Hiroshima Math. J.
45 (2015), 207–230

Galois action on mapping class groups

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(Received May 28, 2014)

Abstract. Let l be a prime number. In the paper, we study the outer Galois action on the profinite and the relative pro-l completions of mapping class groups of pointed orientable topological surfaces. In the profinite case, we prove that the outer Galois action is faithful. In the pro-l case, we prove that the kernel of the outer Galois action has certain stability properties with respect to the genus and the number of punctures. Also, we prove a variant of the above results for arbitrary families of curves.

1. Introduction

Let k be a (commutative) field of characteristic zero, X a smooth geometrically connected curve over k, and (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$ (hyperbolicity). We call X a (g, n)-curve if there exist a proper smooth genus g curve C over k and a closed subscheme $D \subseteq C$ such that $X = C \backslash D$ and the composite $D \hookrightarrow C \to \operatorname{Spec} k$ is a finite étale covering of degree n. Let $\bar{k}$ be an algebraic closure of k. For a (g, n)-curve X, by SGA1 [1], we have a short exact sequence

$$1 \to \pi_1(X \otimes_k \bar{k}) \to \pi_1(X) \to G_k \to 1$$

where π_1 denotes the algebraic fundamental group and $G_k := \operatorname{Gal}(\bar{k}/k)$ is the absolute Galois group of k. Let $\Pi_{g,n}$ denote the profinite completion of the fundamental group $\pi_1(g, n)$ of a compact Riemann surface of genus g with n points punctured. By the comparison theorem, $\pi_1(X \otimes_k \bar{k})$ is isomorphic to $\Pi_{g,n}$. We fix an isomorphism $\pi_1(X \otimes_k \bar{k}) \xrightarrow{\sim} \Pi_{g,n}$. Since $\pi_1(X)$ acts on $\pi_1(X \otimes_k \bar{k})$ by conjugation in the above short exact sequence, $\pi_1(X)$ also acts on $\Pi_{g,n}$. This gives the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \operatorname{Inn}(\Pi_{g,n}) & \longrightarrow & \operatorname{Aut}(\Pi_{g,n}) & \longrightarrow & \operatorname{Out}(\Pi_{g,n}) & \longrightarrow & 1, \end{array}$$

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.
Key words and phrases. mapping class group, outer Galois representation, hyperbolic curve.

where Aut (respectively Inn) denotes the continuous automorphism group (respectively the inner automorphism group) of $\Pi_{g,n}$, and Out denotes the quotient, so that the horizontal sequences are both exact. The right vertical map gives the outer Galois representation

$$\rho_X : G_k \to \mathrm{Out}(\Pi_{g,n}).$$

Note that the kernel of ρ_X is independent of the choice of the isomorphism $\pi_1(X \otimes_k \bar{k}) \xrightarrow{\sim} \Pi_{g,n}$. Belyĭ proved that ρ_X is injective when $X = \mathbf{P}^1_k \backslash \{0, 1, \infty\}$ and k is a number field (Corollary to Theorem 4, [5]). Voevodskiĭ proved the injectivity of ρ_X when the genus of X is 1 and k is a number field, and suggested a conjecture that ρ_X is injective when X is an affine hyperbolic curve and k is a number field ([33]). This conjecture was solved by Matsumoto ([19]). Moreover, the proper case was proved by Hoshi and Mochizuki ([14]). Therefore, we have the following theorem:

Theorem 1.1 (Belyĭ, Voevodskiĭ, Matsumoto, Hoshi-Mochizuki). *The outer Galois representation ρ_X is injective when X is a hyperbolic curve and k is a number field.*

Grothendieck expected that any hyperbolic curve over a number field would be anabelian, i.e., the geometry of any hyperbolic curve X over a number field is determined by ρ_X (the Grothendieck conjecture for algebraic curves, [11]). This conjecture was proved by Mochizuki ([21, 22]) following earlier work of Nakamura and Tamagawa. The above theorem can be regarded as an evidence that ρ_X is highly complicated when k is a number field.

On the other hand, Grothendieck expected that the moduli space of hyperbolic curves would be also anabelian ([11]). Therefore, it is a natural problem whether Voevodskiĭ's conjecture holds in the case when X is the moduli space of hyperbolic curves. Let $\mathcal{M}_{g,n}$ be the moduli stack over k of smooth geometrically connected proper curves of genus g with n (ordered) marked points ([8, 17]). It is known that $\pi_1(\mathcal{M}_{g,n} \otimes \bar{k})$ is isomorphic to the profinite completion $\Gamma_{g,n}$ of the oriented mapping class group $\mathrm{MCG}_{g,n}$ of an n-pointed genus g topological surface ([28]). As above, we have the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Gamma_{g,n}) & \longrightarrow & \mathrm{Aut}(\Gamma_{g,n}) & \longrightarrow & \mathrm{Out}(\Gamma_{g,n}) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are both exact. The right vertical map gives the outer Galois representation

$$\rho_{g,n} : G_k \to \mathrm{Out}(\Gamma_{g,n}).$$

For the injectivity of $\rho_{g,n}$, our result in the present paper is summarized in the following (cf. Theorem 2.3):

THEOREM 1.2. *Let k be a number field and (g,n) a pair of nonnegative integers such that $2g-2+n>0$. Then the homomorphism $\rho_{g,n+1}$ is injective.*

REMARK 1.3. As $\mathcal{M}_{0,4} = \mathbf{P}^1_k \backslash \{0,1,\infty\}$, the injectivity of $\rho_{0,4}$ agrees with the above theorem of Belyĭ (Corollary to Theorem 4, [5]).

The proof of Theorem 1.2 yields a variant, where we consider an arbitrary family of hyperbolic curves instead of the universal family $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. As above, for any geometrically connected locally noetherian scheme X over k, we can consider the outer Galois representation $\rho_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \bar{k}))$ determined by the exact sequence

$$1 \to \pi_1(X \otimes_k \bar{k}) \to \pi_1(X) \to G_k \to 1.$$

Grothendieck expected that hyperbolic polycurves (i.e., successive families of hyperbolic curves) would be also anabelian ([11]). The injectivity of ρ_X is implicit in [14] when X is a hyperbolic polycurve and k is a number field. We can prove the injectivity of ρ_X when X is an arbitrary family of hyperbolic curves (cf. Theorem 4.3):

THEOREM 1.4. *Let k be a number field and (g,n) a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected normal scheme of finite type over k and $X \to S$ a family of (g,n)-curves over S. Then the homomorphism ρ_X is injective.*

Hoshi and Tamagawa informed the author of a different proof of Theorem 1.2. In fact, their proof gave a result stronger than Theorem 1.2, as we will see shortly. By Oda's theory ([28]) and using the Birman exact sequence (Chapter 4, [9])

$$1 \to \pi_1(g,n) \to \mathrm{MCG}_{g,n+1} \to \mathrm{MCG}_{g,n} \to 1,$$

we have the following exact sequence (cf. Lemma 2.1 in [20]):

$$1 \to \Pi_{g,n} \to \pi_1(\mathcal{M}_{g,n+1}) \to \pi_1(\mathcal{M}_{g,n}) \to 1.$$

This exact sequence gives the universal monodromy representation

$$\rho^{univ}_{g,n} : \pi_1(\mathcal{M}_{g,n}) \to \mathrm{Out}(\Pi_{g,n}).$$

It is known that the homomorphism $\rho^{univ}_{g,n}$ is injective if and only if $\rho^{univ}_{g,n}|_{\Gamma_{g,n}}$ is injective (Corollary 6.5, [14]).

REMARK 1.5. The problem of the injectivity of $\rho^{univ}_{g,n}|_{\Gamma_{g,n}}$ is called the congruence subgroup problem for $\mathrm{MCG}_{g,n}$. The congruence subgroup problem was solved in the affirmative for $g \le 1$ by Asada ([4]) and for $g = 2$, $n > 0$ by Boggi ([6]). Boggi called the image of $\rho_{g,n}^{univ}|_{\Gamma_{g,n}}$ the geometric profinite completion of $\mathrm{MCG}_{g,n}$ in [6].

We denote by

$$\rho_{g,n}^{geom} : G_k \to G_k^{g,n} \to \mathrm{Out}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$$

the natural homomorphism determined by the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathscr{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \rho_{g,n}^{univ}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n})) & \longrightarrow & G_k^{g,n} & \longrightarrow & 1, \end{array}$$

where $G_k^{g,n} := \rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))/\rho_{g,n}^{univ}(\Gamma_{g,n})$, and the horizontal sequences are exact.

Theorem 1.6 (Hoshi-Tamagawa). *Let k be a number field and (g,n) a pair of nonnegative integers such that $3g - 3 + n > 0$. Then the homomorphism $\rho_{g,n}^{geom}$ is injective. In particular, $\rho_{g,n}$ is injective.*

We remark that Boggi also announced the same result (Corollary 7.6, [7]). Boggi's proof depends on the theory of complexes of profinite curves developed by him. On the other hand, our proof depends on the combinatorial anabelian geometry developed by Hoshi-Mochizuki.

Next, we consider a pro-l version of Theorem 1.6, where l is a prime number. Let $\Pi_{g,n}^l$ denote the pro-l completion of the fundamental group of a Riemann surface of genus g with n points punctured. For a (g,n)-curve X over k, by the functoriality of pro-l completion, we obtain

$$\rho_X^l : G_k \to \mathrm{Out}(\Pi_{g,n}^l).$$

As above, we have the pro-l universal monodromy representation

$$\rho_{g,n}^{univ,l} : \pi_1(\mathscr{M}_{g,n}) \to \mathrm{Out}(\Pi_{g,n}^l).$$

Therefore, we also have the natural homomorphism

$$\rho_{g,n}^{geom,l} : G_k \to G_k^{l,g,n} \to \mathrm{Out}(\rho_{g,n}^{univ,l}(\Gamma_{g,n}))$$

determined by the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathscr{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \rho_{g,n}^{univ,l}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ,l}(\pi_1(\mathscr{M}_{g,n})) & \longrightarrow & G_k^{l,g,n} & \longrightarrow & 1, \end{array}$$

where $G_k^{l,g,n} := \rho_{g,n}^{univ,l}(\pi_1(\mathscr{M}_{g,n}))/\rho_{g,n}^{univ,l}(\Gamma_{g,n})$, and the horizontal sequences are exact. The field determined by $\mathrm{im}(\ker(\rho_{g,n}^{univ,l}) \to G_k)(= \ker(G_k \to G_k^{l,g,n}))$ can be regarded as a field of definition of the Teichmüller modular function field with l-power level structures. Oda conjectured that this field is independent of (g,n) ([27]). This conjecture was proved by using the weight filtration and the universal deformation of a maximally degenerate stable curve ([26, 25, 19, 16, 32]). We prove the second main result in the present paper by using Oda's conjecture (cf. Theorem 3.4):

Theorem 1.7. *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$, and that either $(g,n) \neq (1,1)$ or $l=2$. Then the kernel of the homomorphism $\rho_{g,n}^{geom,l}$ coincides with the kernel of the homomorphism*

$$\rho^l_{\mathbf{P}^1_k \setminus \{0,1,\infty\}} : G_k \to \mathrm{Out}(\Pi^l_{0,3}).$$

We apply Theorem 1.7 to the relative pro-l representation (Corollary 3.8).

The present paper is organized as follows: In section 2, we study the profinite case. First, we prove a technical lemma (Lemma 2.2) in group theory and we derive Theorem 1.2 from this lemma. Secondly, we explain a proof of Theorem 1.6 due to Hoshi and Tamagawa by using a geometric version of the Grothendieck conjecture. In section 3, we prove Theorem 1.7 by using a geometric version of the Grothendieck conjecture and Oda's conjecture. Finally, we study the kernel of the relative pro-l representation. In section 4, we prove a variant of Theorem 1.2 (including Theorem 1.4) which does not follow from the method of Hoshi and Tamagawa.

Notations and Conventions

Numbers: The notation $\mathbf{Z}$ will be used to denote the set, group, or ring of rational integers and the notation $\mathbf{Q}$ will be used to denote the set, group, or field of rational numbers. We shall refer to a finite extension of $\mathbf{Q}$ as a number field. For a prime number l, the notation $\mathbf{Z}_l$ will be used to denote the set, group, or ring of l-adic integers and the notation $\mathbf{Q}_l$ will be used to denote the set, group, or field of l-adic numbers. We shall refer to a finite extension of $\mathbf{Q}_l$ as an l-adic local field. The notation $\mathbf{C}$ will be used to denote the set, group, or field of complex numbers.

Groups: If G is a topological group, and $H \subseteq G$ is a closed subgroup of G, then we shall write $Z_G(H)$ for the centralizer of H in G, i.e.,

$$Z_G(H) := \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\} \subseteq G,$$

and we shall write $N_G(H)$ for the normalizer of H in G, i.e.,

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\} \subseteq G.$$

If G is a topological group, then we shall denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, by $\mathrm{Inn}(G)$ the group of inner automorphisms of G, by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$.

For a discrete group G and a prime number l, we shall say that G is conjugacy l-separable if, for any $g, h \in G$, it holds that either g is conjugate to h or there exists a homomorphism $\varphi : G \to P$ such that P is an l-group and $\varphi(g)$ is not conjugate to $\varphi(h)$. For a discrete group G, we shall say that G has Property A if, for any $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(g)$ is conjugate to g for any $g \in G$, it holds that $\alpha \in \mathrm{Inn}(G)$.

Surface groups and mapping class groups: For a pair (g, n) of nonnegative integers and a prime number l, the notation $\Pi_{g,n}$ will be used to denote the profinite completion of the fundamental group $\pi_1(g, n)$ of a compact Riemann surface $R_{g,n}$ of genus g with n points punctured, the notation $\Pi_{g,n}^l$ will be used to denote the pro-l completion of $\pi_1(g, n)$, the notation $\mathrm{MCG}_{g,n}$ will be used to denote the mapping class group of (g, n)-type, namely the discrete group of isotopy classes of orientation preserving self-diffeomorphisms of an orientable surface of genus g with n points marked which fix the n points pointwise, the notation $\mathrm{MCG}_{g,[n]}$ will be used to denote the discrete group of isotopy classes of orientation preserving self-diffeomorphisms of an orientable surface of genus g with n points punctured which preserve the set of punctures, and the notation $\Gamma_{g,n}$ will be used to denote the profinite completion of $\mathrm{MCG}_{g,n}$. We shall denote by $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$ the subgroup of $\mathrm{Out}(\Pi_{g,n})$ consisting of elements which preserve the set of conjugacy classes of the cuspidal inertia subgroups of $\Pi_{g,n}$, and by $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n}^l)$ the subgroup of $\mathrm{Out}(\Pi_{g,n}^l)$ consisting of elements which preserve the set of conjugacy classes of the cuspidal inertia subgroups of $\Pi_{g,n}^l$. Here, a conjugacy class of a cuspidal inertia subgroup of $\Pi_{g,n}$ (respectively, $\Pi_{g,n}^l$) means a conjugacy class of the closure of the image of an inertia subgroup of a point punctured of $R_{g,n}$ in $\pi_1(g, n)$ by the natural homomorphism $\pi_1(g, n) \to \Pi_{g,n}$ (respectively, $\pi_1(g, n) \to \Pi_{g,n}^l$).

Curves: Let $f : X \to S$ be a morphism of schemes. Then for a pair (g, n) of nonnegative integers such that $2g - 2 + n > 0$, we shall say that f is a family of (g, n)-curves over S if there exist a proper smooth geometrically connected morphism $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ whose geometric fibers are of dimension one and of genus g, and a relative divisor $D \subseteq X^{\mathrm{cpt}}$ which is finite étale over S of degree n such that X and $X^{\mathrm{cpt}} \backslash D$ are isomorphic over S. We shall say that $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ is a compactification of $f : X \to S$ and $D \subseteq X^{\mathrm{cpt}}$ is a divisor at infinity of $f : X \to S$. We shall say that a family of (g, n)-curves $X \to S$ is split if a finite étale covering $D \to S$ obtained by a divisor at infinity of $X \to S$ is trivial, i.e., D is isomorphic to the disjoint union of n copies of S over S. Note that the pair (X^{cpt}, D) is unique up to canonical isomorphism if S is normal (e.g., Section 0, [23]). In particular, we shall refer to a family of (g,n)-curves over the spectrum of a field k as a (g,n)-curve over k.

Fundamental groups: Let l be a prime number, k a field, and $\bar{k}$ an algebraic closure of k. For a scheme X which is a geometrically connected and of finite type over k, we shall write $\pi_1(X \otimes_k \bar{k})^l$ for the maximal pro-l quotient of $\pi_1(X \otimes_k \bar{k})$, and $\pi_1(X)^{\underline{l}}$ for the quotient of $\pi_1(X)$ by the kernel of the natural surjection $\pi_1(X \otimes_k \bar{k}) \to \pi_1(X \otimes_k \bar{k})^l$.

2. Profinite mapping class groups

In the present section, we prove Theorems 1.2, 1.6. Let k be a field of characteristic zero, (g,n) a pair of nonnegative integers such that $2g-2+n > 0$, $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ determined by a fixed algebraic closure $\bar{k}$ of k, and $G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The following theorem was proved by Matsumoto and Tamagawa (Theorem 1.1, [20]) in the affine case, and more recently by Hoshi and Mochizuki (Corollary 6.4, [14]) in the proper case.

Theorem 2.1. *Let X be a (g,n)-curve over k. Then the subgroup*

$$\rho_X^{-1}(\rho_{g,n}^{univ}(\Gamma_{g,n})) \subseteq G_k$$

of G_k is contained in the kernel of the homomorphism

$$G_k \to G_{\mathbf{Q}}$$

determined by the natural inclusion $\mathbf{Q} \hookrightarrow k$.

Lemma 2.2. *Consider the following commutative diagram of groups where the vertical and horizontal sequences are exact:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & K & \longrightarrow & \Gamma' & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G & \longrightarrow & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1. & & & &
\end{array}$$

Let $\rho_G : H \to \mathrm{Out}(K)$, $\rho_{G'} : H \to \mathrm{Out}(\Gamma')$, $\rho_{\Gamma'} : \Gamma \to \mathrm{Out}(K)$ *denote the natural homomorphisms determined by the above commutative diagram. Then the subgroup*

$$\rho_G(\ker(\rho_{G'})) \subseteq \mathrm{Out}(K)$$

of $\mathrm{Out}(K)$ *is contained in the image of* $\rho_{\Gamma'}$.

Proof. Let h be an element of the kernel of $\rho_{G'}$. Since G surjects onto H, we can take $h' \in G$ mapped to $h \in H$. By the injectivity of the homomorphism $G \to G'$, we may regard h' as an element of G'. Then there exists an element γ of Γ' such that $\mathrm{Inn}(h')$ acts on Γ' by $\mathrm{Inn}(\gamma)$. In particular, $\mathrm{Inn}(h')$ acts on K by $\mathrm{Inn}(\gamma)$. This means $\rho_G(h) \in \mathrm{im}(\rho_{\Gamma'})$. □

Theorem 2.3. *Let* (g, n) *be a pair of nonnegative integers such that* $2g - 2 + n > 0$. *Then the kernel of the homomorphism* $\rho_{g,n+1}$ *is contained in the kernel of the homomorphism*

$$G_k \to G_{\mathbf{Q}}$$

determined by the natural inclusion $\mathbf{Q} \hookrightarrow k$.

In particular, if k *is a number field or an* l*-adic local field, then the homomorphism* $\rho_{g,n+1}$ *is injective.*

Proof. The morphism $\mathscr{M}_{g,n+1} \to \mathscr{M}_{g,n}$ given by forgetting the last marked point induces the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \pi_1(\mathscr{M}_{g,n+1}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Gamma_{g,n} & \longrightarrow & \pi_1(\mathscr{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

where the horizontal sequences are exact, and the vertical arrows are surjective. In particular, by the surjectivity of the left-hand vertical arrow of the above diagram and the right-hand vertical equality of the above diagram, the middle vertical arrow of the above diagram induces an injection

$$Z_{\pi_1(\mathscr{M}_{g,n+1})}(\Gamma_{g,n+1})/Z_{\Gamma_{g,n+1}}(\Gamma_{g,n+1}) \hookrightarrow Z_{\pi_1(\mathscr{M}_{g,n})}(\Gamma_{g,n})/Z_{\Gamma_{g,n}}(\Gamma_{g,n}).$$

Therefore, since the surjection $\pi_1(\mathscr{M}_{g,n}) \twoheadrightarrow G_k$ (respectively, $\pi_1(\mathscr{M}_{g,n+1}) \twoheadrightarrow G_k$) induces the natural isomorphism

$$Z_{\pi_1(\mathscr{M}_{g,n})}(\Gamma_{g,n})/Z_{\Gamma_{g,n}}(\Gamma_{g,n}) \xrightarrow{\sim} \ker(\rho_{g,n})$$

$$(\text{respectively, } Z_{\pi_1(\mathscr{M}_{g,n+1})}(\Gamma_{g,n+1})/Z_{\Gamma_{g,n+1}}(\Gamma_{g,n+1}) \xrightarrow{\sim} \ker(\rho_{g,n+1})),$$

it holds that $\ker(\rho_{g,n+1}) \subseteq \ker(\rho_{g,n})$. Thus, we may assume that n is small, so that there exists a (g, n)-curve X over k such that a divisor at infinity of

$X \to \operatorname{Spec} k$ is split. (Indeed, the case where $g < 2$ is trivial, and the case where $g \geq 2$ follows from the consideration of a hyperelliptic curve.) Since $\mathscr{M}_{g,n+1}$ is the universal curve over $\mathscr{M}_{g,n}$ (see [17]), we obtain a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} k \\ \big\downarrow & \square & \big\downarrow \\ \mathscr{M}_{g,n+1} & \longrightarrow & \mathscr{M}_{g,n}. \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & & & \\ 1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(\mathscr{M}_{g,n+1}) & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ & & G_k & = & G_k & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ & & 1 & & 1, & & & & \end{array}$$

where the vertical and horizontal sequences are exact. Then Lemma 2.2 implies that

$$\rho_X(\ker(\rho_{g,n+1})) \subseteq \operatorname{im}(\Gamma_{g,n} \to \operatorname{Out}(\Pi_{g,n})).$$

By using Theorem 2.1, the result follows. □

Next, we explain a different proof of Theorem 2.3 due to Hoshi and Tamagawa, using a geometric version of the Grothendieck conjecture. In fact, their proof gives a result stronger than Theorem 2.3. The following theorem plays an essential role in their proof.

Theorem 2.4 (Theorem D, [15]). *Let (g,n) be a pair of nonnegative integers such that $3g - 3 + n > 0$ and l a prime number.*

(i) *The group $Z_{\operatorname{Out}^{\mathrm{C}}(\Pi_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$ is isomorphic to*

$$\begin{cases} \mathbf{Z}/2 \times \mathbf{Z}/2 & \text{if } (g,n) = (0,4); \\ \mathbf{Z}/2 & \text{if } (g,n) \in \{(1,1),(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,n) \notin \{(0,4),(1,1),(1,2),(2,0)\}. \end{cases}$$

(ii) *Suppose that*

$$(g,n) \neq (1,1).$$

Then the group $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^l_{g,n})}(\rho^{univ,l}_{g,n}(\Gamma_{g,n}))$ *is isomorphic to*

$$\begin{cases} \mathbf{Z}/2 \times \mathbf{Z}/2 & \text{if } (g,n) = (0,4); \\ \mathbf{Z}/2 & \text{if } (g,n) \in \{(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,n) \notin \{(0,4),(1,2),(2,0)\}. \end{cases}$$

(iii) *Suppose that* $l = 2$. *Then the group* $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^l_{1,1})}(\rho^{univ,l}_{1,1}(\Gamma_{1,1}))$ *is isomorphic to* $\mathbf{Z}/2$.

The proof of Theorem 2.4 is very sophisticated, using the theory of profinite Dehn twists developed in [15].

Theorem 2.5 (Hoshi-Tamagawa). *Let* (g,n) *be a pair of nonnegative integers such that* $3g-3+n>0$. *Then the kernel of the homomorphism* $\rho^{geom}_{g,n}$ *is contained in the kernel of the homomorphism*

$$G_k \to G_{\mathbf{Q}}$$

determined by the natural inclusion $\mathbf{Q} \hookrightarrow k$.

In particular, if k *is a number field or an* l*-adic local field, then the homomorphisms* $\rho^{geom}_{g,n}$ *and* $\rho_{g,n}$ *are injective.*

Proof. We denote by $\mathscr{M}_{g,n/\mathbf{Q}}$ the moduli stack over $\mathbf{Q}$ of smooth geometrically connected proper curves of genus g with n (ordered) marked points. Then the natural morphism $\mathscr{M}_{g,n} \to \mathscr{M}_{g,n/\mathbf{Q}}$ determined by the natural inclusion $\mathbf{Q} \hookrightarrow k$ induces the commutative diagram

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ \rho^{univ}_{g,n}(\Gamma_{g,n}) & = & \rho^{univ}_{g,n}(\Gamma_{g,n}) \\ \downarrow & & \downarrow \\ \pi_1(\mathscr{M}_{g,n})/\ker(\Gamma_{g,n} \twoheadrightarrow \rho^{univ}_{g,n}(\Gamma_{g,n})) & \longrightarrow & \pi_1(\mathscr{M}_{g,n/\mathbf{Q}})/\ker(\Gamma_{g,n} \twoheadrightarrow \rho^{univ}_{g,n}(\Gamma_{g,n})) \\ \downarrow & & \downarrow \\ G_k & \longrightarrow & G_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

where the vertical sequences are exact. In particular, $\rho_{g,n}^{geom}$ factors through $\rho_{g,n}^{geom}$ in the case where k is $\mathbf{Q}$. Therefore, to verify Therem 2.5, it suffices to verify the injectivity of $\rho_{g,n}^{geom}$ in the case where k is $\mathbf{Q}$. Thus, suppose that k is $\mathbf{Q}$. Note that $G_{\mathbf{Q}}^{g,n} := \rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))/\rho_{g,n}^{univ}(\Gamma_{g,n})$ is isomorphic to $G_{\mathbf{Q}}$ by Theorem 2.1. Also, by Theorem 2.3 and the injectivity of $\rho_{g,n}^{univ}$ when g is zero (Theorem 3A, [4]), we may assume that $g > 0$. Then the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \rho_{g,n}^{univ}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n})) & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\rho_{g,n}^{univ}(\Gamma_{g,n})) & \longrightarrow & \mathrm{Aut}(\rho_{g,n}^{univ}(\Gamma_{g,n})) & \longrightarrow & \mathrm{Out}(\rho_{g,n}^{univ}(\Gamma_{g,n})) & \longrightarrow & 1
\end{array}$$

induces an isomorphism

$$\begin{aligned}
&Z_{\rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))}(\rho_{g,n}^{univ}(\Gamma_{g,n}))/Z_{\rho_{g,n}^{univ}(\Gamma_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n})) \\
&\quad \simeq \ker(G_{\mathbf{Q}} \to \mathrm{Out}(\rho_{g,n}^{univ}(\Gamma_{g,n}))).
\end{aligned}$$

Therefore, it is enough to prove

$$Z_{\rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))}(\rho_{g,n}^{univ}(\Gamma_{g,n}))/Z_{\rho_{g,n}^{univ}(\Gamma_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n})) = \{1\}.$$

Note that the image of $\rho_{g,n}^{univ}$ is contained in $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$. By the injectivity of $\mathrm{MCG}_{g,[n]} \to \mathrm{Out}(\pi_1(g,n))$ (e.g., Theorem 8.8, in [9]) and $\mathrm{Out}(\pi_1(g,n)) \to \mathrm{Out}(\Pi_{g,n})$ (Lemma 3.2.1 in [2] for $n > 0$ and [10] for $n = 0$), we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{MCG}_{g,n} & \longrightarrow & \mathrm{Out}(\pi_1(g,n)) \\
\big\downarrow\cap & \nearrow & \big\downarrow \\
\mathrm{MCG}_{g,[n]} & \hookrightarrow & \mathrm{Out}(\Pi_{g,n}).
\end{array}$$

Since an element of $\mathrm{MCG}_{g,[n]}$ induces an action on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$, an element of $\mathrm{MCG}_{g,[n]}$ induces an action on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$. Note that there exits a canonical bijection between the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$ and the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$. Hence, the image of $\mathrm{MCG}_{g,[n]} \hookrightarrow \mathrm{Out}(\Pi_{g,n})$ is contained in $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$. In particular, we have the natural inclusion $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]}) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$. Also, by Theorem 2.4 (i), and section 4 of Chapter 3 in [9], the cardinality of $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]})$ is finite, and is equal to the cardinality of $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$. Thus, the above inclusion $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]}) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n}))$ is an isomorphism. If the image σ' of an element σ of $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]})$ is not contained in $\rho_{g,n}^{univ}(\Gamma_{g,n})$, then σ is not contained in $\mathrm{MCG}_{g,n}$. Since the action of $\mathrm{MCG}_{g,[n]}/\mathrm{MCG}_{g,n}$ on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$ is faithful, σ induces a nontrivial action on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$. Therefore, σ' induces a nontrivial action on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$. Since the action of $\rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))$ on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{g,n}$ is trivial by the definition of $\pi_1(\mathscr{M}_{g,n})$, σ' is not contained in $\rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))$. Hence, we have $Z_{\rho_{g,n}^{univ}(\pi_1(\mathscr{M}_{g,n}))}(\rho_{g,n}^{univ}(\Gamma_{g,n}))/Z_{\rho_{g,n}^{univ}(\Gamma_{g,n})}(\rho_{g,n}^{univ}(\Gamma_{g,n})) = \{1\}$. $\square$

3. Pro-l mapping class groups

In the present section, we prove Theorem 1.7. Let l be a prime number and assume that the base field k is a field of characteristic zero.

Lemma 3.1. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$. Then the natural homomorphism $\pi_1(g,n) \to \Pi_{g,n}^l$ is injective.*

Proof. It follows immediately from the fact that $\pi_1(g,n)$ is conjugacy l-separable (Theorem 3.2, Theorem 4.1 in [29]). $\square$

By the above lemma, we can consider $\pi_1(g,n)$ as a subgroup of $\Pi_{g,n}^l$.

Lemma 3.2. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$. Then the group $N_{\Pi_{g,n}^l}(\pi_1(g,n))$ is equal to $\pi_1(g,n)$. In particular, the natural homomorphism $\mathrm{Out}(\pi_1(g,n)) \to \mathrm{Out}(\Pi_{g,n}^l)$ induced by $\pi_1(g,n) \hookrightarrow \Pi_{g,n}^l$ is injective.*

Proof. It is clear that $N_{\Pi_{g,n}^l}(\pi_1(g,n)) \supseteq \pi_1(g,n)$ by the definition of normalizer. Let a be an element of $N_{\Pi_{g,n}^l}(\pi_1(g,n))$. Then, for any element γ of $\pi_1(g,n)$, γ is conjugate to $a\gamma a^{-1}$ in $\pi_1(g,n)$ by the fact that $\pi_1(g,n)$ is conjugacy l-separable (Theorem 3.2, Theorem 4.1 in [29]). Therefore, since $\pi_1(g,n)$ has Property A (Lemma 1, Theorem 3 in [10] (cf. "Groups" in Notations and Conventions for the definition of Property A)), there exists an element h of $\pi_1(g,n)$ such that $a\gamma a^{-1} = h\gamma h^{-1}$ for any element γ of $\pi_1(g,n)$. Since $\Pi_{g,n}^l$ is center-free (Proposition 1.4 in [24]) and $\pi_1(g,n)$ is dense in $\Pi_{g,n}^l$, we have $a = h \in \pi_1(g,n)$. $\square$

Remark 3.3. These lemmas may be well-known. At least, Lemma 3.2 was proved for special cases by several people (e.g., Proposition 1, [18], Corollary 2 to Proposition B2, [3]).

Theorem 3.4. *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$, and that either $(g,n) \neq (1,1)$ or $l=2$. Then the kernel of the* *homomorphism* $\rho_{g,n}^{geom,l}$ *coincides with the kernel of the homomorphism*

$$\rho^l_{\mathbf{P}^1_k\setminus\{0,1,\infty\}} : G_k \to \mathrm{Out}(\Pi^l_{0,3}).$$

Proof. First, suppose that g is equal to 0. We denote by $\Gamma^l_{g,n}$ the maximal pro-l quotient of $\Gamma_{g,n}$, and by $i^l_{g,n} : \mathrm{Out}(\Gamma_{g,n}) \to \mathrm{Out}(\Gamma^l_{g,n})$ the homomorphism determined by the natural surjection $\Gamma_{g,n} \twoheadrightarrow \Gamma^l_{g,n}$. Then, since $\mathcal{M}_{g,n}$ is isomorphic to a configuration space of $\mathbf{P}^1_k\setminus\{0,1,\infty\}$, the kernel of the composite of $\rho_{g,n}$ and $i^l_{g,n}$ is equal to the kernel of $\rho^l_{\mathbf{P}^1_k\setminus\{0,1,\infty\}}$ (Theorem C, (i), [14]), and there exists an isomorphism $\Gamma^l_{g,n} \xrightarrow{\sim} \rho^{univ,l}_{g,n}(\Gamma_{g,n})$ that is compatible with the outer actions of G_k on either side (Remark to Theorem 1, [4]). Therefore, Theorem 3.4 holds in the case where g is equal to 0. Thus, we may assume that $g > 0$. As in the proof of Theorem 2.5, we can show that the natural homomorphism

$$G_k^{l,g,n} \to \mathrm{Out}(\rho^{univ,l}_{g,n}(\Gamma_{g,n}))$$

is injective. Here, $G_k^{l,g,n}$ is the group

$$\rho^{univ,l}_{g,n}(\pi_1(\mathcal{M}_{g,n}))/\rho^{univ,l}_{g,n}(\Gamma_{g,n}).$$

Indeed, the arguments of the proof of Theorem 2.5 go well as they are, if we replace Theorem 2.4 (i) with Theorem 2.4 (ii), (iii) and the injectivity of $\mathrm{Out}(\pi_1(g,n)) \to \mathrm{Out}(\Pi_{g,n})$ with the injectivity of $\mathrm{Out}(\pi_1(g,n)) \to \mathrm{Out}(\Pi^l_{g,n})$ (Lemma 3.2). Therefore, it is sufficient to prove that

$$\ker(G_k \to G_k^{l,g,n}) = \ker(\rho^l_{\mathbf{P}^1_k\setminus\{0,1,\infty\}}).$$

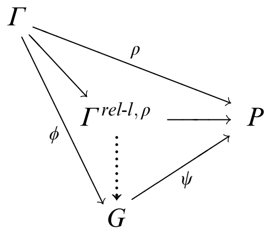
Let $p_{g,n} : \pi_1(\mathcal{M}_{g,n}) \to G_k$ be the natural homomorphism. Then we have

$$\ker(G_k \to G_k^{l,g,n}) = p_{g,n}(\ker(\rho^{univ,l}_{g,n})).$$

However, it is known that $p_{g,n}(\ker(\rho^{univ,l}_{g,n}))$ coincides with $\ker(\rho^l_{\mathbf{P}^1_k\setminus\{0,1,\infty\}})$ (Oda's conjecture, cf. Theorem 3.3, [32]). This completes the proof. □

Next, we consider the relative pro-l case. Since all mapping class groups in genus g are perfect when $g \geq 3$, their pro-l completions are trivial. However, Hain and Matsumoto developed a theory of relative pro-l completion of groups, and showed that the natural relative pro-l completions of mapping class groups are large and more closely reflect their structure ([12]). We explain below their theory.

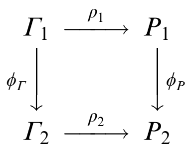
Let Γ be a discrete or profinite group, P a profinite group, and $\rho : \Gamma \to P$ a continuous dense homomorphism. (Here, a dense homomorphism means a homomorphism with dense image.) The relative pro-l completion $\Gamma^{rel\text{-}l,\rho}$ of Γ with respect to ρ is characterized by a universal mapping property: If G is a profinite group, $\psi : G \to P$ a continuous homomorphism with pro-l kernel, and if $\phi : \Gamma \to G$ is a continuous homomorphism whose composition with ψ is ρ, then there is a unique continuous homomorphism $\Gamma^{rel\text{-}l,\rho} \to G$ that extends ϕ:



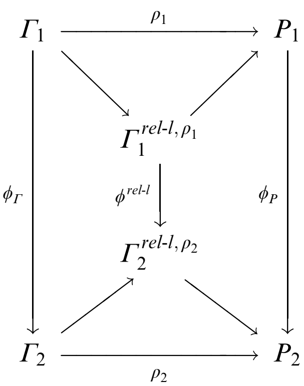
The following properties are direct consequences of the universal mapping property:

Proposition 3.5 (Proposition 2.1, [12]). *Let $\rho : \Gamma \to P$ be a dense homomorphism from a discrete group to a profinite group, and $\bar{\rho} : \hat{\Gamma} \to P$ the homomorphism obtained from the profinite completion of Γ to P. Then the natural homomorphism $\Gamma \to \hat{\Gamma}$ induces a natural isomorphism $\Gamma^{rel\text{-}l,\rho} \to \hat{\Gamma}^{rel\text{-}l,\bar{\rho}}$.*

Proposition 3.6 (Proposition 2.3, [12]). *Let Γ_1, Γ_2 be both discrete groups or both profinite groups, and P_1, P_2 profinite groups. Suppose that $\rho_1 : \Gamma_1 \to P_1$ and $\rho_2 : \Gamma_2 \to P_2$ are continuous dense homomorphisms. If*



is a commutative diagram of topological groups, then there is a unique continuous homomorphism $\phi^{rel\text{-}l} : \Gamma_1^{rel\text{-}l,\rho_1} \to \Gamma_1^{rel\text{-}l,\rho_2}$ such that the diagram



commutes.

Proposition 3.7 (Proposition 2.4, [12]). *Let P_1, P_2, P_3 be profinite groups, and $\rho_1 : \Gamma_1 \to P_1$, $\rho_2 : \Gamma_2 \to P_2$, $\rho_3 : \Gamma_3 \to P_3$ continuous dense homomorphisms of topological groups. Suppose that Γ_1, Γ_2 and Γ_3 are all discrete groups or all profinite groups. If the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 1 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 1 \end{array}$$

of topological groups commutes and has two rows exact, then the sequence

$$\Gamma_1^{rel\text{-}l,\rho_1} \to \Gamma_2^{rel\text{-}l,\rho_2} \to \Gamma_3^{rel\text{-}l,\rho_3} \to 1$$

is exact.

Let $\mathscr{A}_g$ be the moduli stack of principally polarized abelian varieties of dimension g. It is known that the orbifold fundamental groups $\pi_1^{\mathrm{orb}}(\mathscr{M}_{g,n}(\mathbf{C}))$ and $\pi_1^{\mathrm{orb}}(\mathscr{A}_g(\mathbf{C}))$ of $\mathscr{M}_{g,n}(\mathbf{C})$ and $\mathscr{A}_g(\mathbf{C})$ are isomorphic to $\mathrm{MCG}_{g,n}$ and $Sp_g(\mathbf{Z})$ respectively. Here, $Sp_g(A)$ is the group of symplectic $2g \times 2g$ matrices with entries in a commutative ring A. Let

$$\rho^{period} : \mathrm{MCG}_{g,n} \to Sp_g(\mathbf{Z})$$

be the surjective homomorphism determined by the period map $\mathscr{M}_{g,n}(\mathbf{C}) \to \mathscr{A}_g(\mathbf{C})$ which takes the moduli point $[C]$ of a compact Riemann surface C (equipped with n marked points) to that of its jacobian $[\mathrm{Jac}(C)]$ (also see Chapter 6, [9]). Then ρ^{period} induces the continuous dense homomorphism

$$\rho^{period,l} : \mathrm{MCG}_{g,n} \to Sp_g(\mathbf{Z}_l).$$

Hain and Matsumoto defined the relative pro-l completion of the mapping class group by

$$\Gamma_{g,n}^{rel\text{-}l} := \mathrm{MCG}_{g,n}^{rel\text{-}l,\rho^{period,l}}.$$

Let $\bar{\rho}^{period,l} : \Gamma_{g,n} \to Sp_g(\mathbf{Z}/l)$ be the homomorphism determined by ρ^{period}. Then, by using Proposition 3.5 and the universal mapping property, we have the natural isomorphism

$$\Gamma_{g,n}^{rel\text{-}l} \simeq \Gamma_{g,n}^{rel\text{-}l,\bar{\rho}^{period,l}}.$$

This means that $\Gamma_{g,n}^{rel\text{-}l}$ is an almost pro-l group (i.e. there exists a closed subgroup of $\Gamma_{g,n}^{rel\text{-}l}$ with finite index that is a pro-l group). Also, Hain and Matsumoto proved that the natural homomorphism $\mathrm{MCG}_{g,n} \to \Gamma_{g,n}^{rel\text{-}l}$ is injective for $n > 0$ (Proposition 3.1, [12]). (In fact, since the injectivity of

$\mathrm{MCG}_{g,n} \to \Gamma_{g,n}^{rel\text{-}l}$ is reduced to the injectivity of $\mathrm{MCG}_{g,n+1} \to \Gamma_{g,n+1}^{rel\text{-}l}$ by using Lemma 3.2, we also have the injectivity of $\mathrm{MCG}_{g,0} \to \Gamma_{g,0}^{rel\text{-}l}$ (for $g > 1$).)

The functoriality of relative pro-l completion implies that there is an outer Galois action

$$\rho_{g,n}^{rel\text{-}l} : G_k \to \mathrm{Out}(\Gamma_{g,n}^{rel\text{-}l}).$$

Since the representation $\rho_{g,n}^{rel\text{-}l}$ is unramified outside l when k is a number field (Theorem 3, [12]), $\rho_{g,n}^{rel\text{-}l}$ is not injective. By using Theorem 3.4, we have the following corollary.

Corollary 3.8. *Let (g,n) be a pair of natural numbers such that $3g-3+n>0$, and that either $(g,n) \neq (1,1)$ or $l=2$. Then the kernel of the homomorphism $\rho_{g,n}^{rel\text{-}l}$ is contained in the kernel of the homomorphism*

$$\rho_{\mathbf{P}_k^1 \setminus \{0,1,\infty\}}^{l} : G_k \to \mathrm{Out}(\Pi_{0,3}^{l}).$$

Proof. The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{g,n}^{l} & \longrightarrow & \Gamma_{g,n+1}^{rel\text{-}l} & \longrightarrow & \Gamma_{g,n}^{rel\text{-}l} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact (Proposition 3.1 (2), [12]), induces the commutative diagram

$$\begin{array}{ccc} \Gamma_{g,n} & \longrightarrow & \Gamma_{g,n}^{rel\text{-}l} \\ & \searrow^{\rho_{g,n}^{univ,l}} & \downarrow \\ & & \mathrm{Out}(\Pi_{g,n}^{l}). \end{array}$$

Therefore, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{g,n}^{rel\text{-}l} & \longrightarrow & \pi_1(\mathscr{M}_{g,n})/\ker(\Gamma_{g,n} \to \Gamma_{g,n}^{rel\text{-}l}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \rho_{g,n}^{univ,l}(\Gamma_{g,n}) & \longrightarrow & \rho_{g,n}^{univ,l}(\pi_1(\mathscr{M}_{g,n})) & \longrightarrow & G_k^{l,g,n} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and the vertical homomorphisms are surjective. Hence, this induces

$$\ker(\rho_{g,n}^{rel\text{-}l}) \subseteq \ker(\rho_{g,n}^{geom,l}) = \ker(\rho_{\mathbf{P}_k^1 \setminus \{0,1,\infty\}}^{l}).$$

This completes the proof of Corollary 3.8. □

Remark 3.9. It is not clear to the author at the time of writing whether or not a result similar to the results stated in Theorem 3.4 holds in the case where $(g,n) = (1,1)$ and $l > 2$. Nevertheless, Yuichiro Hoshi and the author proved that a result similar to the results stated in Corollary 3.8 holds in the case where $(g,n) = (1,1)$ and $l > 2$ (cf. Theorem 4.3 in [13]).

4. The case of an arbitrary family of hyperbolic curves

In the present section, we prove a variant of Theorem 2.3. Let l be a prime number, k a field of characteristic zero, and $\bar{k}$ an algebraic closure of k. For any geometrically connected normal scheme S of finite type over k and any family $X \to S$ of (g,n)-curves over S, we denote by $\varphi^l_{X/S} : \pi_1(S \otimes_k \bar{k}) \to \mathrm{Aut}(\Pi^{\mathrm{ab}}_{g,n} \otimes_{\mathbf{Z}} (\mathbf{Z}/l))$ the natural monodromy action arising from the family of (g,n)-curves $X \to S$. Here, the group $\Pi^{\mathrm{ab}}_{g,n}$ is the abelianization of $\Pi_{g,n}$.

Proposition 4.1. *Let (g,n) be a pair of nonnegative integers such that $2g - 2 + n > 0$, S a geometrically connected normal scheme of finite type over k, and $X \to S$ a family of (g,n)-curves over S. Then the natural sequence*

$$1 \to \Pi_{g,n} \to \pi_1(X) \to \pi_1(S) \to 1$$

is exact. Moreover, if the image of $\varphi^l_{X/S}$ is an l-group, then the natural sequence

$$1 \to \Pi^l_{g,n} \to \pi_1(X)^{\underline{l}} \to \pi_1(S)^{\underline{l}} \to 1$$

is exact.

Proof. It is enough to prove the case of $k = \bar{k}$. First, we prove the profinite case. Then we have the exact sequence

$$\Pi_{g,n} \to \pi_1(X) \to \pi_1(S) \to 1$$

by [1]. Let $X^{\mathrm{cpt}} \to S$ be the compactification of $X \to S$ and $D \subseteq X^{\mathrm{cpt}}$ the divisor at infinity of $X \to S$. Then we can take a finite étale (connected) Galois covering $S' \to S$ such that the finite étale covering $D \times_S S' \to S'$ is split. We put $X' := X \times_S S'$, $X'^{\mathrm{cpt}} := X^{\mathrm{cpt}} \times_S S'$, $D' := D \times_S S'$. Then the natural projection $X' \to S'$ is a family of (g,n)-curves and X'^{cpt} (respectively D') is the compactification (respectively the divisor at infinity) of $X' \to S'$. Since $D' \to S'$ is split, by Proposition 2.3 in [31], the natural sequence

$$1 \to \Pi_{g,n} \to \pi_1(X') \to \pi_1(S') \to 1$$

is exact. Moreover, by the definition of $X' \to S'$, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X') & \longrightarrow & \pi_1(S') & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \Pi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) & \longrightarrow & 1. \end{array}$$

Now, since the natural projection $X' \to X$ is a finite étale covering, $\pi_1(X') \to \pi_1(X)$ is injective. This completes the proof for the profinite case.

Next, we consider the pro-l case. Since the image of $\varphi_{X/S}^l$ is an l-group, by using Lemma 4.5.5 in [30], the natural homomorphism $\pi_1(S) \to \mathrm{Out}(\Pi_{g,n}) \to \mathrm{Out}(\Pi_{g,n}^l)$ factors through the maximal pro-l quotient $\pi_1(S)^l$ of $\pi_1(S)$. Therefore, the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Pi_{g,n}^l) & \longrightarrow & \mathrm{Aut}(\Pi_{g,n}^l) & \longrightarrow & \mathrm{Out}(\Pi_{g,n}^l) & \longrightarrow & 1 \end{array}$$

induces the commutative diagram

$$\begin{array}{ccccccccc} & & \Pi_{g,n}^l & \longrightarrow & \pi_1(X)^l & \longrightarrow & \pi_1(S)^l & \longrightarrow & 1 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Pi_{g,n}^l) & \longrightarrow & \mathrm{Aut}(\Pi_{g,n}^l) & \longrightarrow & \mathrm{Out}(\Pi_{g,n}^l) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and the left vertical homomorphism is isomorphism by Proposition 1.4 in [24]. This completes the proof for the pro-l case. □

In the notation of the above proposition, we have the natural homomorphisms $\varphi_S : \pi_1(S) \to \mathrm{Out}(\Pi_{g,n})$, $\varphi_S^l : \pi_1(S) \to \mathrm{Out}(\Pi_{g,n}^l)$ determined by the exact sequence

$$1 \to \Pi_{g,n} \to \pi_1(X) \to \pi_1(S) \to 1.$$

Note that $\Gamma_{0,4}$ (respectively $\Gamma_{0,4}^{rel\text{-}l}$) is canonically isomorphic to $\Pi_{0,3}$ (respectively $\Pi_{0,3}^l$). By a similar argument to the one used in the proof of Theorem 2.1 (Theorem 1.1, [20] or Corollary 6.4, [14]), we can prove the following proposition.

Proposition 4.2. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected normal scheme of finite type over*

k with a k-rational point s, $X \to S$ a family of (g,n)-curves over S, X_s the fiber of $X \to S$ at s, and ρ_{X_s} (respectively $\rho^l_{X_s}$) the homomorphism $G_k \to \mathrm{Out}(\Pi_{g,n})$ (respectively $G_k \to \mathrm{Out}(\Pi^l_{g,n})$) associated to the (g,n)-curve X_s over k. Then the subgroup

$$\rho_{X_s}^{-1}(\varphi_S(\pi_1(S \otimes_k \bar{k}))) \subseteq G_k \quad (\textit{respectively } (\rho^l_{X_s})^{-1}(\varphi^l_S(\pi_1(S \otimes_k \bar{k}))) \subseteq G_k)$$

of G_k is contained in the kernel of the homomorphism

$$\rho_{0,4} : G_k \to \mathrm{Out}(\Pi_{0,3}) \quad (\textit{respectively } \rho_{0,4}^{\textit{rel-l}} : G_k \to \mathrm{Out}(\Pi^l_{0,3})).$$

Proof. Since the pro-l case can be proved by exactly the same argument, we prove only the profinite case. Let i_s be the section $G_k \to \pi_1(S)$ induced by the k-rational point s, $k(S)$ the function field of S, $\overline{k(S)}$ an algebraic closure of $k(S)$, $X_{k(S)} := X \times_S \mathrm{Spec}\, k(S)$, $\rho_{X_{k(S)}}$ the homomorphism $G_{k(S)} := \mathrm{Gal}(\overline{k(S)}/k(S)) \to \mathrm{Out}(\Pi_{g,n})$ associated to the (g,n)-curve $X_{k(S)}$ over $k(S)$. Then we have $\varphi_S \circ i_s = \rho_{X_s}$, and, since S is geometrically connected and normal, the natural outer homomorphism $G_{k(S)} \to \pi_1(S)$ (which is determined up to $\pi_1(S)$-inner automorphism) is surjective. Assume that there exist $\gamma \in \pi_1(S \otimes_k \bar{k})$ and $\sigma \in G_k$ such that $\varphi_S(\gamma)$ is equal to $\rho_{X_s}(\sigma)$. By the surjectivity of the above (outer) homomorphism, we can take $\tilde{\gamma}, \tilde{\sigma} \in G_{k(S)}$ mapped to $\gamma, i_s(\sigma) \in \pi_1(S)$, respectively. Since the diagram

$$\begin{array}{ccc} G_{k(S)} & \twoheadrightarrow & \pi_1(S) \\ & {\scriptstyle \rho_{X_{k(S)}}} \searrow & \downarrow {\scriptstyle \varphi_S} \\ & & \mathrm{Out}(\Pi_{g,n}) \end{array}$$

is commutative, $\tilde{\gamma}\tilde{\sigma}^{-1}$ is contained in the kernel of $\rho_{X_{k(S)}}$. Hence, by Corollary 6.2, in [14], $\tilde{\gamma}\tilde{\sigma}^{-1}$ is contained in the kernel of the natural homomorphism $G_{k(S)} \to \mathrm{Out}(\Pi_{0,3})$. Now, since the diagram

$$\begin{array}{ccc} G_{k(S)} & \longrightarrow & G_k \\ & \searrow & \downarrow {\scriptstyle \rho_{0,4}} \\ & & \mathrm{Out}(\Pi_{0,3}) \end{array}$$

is commutative and γ is contained in the kernel of $\pi_1(S) \to G_k$, σ is contained in the kernel of $\rho_{0,4}$. □

For a scheme X which is geometrically connected and of finite type over k, we denote by

$$\rho^l_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \bar{k})^l)$$

the composite of $\rho_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \bar{k}))$ and the natural homomorphism $\mathrm{Out}(\pi_1(X \otimes_k \bar{k})) \to \mathrm{Out}(\pi_1(X \otimes_k \bar{k})^l)$. The following theorem is a variant of Theorem 2.3. (Note that $\mathscr{M}_{g,n+1} \to \mathscr{M}_{g,n}$ is a family of (g,n)-curves.)

Theorem 4.3. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected normal scheme of finite type over k, $X \to S$ a family of (g,n)-curves over S. Then the kernel of the homomorphism ρ_X is contained in the kernel of the homomorphism*

$$\rho_{0,4} : G_k \to \mathrm{Out}(\Pi_{0,3}).$$

Moreover, if the image of $\varphi^l_{X/S}$ is an l-group, then the kernel of the homomorphism ρ^l_X is contained in the kernel of the homomorphism

$$\rho^{\text{rel-}l}_{0,4} : G_k \to \mathrm{Out}(\Pi^l_{0,3}).$$

In particular, if k is a number field or an l-adic local field, then the homomorphism ρ_X is injective.

Proof. First, we prove the profinite case. Let $k(S)$ be the function field of S, $\overline{k(S)}$ an algebraic closure of $k(S)$, $X_{k(S)} := X \times_S \mathrm{Spec}\, k(S)$, $X_{\overline{k(S)}} := X \times_S \mathrm{Spec}\, \overline{k(S)}$, $S_{k(S)} := S \otimes_k k(S)$. Then the diagonal map $S \to S \times_{\mathrm{Spec}\, k} S$ induces a section $\mathrm{Spec}\, k(S) \to S_{k(S)}$ of the natural projection $S_{k(S)} \to \mathrm{Spec}\, k(S)$. Note that we have the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{k(S)}}) & \longrightarrow & \pi_1(X_{k(S)}) & \longrightarrow & G_{k(S)} & \longrightarrow & 1 \\ & & \wr\big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \pi_1(X \otimes_k \bar{k}) & \longrightarrow & \pi_1(X) & \longrightarrow & G_k & \longrightarrow & 1. \end{array}$$

This diagram induces the commutative diagram

$$\begin{array}{ccc} G_{k(S)} & \longrightarrow & G_k \\ & \searrow^{\rho_{X_{k(S)}}} & \big\downarrow \rho_X \\ & & \mathrm{Out}(\pi_1(X \otimes_k \bar{k})). \end{array}$$

Also, since S is geometrically connected over k, the natural (outer) homomorphism $G_{k(S)} = \mathrm{Gal}(\overline{k(S)}/k(S)) \to G_k$ (which is determined up to G_k-inner automorphism) is surjective. In particular, $\ker(\rho_{X_{k(S)}})$ surjects onto $\ker(\rho_X)$. Therefore, if $\ker(\rho_{X_{k(S)}})$ is included in $\ker(G_{k(S)} \to \mathrm{Out}(\Pi_{0,3}))$, $\ker(\rho_X)$ is included in $\ker(G_k \to \mathrm{Out}(\Pi_{0,3}))$ by the commutative diagram

$$
\begin{array}{ccc}
G_{k(S)} & \twoheadrightarrow & G_k \\
 & \searrow & \downarrow \\
 & & \mathrm{Out}(\Pi_{0,3}).
\end{array}
$$

Hence, replacing $X \to S \to \operatorname{Spec} k$ by $X_{k(S)} \to S_{k(S)} \to \operatorname{Spec} k(S)$ if necessary, we may assume that S has a k-rational point. Let s be a k-rational point of S, $\bar{s}$ a $\bar{k}$-rational point over s, X_s the fiber of $X \to S$ at s, $X_{\bar{s}}$ the fiber of $X \to S$ at $\bar{s}$. The above k-rational point s of S induces a cartesian square

$$
\begin{array}{ccc}
X_s & \longrightarrow & \operatorname{Spec} k \\
\downarrow & \square & \downarrow \\
X & \longrightarrow & S.
\end{array}
$$

This induces a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X \otimes_k \bar{k}) & \longrightarrow & \pi_1(S \otimes_k \bar{k}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(X_s) & \longrightarrow & \pi_1(X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_k & = & G_k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}
$$

where the vertical and horizontal sequences are exact. Then Lemma 2.2 implies that

$$\rho_{X_s}(\ker(\rho_X)) \subseteq \operatorname{im}(\varphi_S : \pi_1(S \otimes_k \bar{k}) \to \mathrm{Out}(\Pi_{g,n})).$$

Here, ρ_{X_s} is the homomorphism $G_k \to \mathrm{Out}(\Pi_{g,n})$ associated to the hyperbolic curve X_s over k. Hence, by using Proposition 4.2, the result follows for the profinite case.

For the pro-l case, since we have the commutative diagram

$$\begin{array}{ccc} \pi_1(S) & \twoheadrightarrow & \pi_1(S)^{\underline{l}} \\ & \searrow^{\varphi_S^l} & \downarrow \\ & & \mathrm{Out}(\Pi_{g,n}^l), \end{array}$$

we can prove the assertion by exactly the same argument. □

Remark 4.4. It is trivial that Theorem 2.5 implies Theorem 2.3. However, it seems that Theorem 2.5 (or its proof) does not imply Theorem 4.3.

Acknowledgement

First of all, the author would like to thank *Makoto Matsumoto* for having introduced him problems around the kernel of outer Galois representations. Also, he would like to thank *Yuichiro Hoshi* and *Akio Tamagawa* for informing him of the proof of Theorem 1.6 and allowing him to reproduce it here. Finally, the author would like to thank his advisors, *Toshiro Hiranouchi*, *Hiroki Takahashi* and *Akio Tamagawa* for helpful discussions, warm encouragements, and valuable advice.

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