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Geometry of homogeneous polar foliations of complex hyperbolic spaces

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Abstract. Homogeneous polar foliations of complex hyperbolic spaces have been classified by Berndt and Díaz-Ramos. In this paper, we study geometry of leaves of such foliations: the minimality, the parallelism of the mean curvature vectors, and the congruency of orbits. In particular, we classify minimal leaves.

1. Introduction

An isometric action of a connected Lie group H on a Riemannian manifold M is said to be *polar* if there exists a connected complete submanifold Σ of M such that

(i) Σ meets each orbit of the action, that is, $\Sigma \cap H.p \neq \emptyset$ holds for each $p \in M$,

(ii) Σ intersects the orbits orthogonally, that is, $T_p\Sigma \subset \nu_p(H.p)$ holds for each $p \in \Sigma$.

Note that such a submanifold Σ, called a *section* of the polar action, is always a totally geodesic submanifold of M (for instance, see [4, Theorem 3.2.1]).

Polar actions on Riemannian symmetric spaces have been studied very actively (for instance, refer to [2], [10], and references therein). Above all, it is noteworthy that cohomogeneity one actions on Riemannian symmetric spaces are always polar ([15]). Therefore, one can regard a polar action on a Riemannian symmetric space as a kind of generalizations of cohomogeneity one actions. We also note that polar actions provide a lot of interesting examples of homogeneous submanifolds. For example, a principal orbit of a polar action is an isoparametric submanifold ([14]), and has a parallel mean curvature vector field (refer to [4, Corollary 3.2.5], and also see Remark 3.14).

In this paper, we consider polar actions on a complex hyperbolic space $\mathbf{C}H^n$ having no singular orbits, or equivalently, inducing *homogeneous polar*

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foliations of $\mathbf{C}H^n$. The aim of this paper is to study the geometry of homogeneous polar foliations of $\mathbf{C}H^n$, and to determine the minimality of their leaves. We remark that such polar actions have been classified by Berndt and Díaz-Ramos. More precisely, they have proved that there exist exactly $2n-1$ actions which induce nontrivial homogeneous polar foliations of $\mathbf{C}H^n$ up to orbit equivalence ([5]). Here, a homogeneous foliation of $\mathbf{C}H^n$ is said to be *trivial* if the leaves are points in $\mathbf{C}H^n$ or the leaf coincides with $\mathbf{C}H^n$. According to their result, moreover, the actions can be divided into the following two types:

(i) none of the orbits is contained in horospheres of $\mathbf{C}H^n$,

(ii) all orbits are contained in horospheres of $\mathbf{C}H^n$.

Let us call them *S-type* and *N-type*, respectively. Our main theorem (Theorems 4.6 and 5.1) is as follows.

MAIN THEOREM. *We have that*

(1) *every S-type action has exactly one minimal orbit,*

(2) *every N-type action has the congruency of orbits, and none of the orbits is minimal.*

Here, an isometric action on a Riemannian manifold is said to be having the *congruency of orbits* if all orbits of the action are isometrically congruent to each other.

REMARK 1.1. *Our main theorem includes the known results on cohomogeneity one actions on* $\mathbf{C}H^n$ *in* [1] *and* [6]. *See Remark* 2.5 *for more details.*

This paper is organized as follows. In Section 2, we recall the solvable model of a complex hyperbolic space $\mathbf{C}H^n$, and recall the classification of homogeneous polar foliations of $\mathbf{C}H^n$. In Section 3, we introduce new Lie groups, which play essential roles in the study of homogeneous polar foliations of $\mathbf{C}H^n$. In order to prove the main theorem, we study the geometry of orbits of the S-type actions in Section 4, and deal with the analogue for the N-type actions in Section 5.

2. Preliminaries

In this section, we recall the solvable model of a complex hyperbolic space $\mathbf{C}H^n$ with $n \geq 2$ (refer mainly to [8], [12]). We also recall the classification of homogeneous polar foliations of $\mathbf{C}H^n$ according to [5].

DEFINITION 2.1. We call a triple $(\mathfrak{s}, \langle\ ,\ \rangle, J)$ the *solvable model* of $\mathbf{C}H^n$ if

(1) $\mathfrak{s} := \mathrm{span}_{\mathbf{R}}\{A_0, X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}, Z_0\}$ is a Lie algebra whose bracket relations are defined by

$$[A_0, X_i] = (1/2)X_i, \qquad [A_0, Y_i] = (1/2)Y_i,$$
$$[A_0, Z_0] = Z_0, \qquad [X_i, Y_i] = Z_0, \tag{2.1}$$

(2) $\langle\, , \rangle$ is an inner product on $\mathfrak{s}$ such that the above basis is orthonormal,
(3) J is a complex structure on $\mathfrak{s}$ defined by

$$J(A_0) = Z_0, \qquad J(Z_0) = -A_0, \qquad J(X_i) = Y_i, \qquad J(Y_i) = -X_i. \tag{2.2}$$

Let S be the simply-connected Lie group with Lie algebra $\mathfrak{s}$. Denote by the same symbols $\langle\, , \rangle$ and J the induced left-invariant Riemannian metric and the complex structure on S, respectively.

First of all, we remark that $\mathbf{C}\mathrm{H}^n$ can be identified with $(S, \langle\, , \rangle, J)$, and hence with the solvable model $(\mathfrak{s}, \langle\, , \rangle, J)$. Let us define

$$G := \mathrm{SU}(1, n), \qquad K := \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)). \tag{2.3}$$

One knows that G is the identity component of the isometry group of $\mathbf{C}\mathrm{H}^n$, and K is the isotropy subgroup of G at some point o, called the *origin* of $\mathbf{C}\mathrm{H}^n$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Then, $\mathbf{C}\mathrm{H}^n$ can be realized as a Riemannian symmetric space of noncompact type G/K. It is known that S is isomorphic to the solvable part of the Iwasawa decomposition of G, and that S acts on $\mathbf{C}\mathrm{H}^n$ simply-transitively. Hence, we can naturally identify $\mathbf{C}\mathrm{H}^n$ with the Lie group S. In particular, one can show that $(S, \langle\, , \rangle, J)$ is holomorphically isometric to $\mathbf{C}\mathrm{H}^n$ with the constant holomorphic sectional curvature -1.

We here study the structure of our solvable model $(\mathfrak{s}, \langle\, , \rangle, J)$. Let us define

$$\mathfrak{a} := \mathrm{span}_{\mathbf{R}}\{A_0\}, \tag{2.4}$$
$$\mathfrak{v} := \mathrm{span}_{\mathbf{R}}\{X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}\}, \tag{2.5}$$
$$\mathfrak{z} := \mathrm{span}_{\mathbf{R}}\{Z_0\}, \tag{2.6}$$

and $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$. Then, we have the orthogonal decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z} = \mathfrak{a} \oplus \mathfrak{n}. \tag{2.7}$$

One can easily see that $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, and $\mathfrak{n}$ is the $(2n-1)$-dimensional Heisenberg Lie algebra. In particular, it follows from the definition of the solvable model that, for any $V, W \in \mathfrak{v}$,

$$[V, W] = \langle JV, W \rangle Z_0. \tag{2.8}$$

One can also see that $\mathfrak{v}$ is J-invariant, and hence $\mathfrak{v}$ is an $(n-1)$-dimensional complex vector space. We note that the complex structure J is an isometry of

$(\mathfrak{s}, \langle\,,\rangle)$, that is, for any $X, Y \in \mathfrak{s}$,

$$\langle JX, JV\rangle = \langle X, Y\rangle. \tag{2.9}$$

REMARK 2.2. *Let $\mathfrak{k}_0$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, which is isomorphic to $\mathfrak{u}(n-1)$, and K_0 be the connected Lie subgroup of K with Lie algebra $\mathfrak{k}_0$. Then, one knows that $\mathfrak{k}_0$ normalizes $\mathfrak{s}$, and especially, the adjoint action of K_0 on $\mathfrak{v}$ is isomorphic to the standard action of* $\mathrm{U}(n-1)$ *on* $\mathbf{C}^{n-1}$.

In the rest of this section, we recall the classification of homogeneous polar foliations of $\mathbf{C}\mathrm{H}^n$ according to [5]. We always mean by $\ominus$ the orthogonal complement with respect to $\langle\,,\rangle$. Let us review the Lie groups introduced in [5].

DEFINITION 2.3. Denote by S_b and N_b the connected Lie subgroups of S with Lie algebras

$$\mathfrak{s}_b := \mathfrak{s} \ominus \mathrm{span}_{\mathbf{R}}\{X_1, \ldots, X_b\} \qquad (b \in \{1, \ldots, n-1\}), \tag{2.10}$$

$$\mathfrak{n}_b := \mathfrak{s} \ominus \mathrm{span}_{\mathbf{R}}\{A_0, X_1, \ldots, X_{b-1}\} \qquad (b \in \{1, \ldots, n\}), \tag{2.11}$$

respectively.

REMARK 2.4. *We note that these notations are changed from ones given in* [5]. *Indeed, the Lie groups S_b and N_b are written as $S_{1,b}$ and $S_{0,b-1}$, respectively, in* [5].

One can see that the actions of S_b and N_b on $\mathbf{C}\mathrm{H}^n$ are of cohomogeneity b, and have no singular orbits.

REMARK 2.5. *Consider the case of cohomogeneity one, that is, $b = 1$. Then, the actions of S_1 and N_1 on $\mathbf{C}\mathrm{H}^n$ are well-known. Note that $\mathfrak{n}_1 = \mathfrak{n}$, and hence N_1 is the nilpotent part of the Iwasawa decomposition of $G = \mathrm{SU}(1, n)$. Then, the action of N_1 induces the horosphere foliation on $\mathbf{C}\mathrm{H}^n$. The orbits of N_1, which are nothing but horospheres, are isometrically congruent to each other and not minimal. On the other hand, the action of S_1 induces the so-called solvable foliation. The orbit of S_1 though the origin o, which is the homogeneous ruled minimal hypersurface, is a unique minimal orbit (refer to* [1], *and also see* [6]*).*

Berndt and Díaz-Ramos proved the following theorem.

THEOREM 2.6 ([5]). *Let H be a connected closed subgroup of $G = \mathrm{SU}(1, n)$. Then, the action of H on $\mathbf{C}\mathrm{H}^n$ induces a nontrivial homogeneous polar foliation of $\mathbf{C}\mathrm{H}^n$ if and only if it is orbit equivalent to one of the following:*
(1) *the action of S_b, where $b \in \{1, \ldots, n-1\}$,*
(2) *the action of N_b, where $b \in \{1, \ldots, n\}$.*

We note that the actions of S_b and N_b are of S-type and of N-type mentioned in Section 1, respectively ([5]).

Owing to their result, in order to study geometry of the orbits of polar actions having no singular orbits on $\mathbf{C}\mathrm{H}^n$, it is sufficient to consider the orbits of S_b and N_b.

3. Construction of certain Lie groups and their geometry

In this section, we introduce new Lie subgroups $S_b(\varphi)$ of S, which play essential roles in the study of both of the S_b-orbits and the N_b-orbits. We also study the geometry of the orbits of $S_b(\varphi)$ through the origin o.

Let us define $\mathfrak{w} := \mathrm{span}_{\mathbf{R}}\{X_1, \dots, X_{n-1}\}$, which is an $(n-1)$-dimensional subspace of $\mathfrak{v}$ with $\langle J\mathfrak{w}, \mathfrak{w}\rangle = 0$. For $\varphi \in [0, \pi/2]$, we define

$$\xi_0 := \cos(\varphi)X_1 + \sin(\varphi)A_0. \tag{3.1}$$

Definition 3.1. Denote by $\mathfrak{w}_b$ a $(b-1)$-dimensional subspace of $\mathfrak{w}$ orthogonal to ξ_0. Then, for $\varphi \in [0, \pi/2]$, we define

$$\mathfrak{s}_b(\varphi) := \mathfrak{s} \ominus (\mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b). \tag{3.2}$$

Remark 3.2. *The above definition of $\mathfrak{s}_b(\varphi)$ depends only on φ and b, up to conjugation, because the adjoint action of K_0 on $\mathfrak{v}$ is isomorphic to the standard action of $\mathrm{U}(n-1)$ on $\mathbf{C}^{n-1}$.*

Remark 3.3. *We remark on the range of allowable values of b. Recall that $\mathfrak{w}_b$ is a $(b-1)$-dimensional subspace of $\mathfrak{w}$ orthogonal to ξ_0, and that $\langle \mathfrak{w}, A_0\rangle = 0$. If $\varphi \in [0, \pi/2[$, then we have $\langle \mathfrak{w}_b, X_1\rangle = 0$, and hence $b \in \{1, \dots, n-1\}$. On the other hand, if $\varphi = \pi/2$, then we have $\langle \mathfrak{w}_b, \xi_0\rangle = 0$, and hence $b \in \{1, \dots, n\}$.*

First of all, we shall show that $\mathfrak{s}_b(\varphi)$ is always a subalgebra of $\mathfrak{s}$. Let us define

$$T_0 := \cos(\varphi)A_0 - \sin(\varphi)X_1 \in \mathfrak{s}_b(\varphi), \tag{3.3}$$

which is orthogonal to the normal vector ξ_0, and

$$\mathfrak{v}_0 := \mathfrak{s}_b(\varphi) \ominus (\mathrm{span}_{\mathbf{R}}\{T_0\} \oplus \mathfrak{z}). \tag{3.4}$$

Lemma 3.4. *We have that $\mathfrak{v}_0 \subset \mathfrak{v} \ominus \mathrm{span}_{\mathbf{R}}\{X_1\}$.*

Proof. Note that $\mathfrak{v} \ominus \mathrm{span}_{\mathbf{R}}\{X_1\} = \mathfrak{s} \ominus \mathrm{span}_{\mathbf{R}}\{A_0, X_1, Z_0\}$. Hence, we have only to show

$$\langle \mathfrak{v}_0, A_0\rangle = \langle \mathfrak{v}_0, X_1\rangle = \langle \mathfrak{v}_0, Z_0\rangle = 0. \tag{3.5}$$

By definition, it is clear that $\mathfrak{v}_0$ is orthogonal to Z_0. Meanwhile, one knows that $A_0, X_1 \in \mathrm{span}_{\mathbf{R}}\{T_0, \xi_0\}$. Since $\mathfrak{v}_0$ is orthogonal to T_0 and ξ_0, we have $\langle \mathfrak{v}_0, A_0 \rangle = \langle \mathfrak{v}_0, \xi_0 \rangle = 0$, which completes the proof. $\square$

With the notations above, one has the orthogonal decomposition

$$\mathfrak{s}_b(\varphi) = \mathrm{span}_{\mathbf{R}}\{T_0\} \oplus \mathfrak{v}_0 \oplus \mathfrak{z}, \tag{3.6}$$

which we need hereafter.

Proposition 3.5. *The subspace $\mathfrak{s}_b(\varphi)$ is a subalgebra of $\mathfrak{s}$.*

Proof. Consider the decomposition (3.6) of $\mathfrak{s}_b(\varphi)$. Firstly, it follows from Lemma 3.4 and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ that

$$[\mathfrak{v}_0 \oplus \mathfrak{z}, \mathfrak{v}_0 \oplus \mathfrak{z}] \subset \mathfrak{z} \subset \mathfrak{s}_b(\varphi). \tag{3.7}$$

One also can directly calculate that, for any $V \in \mathfrak{v}_0$,

$$\begin{aligned} [T_0, V] &= (1/2)\cos(\varphi)V - \sin(\varphi)\langle JX_1, V\rangle Z_0, \\ [T_0, Z_0] &= \cos(\varphi)Z_0. \end{aligned} \tag{3.8}$$

This means $[T_0, \mathfrak{v}_0 \oplus \mathfrak{z}] \subset \mathfrak{s}_b(\varphi)$. Hence, we complete the proof. $\square$

We note that $\mathfrak{s}_b(\varphi)$ is a solvable subalgebra of $\mathfrak{s}$ of codimension b.

Definition 3.6. We denote by $S_b(\varphi)$ the connected Lie subgroup of S with Lie algebra $\mathfrak{s}_b(\varphi)$.

Remark 3.7. *In the case where $b = 1$, the Lie groups $S_1(\varphi)$ have been introduced in* [1], *and have played essential roles in the study of cohomogeneity one actions (see* [1], [12] *and* [13]*). We remark that $S_b(\varphi)$ is a natural generalization of $S_1(\varphi)$, and that the propositions mentioned below are natural extensions of the known results in the case where $b = 1$.*

In the rest of this section, we shall study the geometry of the orbit $S_b(\varphi).o$ through the origin o. Recall that we identify $\mathbf{C}H^n$ with the Lie group S. Accordingly, we hereafter identify the submanifold $S_b(\varphi).o$ with the Lie subgroup $S_b(\varphi)$.

We first recall the Levi-Civita connection ∇ of S, which is well-known (see [8] for instance).

Lemma 3.8. *Let $X, Y \in \mathfrak{s}$, and write as*

$$X = x_1 A_0 + V + x_2 Z_0, \qquad Y = y_1 A_0 + W + y_2 Z_0 \tag{3.9}$$

for some $V, W \in \mathfrak{g}_\alpha$. Then, one has

$$2\nabla_X Y = (\langle V, W\rangle + 2x_2 y_2)A_0 - y_1 V - x_2 JW - y_2 JV + (\langle JV, W\rangle - 2x_2 y_1)Z_0. \tag{3.10}$$

Now, we calculate the second fundamental form h of $S_b(\varphi)$. Recall that h is defined by

$$\langle h(X, Y), \xi\rangle = \langle \nabla_X Y, \xi\rangle \tag{3.11}$$

for $X, Y \in \mathfrak{s}_b(\varphi)$ and $\xi \in \mathfrak{s} \ominus \mathfrak{s}_b(\varphi) = \mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b$. Here and hereafter the subscripts indicate the orthogonal projections onto each spaces.

Proposition 3.9. *Let $V, W \in \mathfrak{v}_0$. Then, the second fundamental form h of $S_b(\varphi)$ satisfies that*
(1) $h(T_0, T_0) = (1/2)\sin(\varphi)\xi_0$,
(2) $h(V, W) = (1/2)\langle V, W\rangle \sin(\varphi)\xi_0$,
(3) $h(Z_0, Z_0) = \sin(\varphi)\xi_0$,
(4) $h(V, Z_0) = -(1/2)(JV)_{\mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b}$,
(5) $h(T_0, W) = h(T_0, Z_0) = 0$.

Proof. Let $V, W \in \mathfrak{v}_0$, and put

$$X := x_1 T_0 + V + x_2 Z_0, \qquad Y := y_1 T_0 + W + y_2 Z_0$$

for $x_i, y_i \in \mathbf{R}$. Then, by using Lemma 3.4 and Lemma 3.8, one can directly calculate that, for $\xi \in \mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b$,

$$\begin{aligned} 2\langle h(X, Y), \xi\rangle &= \langle 2\nabla_X Y, \xi\rangle \\ &= (x_1 y_1 \sin^2(\varphi) + \langle V, W\rangle + 2x_2 y_2)\langle A_0, \xi\rangle \\ &\quad + x_1 y_1 \sin(\varphi)\cos(\varphi)\langle X_1, \xi\rangle - \langle x_2 JW + y_2 JV, \xi\rangle \\ &= (\langle X, Y\rangle + x_2 y_2)\sin(\varphi)\langle \xi_0, \xi\rangle - \langle x_2 JW + y_2 JV, \xi\rangle. \end{aligned} \tag{3.12}$$

By using Equation (3.12), one can show the assertions. We here only calculate $h(V, Z_0)$ for $V \in \mathfrak{v}_0$. Let $\{\xi_i\}$ be an orthonormal basis of $\mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b$. In this case, it follows from (3.12) that

$$\begin{aligned} 2h(V, Z_0) &= \sum \langle 2h(V, Z_0), \xi_i\rangle \xi_i \\ &= \sum \langle -JV, \xi_i\rangle \xi_i = -(JV)_{\mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b}, \end{aligned} \tag{3.13}$$

which proves (4). □

Secondly, we calculate the shape operator A_ξ of $S_b(\varphi)$. Recall that A_ξ satisfies

$$\langle A_\xi(X), Y\rangle = \langle h(X, Y), \xi\rangle \tag{3.14}$$

for $X, Y \in \mathfrak{s}_b(\varphi)$ and $\xi \in \mathfrak{s} \ominus \mathfrak{s}_b(\varphi) = \mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b$.

PROPOSITION 3.10. *Let $V, W \in \mathfrak{v}_0$. Then, for each $\xi \in \mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b$, the shape operator A_ξ of $S_b(\varphi)$ satisfies that*
(1) $A_\xi T_0 = (1/2)\sin(\varphi)\langle \xi_0, \xi\rangle T_0$,
(2) $A_\xi V = (1/2)\sin(\varphi)\langle \xi_0, \xi\rangle V + (1/2)\langle V, J\xi\rangle Z_0$,
(3) $A_\xi Z_0 = (1/2)(J\xi)_{\mathfrak{v}_0} + \sin(\varphi)\langle \xi_0, \xi\rangle Z_0$.

PROOF. We only calculate $A_\xi V$ for $V \in \mathfrak{v}_0$ and $\xi \in \mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b$. Let $\{E_i\}$ be an orthonormal basis of $\mathfrak{v}_0$. Then, by Proposition 3.9, one can directly calculate that

$$\begin{aligned} \langle A_\xi V, T_0\rangle &= \langle h(V, T_0), \xi\rangle = 0, \\ \langle A_\xi V, E_i\rangle &= \langle h(V, E_i), \xi\rangle = (1/2)\sin(\varphi)\langle \xi_0, \xi\rangle\langle V, E_i\rangle, \\ \langle A_\xi V, Z_0\rangle &= \langle h(V, Z_0), \xi\rangle = (1/2)\langle V, J\xi\rangle. \end{aligned} \tag{3.15}$$

Altogether, it follows that

$$\begin{aligned} A_\xi V &= \langle A_\xi V, T_0\rangle T_0 + \sum \langle A_\xi V, E_i\rangle E_i + \langle A_\xi V, Z_0\rangle Z_0 \\ &= (1/2)\sin(\varphi)\langle \xi_0, \xi\rangle V + (1/2)\langle V, J\xi\rangle Z_0, \end{aligned} \tag{3.16}$$

which proves (2). The remaining assertions can be obtained by similar calculations. □

An eigenvalue of the shape operator A_ξ is called *a principal curvature in direction* ξ, and the dimension of an eigenspace is called the *multiplicity*.

PROPOSITION 3.11. (1) *The principal curvatures in direction ξ_0 are λ_1, λ_2 and λ_3, and the multiplicities are* 1, $2n - b - 2$, 1, *respectively, where*

$$\begin{aligned} \lambda_1 &:= (3/4)\sin(\varphi) - (1/4)(1 + 3\cos^2(\varphi))^{1/2}, \\ \lambda_2 &:= (1/2)\sin(\varphi), \\ \lambda_3 &:= (3/4)\sin(\varphi) + (1/4)(1 + 3\cos^2(\varphi))^{1/2}. \end{aligned}$$

(2) *If $\xi \in \mathfrak{w}_b$, then the principal curvatures in direction ξ are* $-1/2$, 0, $1/2$, *and the multiplicities are* 1, $2n - b - 2$, 1, *respectively.*

PROOF. Firstly, we consider the case where $\xi = \xi_0$. Note that we have $J\xi_0 = \cos(\varphi)JX_1 + \sin(\varphi)Z_0$, and $JX_1 \in \mathfrak{v}_0$. Then, by Proposition 3.10, one can directly calculate that, for $V \in \mathfrak{v}_0 \ominus \mathrm{span}_{\mathbf{R}}\{JX_1\}$,

$$\begin{aligned} A_{\xi_0} T_0 &= (1/2)\sin(\varphi) T_0, \\ A_{\xi_0} V &= (1/2)\sin(\varphi) V, \\ A_{\xi_0} JX_1 &= (1/2)\sin(\varphi) JX_1 + (1/2)\cos(\varphi) Z_0, \\ A_{\xi_0} Z_0 &= (1/2)\cos(\varphi) JX_1 + \sin(\varphi) Z_0, \end{aligned} \tag{3.17}$$

from which the former assertion follows.

Similarly, we consider the case where $\xi \in \mathfrak{w}_b$, that is, $\langle \xi_0, \xi \rangle = 0$. Note that $J\xi \in \mathfrak{v}_0$. Then, one can also calculate that, for $V \in \mathfrak{v}_0 \ominus \mathrm{span}_{\mathbf{R}}\{J\xi\}$,

$$A_\xi T_0 = A_\xi V = 0, \qquad A_{\xi_0}(J\xi) = (1/2) Z_0, \qquad A_{\xi_0} Z_0 = (1/2) J\xi, \tag{3.18}$$

from which the latter assertion follows. □

Lastly, we calculate the mean curvature vector $\mathscr{H}$. We also study the minimality of $S_b(\varphi)$ and the parallelism of the mean curvature vector. Recall that the *mean curvature vector* is defined by

$$\mathscr{H} := \mathrm{trace}\, h. \tag{3.19}$$

If $\mathscr{H} = 0$, then the submanifold is said to be *minimal*.

Proposition 3.12. *The mean curvature vector $\mathscr{H}$ of $S_b(\varphi)$ is given by*

$$\mathscr{H} = (1/2)(2n - b + 1)\sin(\varphi)\xi_0. \tag{3.20}$$

In particular, $S_b(\varphi)$ is minimal if and only if $\varphi = 0$.

Proof. Let $\{E_i\}$ be an orthonormal basis of $\mathfrak{v}_0$. It follows readily from Proposition 3.9 that

$$\begin{aligned} \mathscr{H} &= h(T_0, T_0) + \sum h(E_i, E_i) + h(Z_0, Z_0) \\ &= (1/2)(2n - b + 1)\sin(\varphi)\xi_0. \end{aligned} \tag{3.21}$$

Therefore, since $\varphi \in [0, \pi/2]$, the remaining assertion is clear. □

Denote by $\nabla^\perp$ the normal part of ∇, namely, the normal connection of $S_b(\varphi)$. The mean curvature vector $\mathscr{H}$ is said to be *parallel* if $\nabla^\perp_X \mathscr{H} = 0$ holds for any $X \in \mathfrak{s}_b(\varphi)$.

Proposition 3.13. *The mean curvature vector $\mathscr{H}$ of $S_b(\varphi)$ is always parallel.*

Proof. It follows from Proposition 3.12 that we have only to calculate $\nabla_{T_0}\xi_0$, $\nabla_{Z_0}\xi_0$, and $\nabla_V \xi_0$ for any $V \in \mathfrak{v}_0$. Take any $V \in \mathfrak{v}_0$. By Lemma 3.8, one can directly calculate that

$$\begin{aligned} \nabla_T \xi_0 &= -(1/2)\sin(\varphi)T_0, \\ \nabla_V \xi_0 &= -(1/2)\sin(\varphi)V + (1/2)\cos(\varphi)\langle JV, X_1\rangle Z_0, \\ \nabla_{Z_0}\xi_0 &= -(1/2)\cos(\varphi)JX_1 - \sin(\varphi)Z_0. \end{aligned} \tag{3.22}$$

It follows that $\nabla_X \xi_0 \in \mathfrak{s}_b(\varphi)$, and hence $\nabla^{\perp}_X \xi_0 = 0$ for any $X \in \mathfrak{s}_b(\varphi)$. □

Remark 3.14. *We note that Proposition* 3.13 *can be shown by the general theory of polar actions. As we mention in the following sections, $S_b(\varphi).o$ is always a principal orbit of some polar action. Therefore, it follows from* [4, Corollary 3.2.5] *that the mean curvature vector field on $S_b(\varphi).o$ is parallel with respect to $\nabla^{\perp}$.*

4. Orbits of the S-type actions

In this section, we consider the S-type actions on $\mathbf{C}\mathrm{H}^n$, namely, the S_b-actions, and study the geometry of their orbits. In particular, we show that, for every S_b-action the orbit through the origin o is a unique minimal orbit.

Throughout this section, we fix $b \in \{1, \dots, n-1\}$. Recall that S_b is the connected Lie subgroup of S with Lie algebra

$$\mathfrak{s}_b := \mathfrak{s} \ominus \mathrm{span}_{\mathbf{R}}\{X_1, \dots, X_b\}. \tag{4.1}$$

Our first aim is to show that every S_b-orbit can be translated into the orbit $S_b(\varphi).o$ for some $\varphi \in [0, \pi/2[$. From now on, we identify the tangent space $T_o\mathbf{C}\mathrm{H}^n$ with $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ through $\mathbf{C}\mathrm{H}^n = S$. Then, for each $k \in K_0$, the differential $(dk)_o$ of k at o satisfies that $(dk)_o = \mathrm{Ad}(k)|_{\mathfrak{s}}$. Recall that K_0 is the connected Lie subgroup of K with Lie algebra $\mathfrak{k}_0$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Lemma 4.1. *Let $N_{K_0}(S_b)$ be the normalizer of S_b in K_0. Then, $N_{K_0}(S_b)$ acts transitively on the unit sphere in $\nu_o(S_b.o) = \mathrm{span}_{\mathbf{R}}\{X_1, \dots, X_b\}$.*

Proof. Recall that the adjoint action of K_0 on $\mathfrak{v}$ is isomorphic to the standard action of $\mathrm{U}(n-1)$ on $\mathbf{C}^{n-1}$. One can see that the action of $N_{K_0}(S_b)$ on the normal space $\nu_o(S_b.o)$ at the origin o is isomorphic to the standard action of $\mathrm{O}(b)$ on $\mathbf{R}^b$. Hence, if $b > 1$, then the assertion is clear. In the case where $b = 1$, one knows that $\mathrm{O}(1) = \{\pm 1\}$ acts on $\mathbf{R}$ naturally, and hence, on its unit sphere $\{\pm 1\}$ transitively. □

Remark 4.2. *Denote by $N^o_K(S_b)$ the identity component of the normalizer $N_K(S_b)$ of S_b in K. Then, the action of $N^o_K(S_b)S_b$ on $\mathbf{C}\mathrm{H}^n$ is of cohomogeneity* *one. If $b > 1$, especially, the orbit $N_K^o(S_b)S_b.o = S_b.o$ is a singular orbit. Refer to* [3], [7] *for more details.*

Let $\gamma_0 : \mathbf{R} \to \mathbf{C}\mathrm{H}^n$ be the unit-speed geodesic defined by

$$\gamma_0(0) = o, \qquad \dot{\gamma}_0(0) = -X_1. \tag{4.2}$$

Lemma 4.3. *Let $p \in \mathbf{C}\mathrm{H}^n$, and $t_0 \geq 0$ be the distance between the orbit $S_b.p$ and the origin o. Then, $S_b.p$ is isometrically congruent to $S_b.\gamma_0(t_0)$.*

Proof. Take any point $p \in \mathbf{C}\mathrm{H}^n$. In the case where $p \in S_b.o$, one knows $t_0 = 0$, and hence we have nothing to prove more.

Thus, we now consider the case where $p \notin S_b.o$. Since the orbit $S_b.p$ is closed, there exists $q \in S_b.p$ such that the distance between o and q is equal to t_0. Since $\mathbf{C}\mathrm{H}^n$ is complete, there exists a unit-speed geodesic γ satisfying $\gamma(0) = o$ and $\gamma(t_0) = q$. A standard variational argument implies that γ intersects the orbit $S_b.q$ perpendicularly. It, hence, follows that γ intersects all orbits of S_b perpendicularly (see for instance [9, p. 78]). Put

$$V := \dot{\gamma}(0) \in \nu_o(S_b.o). \tag{4.3}$$

Then, Lemma 4.1 shows that there exists $k \in N_{K_0}(S_b)$ such that $\mathrm{Ad}(k)V = -X_1$, that is, $(dk)_o\dot{\gamma}(0) = \dot{\gamma}_0(0)$. Since k is an isometry, we have $k.\gamma(t) = \gamma_0(t)$ for any t. Consequently, it follows that

$$k(S_b.p) = kS_b.\gamma(t_0) = S_b k.\gamma(t_0) = S_b.\gamma_0(t_0), \tag{4.4}$$

which completes the proof. $\square$

Recall that $b \in \{1, \ldots, n-1\}$, and let $\varphi \in [0, \pi/2[$. Recall also that $S_b(\varphi)$ is the connected Lie subgroup of S with Lie algebra

$$\mathfrak{s}_b(\varphi) = \mathfrak{s} \ominus (\mathrm{span}_{\mathbf{R}}\{\xi_0\} \oplus \mathfrak{w}_b), \tag{4.5}$$

where $\xi_0 = \cos(\varphi)X_1 + \sin(\varphi)A_0$, and $\mathfrak{w}_b$ is a $(b-1)$-dimensional subspace of $\mathfrak{w}$ orthogonal to ξ_0. In this case, according to Remark 3.2, one may assume that

$$\mathfrak{w}_b = \mathrm{span}_{\mathbf{R}}\{X_2, \ldots, X_b\} \tag{4.6}$$

without loss of generality. Then, we have

$$\mathfrak{s}_b = \mathfrak{s} \ominus (\mathrm{span}_{\mathbf{R}}\{X_1\} \oplus \mathfrak{w}_b) = \mathfrak{s}_b(0). \tag{4.7}$$

Proposition 4.4. *Let $t \geq 0$. Then, the orbit $S_b.\gamma_0(t)$ is isometrically congruent to $S_b(\varphi).o$, where $\varphi := \arcsin(\tanh(t/2)) \in [0, \pi/2[$.*

Proof. Take any $t \geq 0$. Consider the connected Lie subgroup H of S with Lie algebra $\mathfrak{h} := \mathrm{span}_{\mathbf{R}}\{A_0, X_1\}$. Since $H.o$ is a totally geodesic real hyperbolic plane $\mathbf{R}H^2$, the geodesic γ_0 lies in $H.o$. It, hence, follows that there exists $g \in H$ such that $g.o = \gamma_0(t)$ holds. One can readily see that

$$g^{-1}(S_b.\gamma_0(t)) = g^{-1}S_b g.o = I_{g^{-1}}(S_b).o. \tag{4.8}$$

This means that the orbit $S_b.\gamma_0(t)$ is isometrically congruent to $I_{g^{-1}}(S_b).o$, since g^{-1} is an isometry of $\mathbf{C}H^n$. Now it remains to show that $I_{g^{-1}}(S_b) = S_b(\varphi)$, or equivalently, $\mathrm{Ad}(g^{-1})\mathfrak{s}_b = \mathfrak{s}_b(\varphi)$. Since $g \in H \subset S$, one has $\mathrm{Ad}(g^{-1})\mathfrak{s}_b \subset \mathfrak{s}$. For our goal, hence, it suffices to prove that $\mathrm{Ad}(g^{-1})\mathfrak{s}_b$ is orthogonal to ξ_0 and $\mathfrak{w}_b$.

Firstly, we show that $\mathrm{Ad}(g^{-1})\mathfrak{s}_b$ is orthogonal to $\mathfrak{w}_b$. One can see that $\mathfrak{h} \subset \mathfrak{s}_b \oplus \mathrm{span}_{\mathbf{R}}\{X_1\}$, and $\mathfrak{s}_b \oplus \mathrm{span}_{\mathbf{R}}\{X_1\}$ is a subalgebra. It, hence, follows that

$$\mathrm{Ad}(g^{-1})\mathfrak{s}_b \subset \mathfrak{s}_b \oplus \mathrm{span}_{\mathbf{R}}\{X_1\} = \mathfrak{s} \ominus \mathfrak{w}_b. \tag{4.9}$$

Next we show that $\mathrm{Ad}(g^{-1})\mathfrak{s}_b$ is orthogonal to $\xi_0 = \cos(\varphi)X_1 + \sin(\varphi)A_0$. For this purpose, we consider X_1 and A_0 as left-invariant vector fields on S. Since $\dot{\gamma}(t)$ is a unit normal vector of $S_b.\gamma(t)$ at $\gamma(t)$, and the left-translation $L_{g^{-1}}$ is an isometry, one can see that $(dL_{g^{-1}})_e\dot{\gamma}(t)$ is a unit normal vector of $I_{g^{-1}}S_b.o$ at o. On the other hand, by [8, Theorem 2, p. 94] one can obtain that

$$\begin{aligned}\dot{\gamma}(t) &= (1/\cosh(t/2))(-X_1)_g - \tanh(t/2)(A_0)_g \\ &= -(\cos(\varphi)(X_1)_g + \sin(\varphi)(A_0)_g) = -(\xi_0)_g,\end{aligned} \tag{4.10}$$

and hence, $(dL_{g^{-1}})_e\dot{\gamma}(t) = -(\xi_0)_e$. Therefore, we have that $\mathrm{Ad}(g^{-1})\mathfrak{s}_b$ is orthogonal to ξ_0.

Altogether, we have proved that $\mathrm{Ad}(g^{-1})\mathfrak{s}_b \subset \mathfrak{s}_b(\varphi)$, which completes the proof. $\square$

From the arguments above, one can readily obtain the following.

Proposition 4.5. *Let $p \in \mathbf{C}H^n$. Denote by $t \geq 0$ the distance between the orbit $S_b.p$ and the origin o, and set $\varphi := \arcsin(\tanh(t/2))$. Then, $S_b.p$ is isometrically congruent to the orbit $S_b(\varphi).o$.*

Therefore, in order to study the geometry of orbits of the S_b-action, it is sufficient to study $S_b(\varphi).o$ for $\varphi \in [0, \pi/2[$. We conclude this section by proving the first assertion of the main theorem.

Theorem 4.6. *For each $b \in \{1, \dots, n-1\}$, the action of S_b has exactly one minimal orbit, which is through the origin o.*

Proof. It readily follows from Proposition 3.12 that $S_b.o = S_b(0).o$ is minimal. Now we show the uniqueness. Assume that $p \notin S_b.o$, and let $t > 0$

be the distance between the orbit $S_b.p$ and the origin o. Since we have $\varphi = \arcsin(\tanh(t/2)) \neq 0$, it also follows from Proposition 3.12 that $S_b.p = S_b(\varphi).o$ is not minimal. □

Remark 4.7. *In fact, it has been known that the orbit $S_b.o$ through the origin is minimal. In the case where $b = 1$, Berndt has proved its minimality in* [1]. *On the other hands, if $b > 1$, one knows that $S_b.o$ is a singular orbit of a cohomogeneity one action on* $\mathbf{C}H^n$, *as we mentioned in Remark* 4.2. *It has been proved that any singular orbit of a cohomogeneity one action is an austere submanifold, and hence, a minimal submanifold (see* [17] *for more details).*

5. Orbits of the N-type actions

In this section, we consider the N-type actions on $\mathbf{C}H^n$, namely, the N_b-actions, and study the geometry of their orbits. In particular, we show that the action of N_b has the congruency of orbits, and has no minimal orbits.

Throughout this section, we fix $b \in \{1, \ldots, n\}$. Recall that N_b is the connected Lie subgroup of S with Lie algebra

$$\mathfrak{n}_b := \mathfrak{s} \ominus \operatorname{span}_{\mathbf{R}}\{A_0, X_1, \ldots, X_{b-1}\}. \tag{5.1}$$

We consider the case where $\varphi = \pi/2$. In this case, according to Remark 3.2, one may assume that

$$\mathfrak{w}_b = \operatorname{span}_{\mathbf{R}}\{X_1, \ldots, X_{b-1}\}, \tag{5.2}$$

without loss of generality. Note that $\mathfrak{w}_b$ is a $(b-1)$-dimensional subspace of $\mathfrak{w}$ orthogonal to $\xi_0 = A_0$. Then, we have

$$\mathfrak{n}_b = \mathfrak{s} \ominus (\operatorname{span}_{\mathbf{R}}\{A_0\} \oplus \mathfrak{w}_b) = \mathfrak{s}_b(\pi/2). \tag{5.3}$$

Now we show the second assertion of the main theorem.

Theorem 5.1. *For each $b \in \{1, \ldots, n\}$, the action of N_b has the congruency of orbits, that is, all of the N_b-orbits are isometrically congruent to each other. Moreover, the action has no minimal orbits.*

Proof. We first show the congruency of orbits. Recall that S acts transitively on $\mathbf{C}H^n$. One can directly see that $\mathfrak{n}_b$ is an ideal in $\mathfrak{s}$. Hence, it follows from [16, Lemma 2.1] that the action of N_b has the congruency of orbits.

Recall that $N_b.o = S_b(\pi/2).o$ is not minimal by Proposition 3.12. Hence, owing to the congruency, we conclude that the action of N_b has no minimal orbits. □

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