic and δ is continuous, $\delta(t) \to \infty$ as $t \to 1$. The space of lattices being an open subset of $\mathscr{C}(\mathbb{C})$, we may assume that Γ_t is a lattice for any $t \in [0,1]$. We can define generators $g_1(t)$, $g_2(t)$ of Γ_t so that $g_1(t) \to 1$ and $g_2(t) \to i$ as $t \to 1$. For small $\varepsilon > 0$, let N_ε be a neighborhood of Γ_1 in $\mathscr{C}(\mathbb{C})$ such that the following holds: Γ_t lies in N_ε if and only if $|1 - g_1(t)| < \varepsilon$ and $|i - g_2(t)| < \varepsilon$. For each t, there are some integers $k_1(t)$, $k_2(t)$ such that $k_1(t)g_1(t) + k_2(t)g_2(t) = \delta(t)$. But since $t \mapsto k_i(t)$ is continuous and $[0,1]$ is connected, k_1, k_2 are constant maps. This contradicts the fact that δ blows up to ∞ when approaching 1. Hence a continuous path $t \mapsto G_t$ cannot exist, and the proof of Theorem 7 is completed in the first case.

6.4. Non-spiraling cases. There are several subcases that we would like to investigate now. In each case, we will provide conditions for the existence of continuous paths starting at particular points in $\mathbb{X}$.

(1) G is parabolic, isomorphic to $\mathbb{Z}$,
(2) G is parabolic, isomorphic to $\mathbb{R} \times \mathbb{Z}$,
(3) G is parabolic, isomorphic to $\mathbb{R}$,
(4) G is parabolic, isomorphic to $\mathbb{C}$,
(5) G is the trivial subgroup $\{1\}$.

(1) Let G be a parabolic group in $\mathbf{C}_2$ isomorphic to $\mathbb{Z}$. We can assume without loss of generality that its fixed point is 0 and that its associated subgroup Γ_1 is $\mathbb{Z} \subset \mathbb{C}$. For small $\varepsilon > 0$, let N_ε be the neighborhood of Γ_1 in $\mathscr{C}(\mathbb{C})$ consisting of the cyclic groups $\langle g_1 \rangle$ and of the lattices $\langle g_1, g_2 \rangle$ with $|1 - g_1| < \varepsilon$ and $g_2 \in \{|\mathrm{Re}(z)| < 1 \text{ and } \mathrm{Im}(z) > 1/\varepsilon\}$. Take $G_0 \in \mathbb{X}$ in a small ball U around G in $\mathbf{C}_2$ for which every element has its associated subgroup in N_ε. Write Γ_0 for the associated subgroup of G_0; it equals either $g_1(0)\mathbb{Z}$ or $\langle g_1(0), g_2(0) \rangle$. An argument in Subsection 6.3 would show easily that if $\delta(0) = k_1 g_1$ then there are no continuous paths $t \mapsto G_t \in U$ such that $G_1 = G$. Moreover, each different choice for $k_1 > 0$ corresponds to a different connected component for U.

Now if $\delta(0) = k_1g_1 + k_2g_2$ with $k_2 > 0$, the path in U defined by $\Gamma_t = \langle g_1(t), g_2(t)\rangle$, $f_1(t) = (1-t)f_1$, $\delta(t) = k_1g_1(t) + k_2g_2(t)$ with $g_1(t) = (1-t)g_1 + t$ and $g_2(t) = g_2 + \frac{i}{1-t}$ connects G_0 to G_1.

(2) Let G be a parabolic group in $\mathbf{C}_2$ isomorphic to $\mathbb{Z}\times\mathbb{R}$. We may again assume that its fixed point is 0 and its associated subgroup Γ_1 is $\mathbb{R}+i\mathbb{Z}$. For small $\varepsilon > 0$, let N_ε be a neighborhood of Γ_1 in $\mathscr{C}(\mathbb{C})$ consisting of the subgroups $\langle g_1, g_2\rangle$ with $|g_1| < \varepsilon$, $|\mathrm{Arg}(g_1)| < \varepsilon$ and $|i - g_2| < \varepsilon$, and of subgroups of $\mathbb{C}$ isomorphic to $\mathbb{Z}\times\mathbb{R}$ which are close enough to Γ_1. Now, as above, let U be a neighborhood of G for which every element has its associated subgroup in N_ε. Discussions as before show that lattice subgroups in N_ε with a choice $\delta(0) = k_1g_1 + k_2g_2$, $k_2 > 0$ each corresponds to one connected component of U that intersects $\mathbf{P}_2$ non-trivially. The novelty in the case $k_2 = 0$ is that g_1 can be made to converge to 0 continuously. In the process, $\delta(t) = k_1g_1(t)$ has to get close to 0 also, thus G_t needs to exit U at some point.

(3) The case where G is isomorphic to $\mathbb{R}$ is very similar to the case (2) but slightly more complicated, since now it is possible to approach Γ by groups of type $\mathbb{Z}$, and we will not elaborate further.

Case (4) is somewhat wilder. Indeed, assuming again G_1 has fixed point 0 and $\Gamma_1 = \mathbb{C}$, neighborhoods of G_1 include copies of the neighborhood of the wedge point of the D-bouquet in the Chabauty space of $\mathbb{C}^*$ (see [2]). We will only attempt to show that G_1 possesses arbitrary small neighborhoods U such that both U and $\mathbb{X}\cap U$ are arcwise connected; this statement somewhat represent an ultimate non-spiraling behavior.

Lemma 14. *Given any subgroup Γ_0, with marked point $\delta = \delta(0)$ and close enough to $\Gamma_1 = \mathbb{C}$ for the Chabauty topology $d_{\mathscr{C}(\mathbb{C})}$, there is a path $t \mapsto (\Gamma_t, \delta(t)) \in \mathscr{M}$ such that $t \mapsto d_{\mathscr{C}(\mathbb{C})}(\Gamma_t, \Gamma_1)$ is decreasing and $\delta(t)$ is constant.*

Proof. Since Γ_0 is close to $\mathbb{C}$, it has to be of type $\mathbb{R}\times\mathbb{Z}$ or a lattice.

First, consider Γ_0 to be a subgroup of $\mathbb{C}$ of type $\mathbb{R}\times\mathbb{Z}$. Let L_1 be the line passing through 0 and δ, and L_2 any line through 0 distinct from L_1 and not included in Γ_0. Then the path $t \mapsto \Gamma_t$ with Γ_t obtained from Γ_0 by applying the linear shear $T_t(L_1, L_2)$ that leaves L_1 pointwise fixed and contracts the direction L_2 by a factor $1/(1-t)$ satisfies that the distance between two consecutive copies of $\mathbb{R}$ in Γ_t continuously decreases, while $\delta(t) \in L_1$ stays fixed.

Now consider Γ_0 to be a lattice. Let L_1 be the line passing through 0 and δ, and L_2 a line passing through 0 and any point in $\Gamma_0 \backslash L_1$. Then applying $T_t(L_1, L_2)$ to Γ_0 yields a path from Γ_0 to a subgroup of type $\mathbb{R}\times\mathbb{Z}$, with $d_{\mathscr{C}(\mathbb{C})}$ decreasing and $\delta(t)$ fixed. Concatenating this path with the path described above finishes the proof. $\square$

Corollary 3. *There exist arbitrary small neighborhoods $U \subset \mathbf{C}_2$ of G_1 such that both U and $\mathbb{X} \cap U$ are arcwise connected.*

Proof. For $\varepsilon > 0$, consider U_ε to be the neighborhood of G_1 consisting of all groups in $\mathbf{C}_2$ with a fixed point $|f_1| < \varepsilon$, with corresponding Γ either of type $\mathbb{R} \times \mathbb{Z}$ and then the distance between two consecutive copies of $\mathbb{R}$ in Γ is less than ε, or a lattice with generators g_1 and g_2, $|g_1|, |g_2| < \varepsilon$, and in both cases with $\delta \in \Gamma$ verifying $|\delta| > 1/\varepsilon$.

Now let $G, G' \in U_\varepsilon$. We want to show there exists a path in U_ε from G to G', with the additionnal requirement that if both groups are in $\mathbb{X}$, the path is in $\mathbb{X}$. By Lemma 14, we can assume that both G and G' have corresponding group $\Gamma = \mathbb{C}$. Then, by concatenating if necessary with a path that moves a fixed point to 0 while leaving Γ and δ constant, we can assume that G and G' have $f_1 = 0$. But then, by concatenating with a path that connects the δ of G and G' via a segment, we see that the claim holds. $\square$

(5) $\{\mathrm{Id}\}$ has arbitrary small arcwise connected neighborhoods U, but $\mathbb{X} \cap U$ always has infinitely many arcwise connected components. First, for $\varepsilon > 0$, let U_ε be the neighborhood consisting of all groups in $\mathbf{C}_2$ with fixed points $|f_i| < \varepsilon$ and with $\Gamma \cap B_{\mathbb{C}}(0, 1/\varepsilon) = \{0\}$. Then for any group G_0 in U_ε, the path in U_ε that moves f_1 to 0 (for instance in a straight line) while expanding Γ_0 by a factor $1/(1-t)$ continuously deforms G_0 into Id. Therefore U_ε is path-connected. The second claim holds because for any given neighborhood U of Id, and any integer $k > 0$ there are groups $G \in U$ such that $[0, \delta] \cap \Gamma$ contains exactly $k+1$ points. As argued before, different choices for k yield different connected components of $U \cap \mathbb{X}$.

7. Summary statement

The following statement collects and summarizes all results in this paper.

Summary Statement The space $\mathbb{X}$ of non-trivial, non-parabolic, closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is homeomorphic to

$$\Theta \times (\mathscr{C}(\mathbb{C}^*) \backslash \{1\}),$$

where $\Theta \cong \mathbb{CP}^2 \backslash \mathbb{CP}^1$ is the space of pairs of points of a 2-sphere (see Subsection 3.3) and $\mathscr{C}(\mathbb{C}^*)$ is the Chabauty space of $\mathbb{C}^*$ (see for instance [2]). See Proposition 1.

Moreover, the space of non-trivial discrete cyclic subgroups of $\mathrm{PSL}_2(\mathbb{C})$ generated by an elliptic generator is homeomorphic to

$$\Theta \times \mathbb{N}_{\geq 2},$$

and the space of non-trivial discrete cyclic subgroups of $\mathrm{PSL}_2(\mathbb{C})$ generated by a hyperbolic generator is homeomorphic to

$$\Theta \times (\mathbb{C} \backslash \overline{\mathbb{D}}).$$

See Proposition 2.

The closure $\mathbf{P}_1$ of the space of cyclic parabolic subgroups in $\mathbf{C}_1 \subset \mathbf{C}_2$ is the one-point compactification of a 4-twist SO_2-bundle of $\overline{\mathbb{D}} \backslash \{0\}$ over S^2 (see Corollary 1). It lies inside the space $\mathbf{P}_2$ of parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$; $\mathbf{P}_2$ is homeomorphic to the one-point compactification of $S^2 \times \mathbb{R}^4$ (see Corollary 2).

Reducing arguments (Theorems 2 and 3) show that the problem of convergence of sequences of elements of $\mathbf{C}_2$ can be reduced to a problem about convergence of the associated closed subgroups of $\mathbb{C}$.

The way $\mathbb{X} = \mathbf{C}_2 \backslash \mathbf{P}_2$ is attached to $\mathbf{P}_2$ derives from Theorem 2. It results from a blow-up phenomenon corresponding to cylinders getting wider and wider; see Subsection 4.3. In spirit, this attachment is very similar to the bending described in [3] (recall Theorem 1).

The whole exhaustion of cases for sequences of closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is described in Section 5. The last case, englobing in particular the case of cyclic hyperbolic subgroups H_n of $\mathbb{X}$ converging to a point in $\mathbf{P}_2$, involves the expansion in continued fraction of the multiplier θ_n of a generator of H, by stopping at some index and then reading the expansion from right to left, and from left to right starting at this index. See Subsection 5.3.

Finally, the description of local models for neighborhoods in $\mathbf{C}_2$ of parabolic groups $G \in \mathbf{C}_2$ is reduced to a problem about marked subgroups of $\mathbb{C}$; see Subsection 6.1. In the generic case where G is isomorphic to $\mathbb{Z}^2$, $\mathbb{X}$ accumulates towards G in a spiraling way. When G is not isomorphic to $\mathbb{Z}^2$, $\mathbb{X}$ accumulates toward G in a non-spiraling way (see Section 6 for more details).

An interesting direction in generalizing this work would be to study the case of elementary subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Since those are precisely the subgroups of $\mathrm{PSL}_2(\mathbb{C})$ with finite index abelian groups, it seems reasonable to expect that the Chabauty space of elementary groups is not too much more complicated.

Also, along the way of Subsection 5.3 we discovered that it was possible to relate some aspects of geometric limits to the continued fraction of some quantity (namely θ_n) by reading its expansion, first backwards starting from some index $j+1$, then forwards starting at the index $j+2$. We would be very interested in finding other occurences of these relations in other parts of mathematics.

Acknowledgement

We thank John H. Hubbard for introducing us to this problem and fora lot of encouragement. We also thank William P. Thurston for helpful discussions. Finally, we gratefully thank Makoto Sakuma and the referee for numerous helpful comments and suggestions on a previous draft which greatly improved the readability of the paper.

References

[1] Ismaël Bachy, Enrichissements de Siegel, Ph.D. thesis, Université de Provence (2011).
[2] Hyungryul Baik & Lucien Clavier, Pictures of the Chabauty space of $\mathbb{C}^*$, arXiv:1209.0221v1 [math.GT] (2012).
[3] Hyungryul Baik & Lucien Clavier, The Space of Geometric Limits of One-generator Closed Subgroups of $\mathrm{PSL}(2, \mathbb{R})$, Algebr. Geom. Topol., **13** (2013), 549–576.
[4] Claude Chabauty, Limite d'ensembles et géométrie des nombres, Bull. Soc. Math. France, **78** (1950), 143–151.
[5] Pierre de la Harpe, Spaces of closed subgroups of locally compact groups, arXiv:0807.2030 [math.GR] (2008).
[6] Allen Hatcher, Topology of Numbers, http://www.math.cornell.edu/~hatcher/#TN.
[7] John Hubbard & Ibrahim Pourezza, The Space of closed subgroups of $\mathbb{R}^2$, Topology, **18** (1979), 2:143–146.
[8] John Hubbard, Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, 2: Four Theorems by William Thurston, Matrix Edition (2014) (unpublished).
[9] Troels Jorgensen & Albert Marden, Algebraic and geometric convergence of Kleinian groups, Math. Scand., **66** (1990), 1:47–72.
[10] Charles G. Moore, An Introduction to Continued Fraction, National Council of Teachers of Mathematics (1964).
[11] William Thurston, The geometry and topology of three-manifolds, Princeton University Lecture Notes (1980).

Hyungryul Baik
Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60
53115 *Bonn, Germany*
E-mail: baik@math.uni-bonn.de
URL: http://www.hbaik.org

Lucien Clavier
Department of Mathematics
Cornell University
111 *Malott Hall*
Ithaca, NY 14853, *USA*
E-mail: lpc49@cornell.edu