Hiroshima Math. J.
44 (2014), 261–266

A lower bound on WAFOM

Takehito Yoshiki

(Received December 20, 2013)
(Revised March 10, 2014)

Abstract. We give a lower bound on the Walsh figure of merit (WAFOM), which estimates the integration error for quasi-Monte Carlo (QMC) integration by a point set called a digital net. The logarithm of this lower bound is optimal up to a constant multiple, because the existence of point sets attaining the order was proved in [K. Suzuki, An explicit construction of point sets with large minimum Dick weight, to appear in J. Complexity].

1. Introduction

We explain the relation between quasi-Monte Carlo (QMC) integration and the Walsh figure of merit (WAFOM) (see [3] for details). QMC integration is one of the methods for numerical integration (see [2], [5] and [7] for details). Let Q be a point set in the s-dimensional cube $[0,1)^s$ with finite cardinality $\#(Q) = N$, and $f : [0,1)^s \to \mathbf{R}$ be a Riemann integrable function. The QMC integration by Q is the approximation of $I(f) := \int_{[0,1)^s} f(x)dx$ by the average $I_Q(f) := \frac{1}{\#(Q)} \sum_{x \in Q} f(x)$.

WAFOM bounds the error of QMC integration for a certain class of functions by a point set P called a digital net, which is defined by the following identification (see [3] and [5] for details): Let $\mathscr{P}$ be a subspace of $s \times n$ matrices over the finite field $\mathbf{F}_2$ of order two. We define the function $\varphi : \mathscr{P} \ni X = (x_{i,j}) \mapsto x = (\sum_{j=1}^{n} x_{i,j} \cdot 2^{-j})_{i=1}^{s} \in \mathbf{R}^s$, where $x_{i,j}$ is considered to be 0 or 1 in $\mathbf{Z}$ and the sum is taken in $\mathbf{R}$. The digital net P in $[0,1)^s$ is defined by $\varphi(\mathscr{P})$. We identify the digital net P with a linear space $\mathscr{P}$. If $\mathscr{P}$ is an m-dimensional space, the cardinality of P is 2^m.

Let f be a function whose mixed partial derivatives up to order $\alpha \geq 1$ in each variable are square integrable (see [1] and [3] for details). We say that such a function f is an α-smooth function or the smoothness of a function f is α here. By using 'n-digit discretization f_n' (see [3] for details), we approximate $I(f)$ by $I_P(f_n) := \frac{1}{\#(P)} \sum_{x \in P} f_n(x)$ for an n-smooth function f, that is, we can

This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.
2010 *Mathematics Subject Classification*. 65C05.
Key words and phrases. Quasi-Monte Carlo (QMC) integration, Walsh figure of merit, digital net.

evaluate the integration error by the following Koksma-Hlawka type inequality of WAFOM:

$$|I(f) - I_P(f_n)| \leq C_{s,n} \|f\|_n \times \mathrm{WAFOM}(P),$$

where $\|f\|_n$ is the norm of f defined in [1] and $C_{s,n}$ is a constant independent of f and P. If the difference between $I_P(f_n)$ and $I_P(f)$ is negligibly small, we see that $|I(f) - I_P(f)| \leq C_{s,n} \|f\|_n \times \mathrm{WAFOM}(P)$ approximately holds (see [3] for details). In [4], we proved that there is a digital net P of size 2^m with $\mathrm{WAFOM}(P) < 2^{-Cm^2/s}$ for sufficiently large m by a probabilistic argument. (Suzuki [8] gave a constructive proof.) In this paper, we prove that $\mathrm{WAFOM}(P) > 2^{-C'm^2/s}$ holds for large m and any digital net P with $\#(P) = 2^m$ (see Theorem 3.1 for a precise statement, which is formulated for a linear subspace $\mathscr{P}$, instead of a digital net P). Thus, the order m^2/s of the logarithm of this lower bound is optimal.

This paper is organized as follows: We introduce some definitions in Section 2. We prove a lower bound on WAFOM in Section 3.

2. Definition and notation

In this section, we introduce WAFOM and the minimum weight which will be needed later on.

Let s and n be positive integers. $\mathrm{M}_{s,n}(\mathbf{F}_2)$ denotes the set of $s \times n$ matrices over the finite field $\mathbf{F}_2$ of order 2. We regard $\mathrm{M}_{s,n}(\mathbf{F}_2)$ as an sn-dimensional inner product space under the inner product $A \cdot B = (a_{i,j}) \cdot (b_{i,j}) = \sum_{i,j} a_{i,j} b_{i,j} \in \mathbf{F}_2$.

WAFOM is defined using a Dick weight in [3].

Definition 2.1. Let $X = (x_{i,j})$ be an element of $\mathrm{M}_{s,n}(\mathbf{F}_2)$. The Dick weight of X is defined by

$$\mu(X) := \sum_{1 \leq i \leq s, 1 \leq j \leq n} j \cdot x_{i,j},$$

where we regard $x_{i,j} \in \{0, 1\}$ as the element of $\mathbf{Z}$ and take the sum in $\mathbf{Z}$, not in $\mathbf{F}_2$.

Definition 2.2. Let $\mathscr{P}$ be a subspace of $\mathrm{M}_{s,n}(\mathbf{F}_2)$. WAFOM of $\mathscr{P}$ is defined by

$$\mathrm{WAFOM}(\mathscr{P}) := \sum_{X \in \mathscr{P}^{\perp} \setminus \{O\}} 2^{-\mu(X)}, \tag{1}$$

where $\mathscr{P}^{\perp}$ denotes the orthogonal space to $\mathscr{P}$ in $\mathrm{M}_{s,n}(\mathbf{F}_2)$ and O denotes the zero matrix.

In order to estimate a lower bound on WAFOM, we use the minimum weight introduced in [4].

DEFINITION 2.3. Let $\mathscr{P}$ be a proper subspace of $\mathrm{M}_{s,n}(\mathbf{F}_2)$. The minimum weight of $\mathscr{P}^\perp$ is defined by

$$\delta_{\mathscr{P}^\perp} := \min_{X \in \mathscr{P}^\perp \setminus \{O\}} \mu(X). \tag{2}$$

3. A lower bound on WAFOM

Now we state a lower bound on WAFOM. The theorem is mentioned for a linear subspace identified with a digital net (see Section 1).

THEOREM 3.1. *Let n, s and m be positive integers such that $m < ns$, and let C' be an arbitrary real number greater than $1/2$. If $m/s \geq (\sqrt{C' + 1/16} + 3/4)/(C' - 1/2)$, then for any m-dimensional subspace $\mathscr{P}$ of $\mathrm{M}_{s,n}(\mathbf{F}_2)$ we have*

$$\mathrm{WAFOM}(\mathscr{P}) \geq 2^{-C'm^2/s}.$$

PROOF. Let n, s, m and C' be defined as above. The following inequality immediately results from (1), (2) in Section 2:

$$\mathrm{WAFOM}(\mathscr{P}) = \sum_{X \in \mathscr{P}^\perp \setminus \{O\}} 2^{-\mu(X)} \geq 2^{-\delta_{\mathscr{P}^\perp}}. \tag{3}$$

By an upper bound on $\delta_{\mathscr{P}^\perp}$ in Lemma 3.1 (b) below and the inequality (3), for any m-dimensional subspace $\mathscr{P}$ of $\mathrm{M}_{s,n}(\mathbf{F}_2)$, we have

$$\mathrm{WAFOM}(\mathscr{P}) = \sum_{X \in \mathscr{P}^\perp \setminus \{O\}} 2^{-\mu(X)} \geq 2^{-\delta_{\mathscr{P}^\perp}} \geq 2^{-C'm^2/s}.$$

Thus Theorem 3.1 follows.

We prove an upper bound on the minimum weight $\delta_{\mathscr{P}^\perp}$ to complete the proof of Theorem 3.1.

LEMMA 3.1. *Let n, s and m be positive integers such that $m < ns$. Then we have the following statements:*

(a) *Let q and r be non-negative integers satisfying $q = (m - r)/s$ and $r < s$. Then we obtain*

$$\delta_{\mathscr{P}^\perp} \leq \frac{sq(q+1)}{2} + (q+1)(r+1)$$

for any m-dimensional subspace $\mathscr{P}$ of $\mathrm{M}_{s,n}(\mathbf{F}_2)$.

(b) *Let C' be an arbitrary positive real number greater than $1/2$. If $m/s \geq (\sqrt{C'+1/16}+3/4)/(C'-1/2)$, then we have*

$$\delta_{\mathscr{P}^\perp} \leq C'm^2/s$$

for any m-dimensional subspace $\mathscr{P}$ of $\mathrm{M}_{s,n}(\mathbf{F}_2)$.

Proof. (a) If there exists a subspace W of $\mathrm{M}_{s,n}(\mathbf{F}_2)$ such that for any m-dimensional subspace $\mathscr{P}$ of $\mathrm{M}_{s,n}(\mathbf{F}_2)$ we have $\mathscr{P}^\perp \cap W \neq \{O\}$, then $\delta_{\mathscr{P}^\perp} \leq \max_{X \in W} \mu(X)$ holds. Therefore in order to obtain a sharp upper bound on $\delta_{\mathscr{P}^\perp}$, we need a subspace W with $\max_{X \in W} \mu(X)$ small. We can construct W as follows:

$$W := \left\{ X = (x_{i,j}) \in \mathrm{M}_{s,n}(\mathbf{F}_2) \;\middle|\; x_{i,j} = 0 \quad \begin{matrix} (i \leq r+1 \text{ and } q+2 \leq j) \\ \text{or} \\ (r+2 \leq i \text{ and } q+1 \leq j) \end{matrix} \right\},$$

that is, W consists of the following type of matrices:

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,q} & x_{1,q+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\ x_{r+1,1} & \cdots & x_{r+1,q} & x_{r+1,q+1} & 0 & \cdots & 0 \\ x_{r+2,1} & \cdots & x_{r+2,q} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\ x_{s,1} & \cdots & x_{s,q} & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad (x_{i,j} \in \mathbf{F}_2). \tag{4}$$

The subspace W satisfies $\mathscr{P}^\perp \cap W \neq \{O\}$ for any m-dimensional subspace $\mathscr{P}$ of $\mathrm{M}_{s,n}(\mathbf{F}_2)$. Indeed we can see that

$$\begin{aligned} \dim(\mathscr{P}^\perp \cap W) &\geq \dim \mathscr{P}^\perp + \dim W - \dim \mathrm{M}_{s,n}(\mathbf{F}_2) \\ &= (sn - m) + (sq + r + 1) - sn = 1. \end{aligned}$$

Hence there exists a non-zero matrix $X_{\mathscr{P}} \in W \cap \mathscr{P}^\perp$. This yields

$$\delta_{\mathscr{P}^\perp} = \min_{X \in \mathscr{P}^\perp \setminus \{O\}} \mu(X) \leq \mu(X_{\mathscr{P}}) \leq \max_{X \in W} \mu(X).$$

Let us estimate $\max_{X \in W} \mu(X)$ of W. Let $X_{\max}$ of W be a matrix whose entries $x_{i,j}$ in (4) are all 1. The function μ attains its maximum at $X_{\max}$ in W. Thus it follows that

$$\max_{X \in W} \mu(X) = \mu(X_{\max}) = \frac{sq(q+1)}{2} + (q+1)(r+1).$$

We obtain that

$$\delta_{\mathscr{P}^\perp} = \min_{X \in \mathscr{P}^\perp \setminus \{O\}} \mu(X) \leq \mu(X_{\mathscr{P}}) \leq \max_{X \in W} \mu(X) = \frac{sq(q+1)}{2} + (q+1)(r+1),$$

where $\mathscr{P}$ is an arbitrary m-dimensional subspace of $\mathbf{M}_{s,n}(\mathbf{F}_2)$.

(b) Let C' be a real number greater than $1/2$ and assume $m/s \geq (\sqrt{C' + 1/16} + 3/4)/(C' - 1/2)$. By combining $r + 1 \leq s$, $q \leq m/s$ and the assertion (a), we have

$$\delta_{\mathscr{P}^\perp} \leq \frac{m}{2}\left(\frac{m}{s} + 1\right) + \left(\frac{m}{s} + 1\right) \cdot s = \frac{m^2}{s}\left(\frac{1}{2} + \frac{3s}{2m} + \frac{s^2}{m^2}\right) \leq C' \frac{m^2}{s},$$

where the last inequality follows from the assumption by completing the square with respect to s/m.

Remark 3.1. This remark is to clarify relations between the above result and existing results. Fix α, and consider the space of α-smooth functions. For this (and even a larger) function class, Dick [1, Corollary 5.5 and the comment after its proof] gave digital nets for which the QMC integration error is bounded from above by the order of $2^{-\alpha m} m^{\alpha s + 1}$. This is optimal, since for any point set of size 2^m, Sharygin [6] constructed an α-smooth function whose QMC integration error is at least of this order.

Since WAFOM gives only an upper bound of the QMC integration error, our lower bound $2^{-C'm^2/s}$ on WAFOM in Theorem 3.1 implies nothing on the lower bound of the integration error.

A merit of WAFOM is that the value depends only on the point set, not on the smoothness α such as [1]. On the other hand, WAFOM depends on the degree n of discretization. Thus, it seems not easy to compare directly the upper bound on the integration error given in [1] and that by WAFOM. However, we might consider that our lower bound $2^{-C'm^2/s}$, which is independent of n and α, shows a kind of limitation of the method in bounding the integration error in [1] in the limit $\alpha \to \infty$.

References

[1] J. Dick, Walsh spaces containing smooth functions and Quasi-Monte Carlo rules of arbitrary high order. SIAM J. Numer. Anal., 46 (2008), 1519–1553.

[2] J. Dick and F. Pillichshammer, Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.

[3] M. Matsumoto, M. Saito, and K. Matoba, A computable figure of merit for Quasi-Monte Carlo point sets. Math. Comp., 83 (2014), 1233–1250.

[4] M. Matsumoto and T. Yoshiki, Existence of Higher Order Convergent Quasi-Monte Carlo Rules via Walsh Figure of Merit. Monte Carlo and Quasi-Monte Carlo Methods 2012, Springer, Berlin, (2013), 569–579.

[5] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods. CBMS-NSF, Philadelphia, Pennsylvania, 1992.

[6] I. F. Sharygin, A lower estimate for the error of quadrature formulas for certain classes of functions. Zh. Vychisl. Mat. i Mat. Fiz., 3 (1963), 370–376.

[7] I. H. Sloan and S. Joe, Lattice Methods for Multiple Integration. Clarendon Press, Oxford, 1994.

[8] K. Suzuki, An explicit construction of point sets with large minimum Dick weight, to appear in J. Complexity.

Takehito Yoshiki
Graduate School of Mathematical Sciences
University of Tokyo
Tokyo 153-8914, *Japan*
E-mail: yosiki@ms.u-tokyo.ac.jp