HIROSHIMA MATH. J.
44 (2014), 193–215

On Hardy spaces of local and nonlocal operators

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(Received December 20, 2012)
(Revised December 17, 2013)

ABSTRACT. We characterize (conditional) Hardy spaces of the Laplacian and the fractional Laplacian by using Hardy-Stein type identities.

1. Introduction

We fix an arbitrary open set $D \subset \mathbf{R}^d$ and a point $x_0 \in D$. For $p > 0$ and $0 < \alpha < 2$ we consider the Hardy space $\mathscr{H}^p(D, \alpha)$ of the fractional Laplacian $\Delta^{\alpha/2}$. Here

$$\Delta^{\alpha/2} u(x) = \lim_{\eta \to 0^+} \int_{|y-x|>\eta} \mathscr{A} \frac{u(y) - u(x)}{|y - x|^{d+\alpha}} \, dy, \tag{1}$$

$\mathscr{A} = \Gamma((d+\alpha)/2)/(2^{-\alpha}\pi^{d/2}|\Gamma(-\alpha/2)|)$ and $\mathscr{H}^p(D, \alpha)$ is defined as follows. Let X be the isotropic α-stable Lévy process, i.e. the symmetric Lévy process on $\mathbf{R}^d$ with the Lévy measure $\nu(dy) = \mathscr{A}|y|^{-d-\alpha}dy$ and zero Gaussian part ([11]). Let $\mathbf{E}_x$ be the expectation for X starting at $x \in \mathbf{R}^d$. We define

$$\tau_D = \inf\{t > 0 : X_t \notin D\},$$

the first exit time of X from D. A Borel function $u : \mathbf{R}^d \to \mathbf{R}$ is called α-harmonic on D if for every open U relatively compact in D (denoted $U \subset\subset D$) we have

$$u(x) = \mathbf{E}_x u(X_{\tau_U}), \qquad x \in U. \tag{2}$$

We assume that the expectation is absolutely convergent, in particular–finite. Equivalently, u is α-harmonic on D if u is twice continuously differentiable

The first author was partially supported by MNiSW through grant N N201 397137.
The second author was supported by MNiSW through grant N N201 397137 and by the DFG through SFB-701 'Spectral Structures and Topological Methods in Mathematics'.
The third author was supported by Agence Nationale de la Recherche through grant ANR-09-BLAN-0084-01 and by MNiSW through grant N N201 373136.
2010 *Mathematics Subject Classification*. Primary 60J75; Secondary 60J50, 42B30, 31B25.
Key words and phrases. fractional Laplacian, Hardy spaces, conditional stable Lévy process.

on D, $\int_{\mathbf{R}^d} |u(y)|(1+|y|)^{-d-\alpha} dy < \infty$, and

$$\Delta^{\alpha/2} u(x) = 0, \qquad x \in D. \tag{3}$$

We refer to [10, 14, 17, 21] for this characterization and detailed discussion of α-harmonic functions, including structure theorems for nonnegative α-harmonic functions, and explicit formulas for the Green function, Poisson kernel and Martin kernel of $\Delta^{\alpha/2}$ for the ball. The equivalence of various notions of harmonicity for more general Markov processes is proved in [18]. We also refer to [15, p. 120], which shows by means of an example why the mean value property (2) is preferred over analogues of (3) for harmonic functions of Markov processes. The reader may verify, using (2) and the strong Markov property of X, that $\{u(X_{\tau_U})\}_{U \subset\subset D}$ is a martingale ordered by inclusion of sets U. In particular, $\mathbf{E}_x |u(X_{\tau_U})|^p$ is non-decreasing in U, if $p \geq 1$.

DEFINITION 1. *Let $0 < p < \infty$. We write $u \in \mathscr{H}^p = \mathscr{H}^p(D, \alpha)$, if u is α-harmonic on D and*

$$\|u\|_p := \sup_{U \subset\subset D} (\mathbf{E}_{x_0} |u(X_{\tau_U})|^p)^{1/p} < \infty. \tag{4}$$

The finiteness condition does not depend on the choice of $x_0 \in D$, because the function $U \ni x \mapsto \mathbf{E}_x |u(X_{\tau_U})|^p$ satisfies Harnack inequality for arbitrary (Borel) function u, see [48, p. 17] or [4, Lemma 2.1]. If $p \leq q$, then $\mathscr{H}^p \supset \mathscr{H}^q$.

We say that nonnegative functions $f(u)$ and $g(u)$ are *comparable*, and write $f(u) \asymp g(u)$, if numbers $0 < c \leq C < \infty$ exist such that $cf(u) \leq g(u) \leq Cf(u)$ for every u.

Let $G_D(x, y)$ be the Green function of $\Delta^{\alpha/2}$ for the Dirichlet problem on D. The function is defined as follows. We let

$$p_t(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-t|\xi|^\alpha} e^{ix\cdot\xi}\, d\xi, \qquad t > 0,\ x \in \mathbf{R}^d,$$

so that $p_t(y - x)$ is the time-homogeneous transition density function of X. Then we use Hunt's formula to define the Dirichlet heat kernel of $\Delta^{\alpha/2}$ for D:

$$p_D(t, x, y) = p_t(y - x) - \mathbf{E}^x[\tau_D < t; p_{t-\tau_D}(y - X_{\tau_D})], \qquad t > 0,\ x, y \in \mathbf{R}^d,$$

cf. [24, Section 2.2] or [19, Section 3]. Finally, we let

$$G_D(x, y) = \int_0^\infty p_D(t, x, y) dt, \qquad x, y \in \mathbf{R}^d.$$

It may happen that $G_D \equiv \infty$ on D. This is the case, e.g., if $d = 1 \leq \alpha$ and $D = (-\infty, \infty)$. Such sets D are not excluded from our considerations. We also remark that purely analytic definition of G_D may be found in [38].

The reader may notice that (4) is far from being explicit because it involves the distribution of X_{τ_U} for all $U \subset D$. The following result and the exact formula for $\|u\|_p$ given in (16) below simplify this perspective.

THEOREM 2. *If* $1 < p < \infty$, *then* $\|u\|_p^p$ *is* comparable *on* $\mathscr{H}^p$ *with*

$$|u(x_0)|^p + \int_D G_D(x_0, y) \int_{\mathbf{R}^d} \frac{[u(z) - u(y)]^2 (|u(z)| \vee |u(y)|)^{p-2}}{|z - y|^{d+\alpha}} \, dzdy. \tag{5}$$

In fact, $u \in \mathscr{H}^p$ *if and only if* u *is* α*-harmonic in* D *and the integral is finite.*

Incidentally, if $G_D \equiv \infty$ on D, $1 < p < \infty$ and $u \in \mathscr{H}^p$, then u must be constant on D. We also describe conditional Hardy spaces $\mathscr{H}_h^p = \mathscr{H}_h^p(D, \alpha)$, where h is a fixed α-harmonic function positive on D and vanishing on D^c. The class $\mathscr{H}_h^p$ is of considerable interest because it directly relates to *ratios* of α-harmonic functions, weighted L^p integrability of α-harmonic functions and Doob's h-transform. We note in passing that Doob's conditioning also plays an important role in the study of the relative Fatou theorem for α-harmonic functions [42, 35, 12], and in the theory of conditional α-stable Lévy processes [10, 16].

We give similar characterizations for Hardy spaces of the classical Laplacian Δ, too: formula (28) below is the celebrated Hardy-Stein identity but Theorem 17, which may be considered a *conditional* Hardy-Stein identity, is new, and may be interesting for its own sake.

The paper is composed as follows. In Section 2 we observe the formula

$$\sup_{U \subset\subset D} \mathbf{E}_{x_0} u^2(X_{\tau_U}) = |u(x_0)|^2 + \int_D G_D(x_0, y) \int_{\mathbf{R}^d} \mathscr{A} \frac{[u(z) - u(y)]^2}{|z - y|^{d+\alpha}} \, dzdy, \tag{6}$$

for the norm of $\mathscr{H}^2$, and we extend it in Lemma 8 and Theorem 2 to $\mathscr{H}^p$ for $p > 1$. The conditional Hardy spaces $\mathscr{H}_h^p$ are characterized in Lemma 15, Theorem 16 and formula (27) in Section 3, see also Remark 11. In Section 4 we state the results for the Laplacian: formula (28) and Theorem 17. In Section 5 we describe the norm of the Hardy spaces in terms of the Krickeberg decomposition for $p \geq 1$, and we prove a classical Littlewood-Paley inequality.

Formula (6) and its modifications (16, 27, 29) below are the main subject of the paper, and they may be considered nonlocal or conditional extensions of the classical Hardy-Stein equality, for which we refer the reader to (28) in Section 4 and to [53, 45, 46].

Our work was motivated by the notion of the quadratic variation of martingales, operator carré du champ, and the characterization of the classical and martingale Hardy and Bergman spaces ([28, 41, 47, 46, 55, 53, 39, 56]). The resulting technique should apply to Hardy spaces of operators and Markov processes much more general than the fractional Laplacian and the isotropic stable Lévy process. The style of the presentation and the inclusion of both jump and continuous processes in the present paper is intended to clarify the methodology and indicate such extensions. Our development is mostly analytic. In fact, the definitions of the Hardy spaces can be easily formulated analytically by using the harmonic measures of the Laplacian and the fractional Laplacian ([6, 38]). A clarifying comparison of the conditional and the non-conditional cases is made at the end of Section 4.

2. Characterization of $\mathscr{H}^p$

Consider an open set $U \subset\subset D$ and a real-valued function $\phi : \mathbf{R}^d \to \mathbf{R}$ which is C^2 in a neighborhood of $\overline{U}$ and satisfies $\int_{\mathbf{R}^d} |\phi(y)|(1+|y|)^{-d-\alpha} dy < \infty$. Then $\Delta^{\alpha/2}\phi$ is bounded on $\overline{U}$, and for every $x \in \mathbf{R}^d$ we have

$$\phi(x) = \mathbf{E}_x \phi(X_{\tau_U}) - \int_U G_U(x,y) \Delta^{\alpha/2}\phi(y) dy. \tag{7}$$

Indeed, if ϕ is compactly supported and smooth in $\mathbf{R}^d$, i.e. it is a *test function*, then (7) follows from Dynkin's formula, see also a brief semi-analytic proof given in [13, Lemma 8 with $b=0$]. For arbitrary function ϕ satisfying the assumptions stated before (7), let test functions ϕ_n converge to ϕ in $L^1(\mathbf{R}^d, (1+|y|)^{-d-\alpha}dy)$ and in C^2 on a neighborhood of $\overline{U}$. Then $\Delta^{\alpha/2}\phi_n \to \Delta^{\alpha/2}\phi$ uniformly on U because we can use Taylor expansion with remainder of the second-order for the integrand in (1) in a neighborhood of $\overline{U}$, and we also have $\Delta^{\alpha/2}u(x) = \int_{U^c} u(y)\mathscr{A}|y-x|^{-d-\alpha}dy$ if $x \in U$ and u is supported in U^c. We also note that the distribution of X_{τ_U} for the process X starting at x has the density function $z \mapsto \int_U G_U(x,y)\mathscr{A}|z-y|^{-d-\alpha}dy$ in the complement of $\overline{U}$. The fact is known as the Ikeda-Watanabe formula and follows immediately from (7) for test functions (we also refer to [33] for the original contribution and to [13, Lemma 6 with $b=0$] for a brief semi-analytic proof). We note that $\int_U G_U(x,y)\mathscr{A}|z-y|^{-d-\alpha}dy \le c(1+|z|)^{-d-\alpha}$ in the complement of each neighborhood of $\overline{U}$, see also [7, Lemma 7]. By bounded convergence in a neighborhood of $\overline{U}$ and by L^1 convergence elsewhere we extend (7) from ϕ_n to ϕ. The reader interested in proving (7) by means of the maximum principle of $\Delta^{\alpha/2}$ may also do so by using [10, Lemma 3.8 and the proof of Theorem 3.9].

LEMMA 3. *If u is α-harmonic on D and $U \subset\subset D$, then*

$$\mathbf{E}_{x_0} u^2(X_{\tau_U}) = u(x_0)^2 + \int_U G_U(x_0,y) \int_{\mathbf{R}^d} \mathscr{A} \frac{[u(z)-u(y)]^2}{|z-y|^{d+\alpha}} dz dy. \tag{8}$$

Proof. If $\int_{\mathbf{R}^d} u(y)^2(1+|y|)^{-d-\alpha}dy = \infty$, then $\int_{\mathbf{R}^d}[u(z)-u(y)]^2/|z-y|^{d+\alpha}dz = \infty$ for every y. Also $\mathbf{E}_{x_0}u^2(X_{\tau_U}) = \infty$, because the distribution of X_{τ_U} has density function bounded below by a multiple of $(1+|y|)^{-d-\alpha}$ in the complement of the neighborhood of $\overline{U}$, see the discussion of (7). Therefore in what follows we may assume that $\int_{\mathbf{R}^d} u(y)^2(1+|y|)^{-d-\alpha}dy < \infty$. Since u^2 is C^2 on D, $\Delta^{\alpha/2}(u^2)$ is bounded on $\overline{U}$. By (7) with $\phi = u^2$, for $x \in \mathbf{R}^d$ we have

$$\mathbf{E}_x u^2(X_{\tau_U}) = u^2(x) + \int_U G_U(x,y)\Delta^{\alpha/2}(u^2)(y)dy.$$

For $y \in \overline{U}$, $z \in \mathbf{R}^d$, we have $u^2(z) - u^2(y) - 2u(y)[u(z)-u(y)] = [u(z)-u(y))]^2$. Since $\Delta^{\alpha/2}u(y) = 0$, we have

$$\begin{aligned}
\Delta^{\alpha/2}u^2(y) &= \Delta^{\alpha/2}u^2(y) - 2u(y)\Delta^{\alpha/2}u(y) \\
&= \lim_{\eta\to 0^+}\int_{\{z\in\mathbf{R}^d:|z-y|>\eta\}} \mathscr{A}\frac{u^2(z)-u^2(y)-2u(y)[u(z)-u(y)]}{|z-y|^{d+\alpha}}\,dz \\
&= \int_{\mathbf{R}^d}\mathscr{A}\frac{[u(z)-u(y)]^2}{|z-y|^{d+\alpha}}\,dz,
\end{aligned}$$

and (8) follows. $\square$

We obtain the description of $\mathscr{H}^2$ aforementioned in Introduction.

Corollary 4. *If u is α-harmonic in D, then* (6) *holds.*

Proof. Recall that $G_U(x,y) \uparrow G_D(x,y)$ as $U \uparrow D$. By the monotone convergence theorem we obtain the result, also if $G_D \equiv \infty$ on D. $\square$

We conclude that $\mathscr{H}^2$ consists of precisely all those functions α-harmonic on D for which the quadratic form on the right hand side of (6) is finite.

We now consider arbitrary real number $p > 1$. We note that $x \mapsto |x|^p$ is convex on $\mathbf{R}$, with the derivative $pa|a|^{p-2}$ at $x = a$. For $a, b \in \mathbf{C}$ we let

$$F(a,b) = |b|^p - |a|^p - p\bar{a}|a|^{p-2}(b-a). \tag{9}$$

We have $F(a,b) = |b|^p$ if $a = 0$, and $F(a,b) = (p-1)|a|^p$ if $b = 0$. If $a, b \in \mathbf{R}$, then $F(a,b)$ is the second-order Taylor remainder of $\mathbf{R} \ni x \mapsto |x|^p$, and, by convexity, $F(a,b) \geq 0$.

Example 5. For (even) $p = 2, 4, \ldots$ and $a, b \in \mathbf{R}$, we have

$$F(a,b) = b^p - a^p - pa^{p-1}(b-a) = (b-a)^2\sum_{k=0}^{p-2}(k+1)b^{p-2-k}a^k.$$

Let ε, b and a be real numbers. For $p > 1$ we define

$$F_\varepsilon(a,b) = \mathrm{Re}\, F(a+i\varepsilon, b+i\varepsilon) = |b+i\varepsilon|^p - |a+i\varepsilon|^p - pa|a+i\varepsilon|^{p-2}(b-a). \quad (10)$$

$F_\varepsilon(a,b)$ is the second-order Taylor remainder of $\mathbf{R} \ni x \mapsto (x^2+\varepsilon^2)^{p/2}$, and, by convexity, $F_\varepsilon(a,b) \ge 0$ (see below). Of course, $F_\varepsilon(a,b) \to F(a,b)$ as $\varepsilon \to 0$.

LEMMA 6. *For every $p > 1$, we have*

$$F(a,b) \asymp (b-a)^2(|b| \vee |a|)^{p-2}, \qquad a,b \in \mathbf{R}. \quad (11)$$

If $p \in (1,2)$, then

$$0 \le F_\varepsilon(a,b) \le \frac{1}{p-1} F(a,b), \qquad \varepsilon, a, b \in \mathbf{R}. \quad (12)$$

PROOF. We denote $K(a,b) = (b-a)^2(|b| \vee |a|)^{p-2}$. For every $k \in \mathbf{R}$, $F(ka,kb) = |k|^p F(a,b)$ and $K(ka,kb) = |k|^p K(a,b)$. If $a = 0$, then (11) becomes equality, hence we may assume that $a \ne 0$, in fact—that $a = 1$. Let $f(b) = F(1,b) = |b|^p - 1 - p(b-1)$. We will prove that

$$c_p(b-1)^2(|b| \vee 1)^{p-2} \le f(b) \le C_p(b-1)^2(|b| \vee 1)^{p-2}. \quad (13)$$

Since $f(1) = f'(1) = 0$ and $f''(y) = p(p-1)|y|^{p-2}$ for $y \ne 0$, we obtain

$$f(b) = p(p-1)\int_1^b \int_1^x |y|^{p-2} dy dx = p(p-1)\int_1^b |y|^{p-2}(b-y) dy.$$

The first integral is over a simplex of area $(b-1)^2/2$, and it is a monotone function of the simplex (as ordered by inclusion). For b close to 1 the integral is comparable to $(b-1)^2$. For large $|b|$ the (second) integral is comparable to $|b|^p$. This proves (13), hence (11). We now consider F_ε for $\varepsilon \ne 0$ and $p > 1$. Let

$$f_\varepsilon(b) = F_\varepsilon(1,b) = (b^2+\varepsilon^2)^{p/2} - (1+\varepsilon^2)^{p/2} - p(1+\varepsilon^2)^{(p-2)/2}(b-1).$$

We have $f_\varepsilon(1) = f_\varepsilon'(1) = 0$ and

$$f_\varepsilon''(y) = (y^2+\varepsilon^2)^{(p-4)/2} p[y^2(p-1)+\varepsilon^2] \ge 0, \qquad y \in \mathbf{R}.$$

Therefore,

$$f_\varepsilon(b) = \int_1^b \int_1^x f_\varepsilon''(y) dy dx \ge 0.$$

In fact we have

$$f_\varepsilon''(y) \le p[1 \vee (p-1)](y^2+\varepsilon^2)^{(p-2)/2}, \qquad y \in \mathbf{R}.$$

We now let $1 < p \le 2$. For $y \in \mathbf{R}$ we obtain $f_\varepsilon''(y) \le p|y|^{p-2}$, hence

$$f_\varepsilon(b) \le p \int_1^b \int_1^x |y|^{p-2} dy dx = \frac{1}{p-1} f(b).$$

If $a \neq 0$, then by (13),

$$\begin{aligned} F_\varepsilon(a,b) &= |a|^p[|b/a + i\varepsilon/a|^p - |1 + i\varepsilon/a|^p - p|1 + i\varepsilon/a|^{p-2}(b/a - 1)] \\ &= |a|^p F_{\varepsilon/a}(1, b/a) = |a|^p f_{\varepsilon/a}(b/a) \\ &\le \frac{1}{p-1}|a|^p f(b/a) = \frac{1}{p-1} F(a,b). \end{aligned}$$

If $a = 0$, then

$$F_\varepsilon(a,b) = (b^2 + \varepsilon^2)^{p/2} - |\varepsilon|^p \le |b|^p = F(a,b),$$

too, since $(\rho + \eta)^{p/2} - \rho^{p/2} = \int_\rho^{\rho+\eta} \frac{p}{2}(y + \eta)^{p/2-1} dy \le \frac{p}{2}\eta^{p/2} \le \eta^{p/2}$ for $\rho, \eta \ge 0$. The proof of (12) is complete. □

To prepare for limiting arguments we make the following observation, which follows from Fatou's lemma and dominated convergence theorem.

Remark 7. If $0 \le f_n \to f$ μ-a.e., $\int f_n \, d\mu$ is bounded and $f_n \le cf$ for some constant c, then $\int f_n \, d\mu \to \int f \, d\mu$ as $n \to \infty$.

Lemma 8. *If u is α-harmonic in $D, U \subset\subset D$, and $p > 1$, then*

$$\mathbf{E}_{x_0}|u(X_{\tau_U})|^p = |u(x_0)|^p + \int_U G_U(x_0, y) \int_{\mathbf{R}^d} \mathscr{A} \frac{F(u(y), u(z))}{|z - y|^{d+\alpha}} \, dz dy. \tag{14}$$

Proof. We proceed as in Lemma 3. In particular, if

$$\int_{\mathbf{R}^d} |u(y)|^p (1 + |y|)^{-d-\alpha} dy = \infty,$$

then $\mathbf{E}_{x_0}|u(X_{\tau_U})|^p = \infty$ and also, by Lemma 6,

$$\int_{\mathbf{R}^d} \frac{F(u(y), u(z))}{|z - y|^{d+\alpha}} \, dz = \infty$$

for every $y \in \mathbf{R}^d$. Therefore in what follows, we assume that $\int_{\mathbf{R}^d} |u(y)|^p (1 + |y|)^{-d-\alpha} dy < \infty$, or $\mathbf{E}_{x_0}|u(X_{\tau_U})|^p < \infty$. We first consider the case of $p \ge 2$ and apply (7) to $\phi = |u|^p \in C^2(D)$. For $y \in D$ we have $\Delta^{\alpha/2} u(y) = 0$, hence

$$\begin{aligned}
&\Delta^{\alpha/2}|u|^p(y)\\
&\quad = \Delta^{\alpha/2}|u|^p(y) - pu(y)|u(y)|^{p-2}\Delta^{\alpha/2}u(y)\\
&\quad = \lim_{\eta\to 0^+}\int_{\{z\in\mathbf{R}^d:|z-y|>\eta\}} \mathscr{A}\frac{|u(z)|^p - |u(y)|^p - pu(y)|u(y)|^{p-2}[u(z)-u(y)]}{|z-y|^{d+\alpha}}\,dz\\
&\quad = \int_{\mathbf{R}^d}\mathscr{A}\frac{F(u(y),u(z))}{|z-y|^{d+\alpha}}\,dz.
\end{aligned}$$

This and (7) yield (14) for $p \geq 2$. We now consider $1 < p < 2$. We note that $|u+i\varepsilon|^p \in C^2(D)$. As in the first part of the proof,

$$\begin{aligned}
\Delta^{\alpha/2}|u+i\varepsilon|^p(y) &= \Delta^{\alpha/2}|u+i\varepsilon|^p(y) - pu(y)|u(y)+i\varepsilon|^{p-2}\Delta^{\alpha/2}u(y)\\
&= \int_{\mathbf{R}^d}\mathscr{A}\frac{F_\varepsilon(u(y),u(z))}{|z-y|^{d+\alpha}}\,dz,
\end{aligned}$$

hence

$$\mathbf{E}_{x_0}|u(X_{\tau_U})+i\varepsilon|^p = |u(x_0)+i\varepsilon|^p + \int_U G_U(x_0,y)\int_{\mathbf{R}^d}\mathscr{A}\frac{F_\varepsilon(u(y),u(z))}{|z-y|^{d+\alpha}}\,dzdy. \tag{15}$$

By Jensen's inequality,

$$\mathbf{E}_{x_0}|u(X_{\tau_U})+i\varepsilon|^p \leq \mathbf{E}_{x_0}(|u(X_{\tau_U})|+|\varepsilon|)^p \leq 2^{p-1}\mathbf{E}_{x_0}(|u(X_{\tau_U})|^p+|\varepsilon|^p),$$

which remains bounded as $\varepsilon \to 0$. By Remark 7 and Lemma 6 applied to the right-hand side of (15) and by the dominated convergence theorem applied to its left-hand side we obtain (14). □

PROOF (Proof of Theorem 2). Lemma 6, Lemma 8 and monotone convergence imply the comparability of $\|u\|_p^p$ and (5) with the same constants as in (13), under the mere assumption that u be α-harmonic on D. In fact,

$$\|u\|_p^p = |u(x_0)|^p + \int_D G_D(x_0,y)\int_{\mathbf{R}^d}\mathscr{A}\frac{F(u(y),u(z))}{|z-y|^{d+\alpha}}\,dzdy. \tag{16}$$

□

We note that in many cases sharp two-sided estimates of G_D are known. For instance, if D is a bounded open $C^{1,1}$ set in $\mathbf{R}^d$ and $d > \alpha$, then

$$G_D(x_0,y) \asymp \delta_D(y)^{\alpha/2}|y-x_0|^{\alpha-d},$$

where $\delta_D(y) := \mathrm{dist}(y, D^c)$, see [20, 37, 22].

Recall the definition of F_ε, (10), and the fact that $F_0 = F$ of (9). Before moving to conditional Hardy spaces we record the following observation.

Lemma 9. *For every* $p > 1$ *and* $a_1, a_2, b_1, b_2, \varepsilon \in \mathbf{R}$, *we have*

$$F_\varepsilon(a_1 \wedge a_2, b_1 \wedge b_2) \le F_\varepsilon(a_1, b_1) \vee F_\varepsilon(a_2, b_2), \tag{17}$$

$$F_\varepsilon(a_1 \vee a_2, b_1 \vee b_2) \le F_\varepsilon(a_1, b_1) \vee F_\varepsilon(a_2, b_2). \tag{18}$$

In particular, $F(a \wedge 1, b \wedge 1) \le F(a,b)$ *and* $F(a \vee (-1), b \vee (-1)) \le F(a,b)$, *for all* $a, b \in \mathbf{R}$. *The latter also extends to* $K(a,b) = (b-a)^2(|b| \vee |a|)^{p-2}$.

Proof. Let $\varepsilon \neq 0$. We claim that the function $b \mapsto F_\varepsilon(a,b)$ decreases on $(-\infty, a]$ and increases on $[a, \infty)$. To see this, we consider

$$\frac{\partial F_\varepsilon}{\partial b}(a,b) = pb(b^2+\varepsilon^2)^{p/2-1} - pa(a^2+\varepsilon^2)^{p/2-1}. \tag{19}$$

The function $h(x) = px(x^2+\varepsilon^2)^{p/2-1}$ has derivative $h'(x) = p(x^2+\varepsilon^2)^{p/2-2}(x^2(p-1)+\varepsilon^2) > 0$. If follows that the difference in (19) is positive if $b > a$ and negative if $b < a$. This proves our claim.

Furthermore the function $a \mapsto F_\varepsilon(a,b)$ decreases on $(-\infty, b]$ and increases on $[b, \infty)$, as follows from calculating the derivative,

$$\frac{\partial F_\varepsilon}{\partial a}(a,b) = p(a-b)(a^2+\varepsilon^2)^{p/2-2}(a^2(p-1)+\varepsilon^2).$$

We now prove (17). If $b_1 \wedge b_2 = b_1$ and $a_1 \wedge a_2 = a_1$ (or $b_1 \wedge b_2 = b_2$ and $a_1 \wedge a_2 = a_2$), then (17) is trivial. If $b_1 \wedge b_2 = b_1$ and $a_1 \wedge a_2 = a_2$, then the monotonicity of F_ε yields

$$F_\varepsilon(a_2, b_1) \le F_\varepsilon(a_1, b_1), \qquad \text{if } b_1 < a_2,$$

$$F_\varepsilon(a_2, b_1) \le F_\varepsilon(a_2, b_2), \qquad \text{if } b_1 \ge a_2.$$

The case $b_1 \wedge b_2 = b_2$ and $a_1 \wedge a_2 = a_1$ obtains by renaming the arguments. This proves inequality (17). (18) follows from (17) and the identity

$$F_\varepsilon(-a, -b) = F_\varepsilon(a,b).$$

The case $\varepsilon = 0$ obtains by passing to the limit. When $a = b$, we have $F(a,b) = 0$, which yields the second last statement of the lemma. For K we also get $(b \wedge 1 - a \wedge 1)^2(|b \wedge 1| \vee |a \wedge 1|)^{p-2} \le (b-a)^2(|b| \vee |a|)^{p-2}$ and $(b \vee (-1) - a \vee (-1))^2(|b \vee (-1)| \vee |a \vee (-1)|)^{p-2} \le (b-a)^2(|b| \vee |a|)^{p-2}$.

□

In passing we note that if the right-hand side of (14) is finite for u, then it is also finite (in fact–smaller) for $u \wedge 1$ and $u \vee (-1)$. The latter functions have smaller values and increments than u, a property defining *normal contractions* for Dirichlet forms ([30]).

3. Characterization of $\mathscr{H}_h^p$

The fractional Laplacian is a nonlocal operator and the corresponding stochastic process X has jumps. In consequence the definitions of α-harmonicity (2) and (3) involve the values of the function on the whole of D^c ([14]). This is somewhat unusual compared with the classical theory of the Laplacian and the Brownian motion, and efforts were made by various authors to ascribe genuine boundary conditions to such processes and functions ([9, 12, 32, 35, 42, 40], see also [5, 1]). One possibility is to study the boundary behavior of α-harmonic functions after an appropriate normalization. We shall use Doob's conditioning to normalize. The procedure was proposed for classical harmonic functions in [26], and [23, Chapter 11] treats a general case. We shall focus on α-harmonic functions vanishing on D^c, so that D^c may be ignored. Namely, let h be α-harmonic and positive on D, and let h vanish on D^c. Such functions are called singular α-harmonic on D ([14]). We consider the transition semigroup

$$P_t^h f(x) = \frac{1}{h(x)} \int p_D(t,x,y) f(y) h(y) dy, \tag{20}$$

where p_D, defined in Section 1, is the time-homogeneous transition density of X *killed* on leaving D ([10]). The semigroup property of P_t^h follows directly from the Chapman-Kolmogorov equations for p_D (cf. [24, Section 2.2] or [19, Section 3]),

$$\int_{\mathbf{R}^d} p_D(s,x,y) p_D(t,y,z) dy = p_D(s+t,x,z).$$

By α-harmonicity and the optional stopping theorem, $\mathbf{E}_x h(X_{\tau_U \wedge t}) = h(x)$, if $x \in U \subset\subset D$. Letting $U \uparrow D$, by Fatou's lemma we obtain $\int p_D(t,x,y)h(y)dy = \mathbf{E}_x\{t < \tau_D : h(X_t)\} \le h(x)$, i.e. P_t^h is subprobabilistic.

The conditional process is defined as the Markov process with the transition semigroup P^h, and it will be denoted by the same symbol X. We let $\mathbf{E}_x^h$ be the corresponding expectation for X starting at $x \in D$,

$$\mathbf{E}_x^h f(X_t) = \frac{1}{h(x)} \mathbf{E}_x[t < \tau_D; f(X_t)h(X_t)],$$

see also [10]. A Borel function $r : D \to \mathbf{R}$ is h-harmonic (on D) if for every open $U \subset\subset D$ we have

$$r(x) = \mathbf{E}_x^h r(X_{\tau_U}) = \frac{1}{h(x)} \mathbf{E}_x[X_{\tau_U} \in D; r(X_{\tau_U})h(X_{\tau_U})], \qquad x \in U.$$

Here we assume that the expectation is absolutely convergent, in particular– finite. It is evident that r is h-harmonic if and only if $r = u/h$ on D, where u is

α-harmonic on D and vanishes on D^c. Below we call such functions u singular α-harmonic on D, too, without requiring nonnegativity. We are interested in L^p integrability of u/h, which amounts to weighted L^p integrability of u. The following definition is adapted from [43].

Definition 10. *For $0 < p < \infty$ we define $\mathscr{H}_h^p = \mathscr{H}_h^p(D, \alpha)$ as the class of all the functions $u : \mathbf{R}^d \to \mathbf{R}$, singular α-harmonic on D and such that*

$$\|u\|_{\mathscr{H}_h^p}^p := \sup_{U \subset\subset D} \mathbf{E}_{x_0}^h \left|\frac{u(X_{\tau_U})}{h(X_{\tau_U})}\right|^p = \frac{1}{h(x_0)} \sup_{U \subset\subset D} \mathbf{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} < \infty,$$

where $\mathbf{E}_{x_0}|u(X_{\tau_U})|^p/h(X_{\tau_U})^{p-1}$ means $\mathbf{E}_{x_0}[X_{\tau_U} \in D; |u(X_{\tau_U})|^p/h(X_{\tau_U})^{p-1}]$.

By Harnack inequality, $\mathscr{H}_h^p$ does not depend on the choice of $x_0 \in D$. In what follows we adopt the convention that $u(z)/h(z) = 0$ if u is singular α-harmonic on D and $z \in D^c$.

Remark 11. Note that the elements of this $\mathscr{H}_h^p$ are α-harmonic, rather than h-harmonic. In view of Definition 10, the genuine conditional Hardy space of $\Delta^{\alpha/2}$ and h is $\{u/h : u \in \mathscr{H}_h^p\}$, with the norm $\|u/h\| = \|u\|_{\mathscr{H}_h^p}$. $\mathscr{H}_h^p$ may be considered a weighted Hardy space of $\Delta^{\alpha/2}$, but it is convenient to call it conditional Hardy space, too. Below we focus on $\|u\|_{\mathscr{H}_h^p}$, which yields description of both spaces.

Lemma 12. *If u is singular α-harmonic on D and $U \subset\subset D$, then*

$$\mathbf{E}_{x_0} \frac{u^2(X_{\tau_U})}{h(X_{\tau_U})} = \frac{u(x_0)^2}{h(x_0)} + \int_U G_U(x_0, y) \int_{\mathbf{R}^d} \mathscr{A} \left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)}\right]^2 \frac{h(z)dzdy}{|z-y|^{d+\alpha}}. \tag{21}$$

Proof. As in Lemma 3 we assume that $\mathbf{E}_{x_0} u^2(X_{\tau_U})/h(X_{\tau_U}) < \infty$, equivalently $\int_D u^2(y)h(y)^{-1}(1+|y|)^{-d-\alpha}dy < \infty$, else both sides of (21) are infinite. We also note that u^2/h is C^2 on D. Let $y \in D$. For arbitrary $z \in \mathbf{R}^d$ we have

$$\begin{aligned} &\frac{u^2(z)}{h(z)} - \frac{u^2(y)}{h(y)} - 2\frac{u(y)}{h(y)}(u(z) - u(y)) + \frac{u^2(y)}{h^2(y)}(h(z) - h(y)) \\ &\quad = \left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)}\right]^2 h(z). \end{aligned} \tag{22}$$

By (22) and α-harmonicity,

$$\begin{aligned} \Delta^{\alpha/2}\left(\frac{u^2}{h}\right)(y) &= \Delta^{\alpha/2}\left(\frac{u^2}{h}\right)(y) - 2\frac{u(y)}{h(y)}\Delta^{\alpha/2}u(y) + \frac{u^2(y)}{h^2(y)}\Delta^{\alpha/2}h(y) \\ &= \int_{\mathbf{R}^d} \mathscr{A}\left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)}\right]^2 |z-y|^{-d-\alpha}h(z)dz. \end{aligned} \tag{23}$$

Noteworthy, the integrand is nonnegative. Following (7), for u^2/h we get

$$\mathbf{E}_{x_0}\frac{u^2(X_{\tau_U})}{h(X_{\tau_U})} = \frac{u(x_0)^2}{h(x_0)} + \int_U G_U(x_0, y)\Delta^{\alpha/2}\left(\frac{u^2}{h}\right)(y)dy.$$

By using (23) we obtain (21). □

We can interpret (21) in terms of h-conditioning and $r = u/h$ as follows,

$$\mathbf{E}^h_{x_0} r(X_{\tau_U})^2 = r(x_0)^2 + \int_U \frac{G_U(x_0, y)}{h(x_0)h(y)} \int_{\mathbf{R}^d} \mathscr{A}\frac{[r(z) - r(y)]^2}{|z - y|^{d+\alpha}} \frac{h(z)}{h(y)}\, dzh^2(y)dy.$$

This is an analogue of (6), and also indicates the general situation. For $p > 1$ we consider the expressions of the form

$$F\left(\frac{a}{s}, \frac{b}{t}\right), \qquad a, b \in \mathbf{C},\ s, t > 0,$$

see (9). By Lemma 6 we have

$$0 \le F\left(\frac{a}{s}, \frac{b}{t}\right) \asymp \left(\frac{b}{t} - \frac{a}{s}\right)^2 \left(\frac{|b|}{t} \vee \frac{|a|}{s}\right)^{p-2}, \qquad a, b \in \mathbf{R},\ s, t > 0, \tag{24}$$

and the comparisons on the right of (24) hold with the constants c_p and C_p of (13). If $1 < p < 2$, then we also consider

$$F_\varepsilon\left(\frac{a}{s}, \frac{b}{t}\right), \qquad \varepsilon, a, b \in \mathbf{R},\ s, t > 0,$$

where F_ε is defined in (10). By Lemma 6 we have

$$0 \le F_\varepsilon\left(\frac{a}{s}, \frac{b}{t}\right) \le \frac{1}{p-1} F\left(\frac{a}{s}, \frac{b}{t}\right), \qquad a, b \in \mathbf{R},\ s, t > 0. \tag{25}$$

LEMMA 13. *For $p > 1$, $a, b \in \mathbf{C}$ and $s, t > 0$, we have*

$$F\left(\frac{a}{s}, \frac{b}{t}\right) = \frac{|b|^p}{t^p} - \frac{|a|^p}{ts^{p-1}} - \frac{p|a|^{p-2}\bar{a}(b-a)}{ts^{p-1}} + \frac{(p-1)|a|^p(t-s)}{ts^p}. \tag{26}$$

PROOF. By the definition of F,

$$F\left(\frac{a}{s}, \frac{b}{t}\right) = \frac{|b|^p}{t^p} - \frac{|a|^p}{s^p} - \frac{p|a|^{p-2}\bar{a}b}{ts^{p-1}} + \frac{p|a|^p}{s^p}.$$

We get the same quantity expanding the right-hand side of (26):

$$\frac{|b|^p}{t^p} - \frac{|a|^p}{ts^{p-1}} - \frac{p|a|^{p-2}\bar{a}b}{ts^{p-1}} + \frac{p|a|^p}{ts^{p-1}} + \frac{p|a|^p}{s^p} - \frac{|a|^p}{s^p} - \frac{p|a|^p}{ts^{p-1}} + \frac{|a|^p}{ts^{p-1}}$$
$$= \frac{|b|^p}{t^p} - \frac{|a|^p}{s^p} - \frac{p|a|^{p-2}\bar{a}b}{ts^{p-1}} + \frac{p|a|^p}{s^p}.$$

□

The homogeneity seen on the left-hand side of (26) is an interesting feature for the right-hand side of (26). We also like to note that for real arguments $tF(a/s, b/t)$ is the second-order Taylor remainder for $(a,s) \mapsto |a|^p/s^{p-1}$ at (b,t) and, of course, $F_\varepsilon(a/s, b/t) \to F(a/s, b/t)$ as $\varepsilon \to 0$.

Corollary 14. *For $p > 1$, $a, b \in \mathbf{R}$, and $s, t, \varepsilon > 0$, we have*

$$F_\varepsilon\left(\frac{a}{s}, \frac{b}{t}\right) = \frac{|b + i\varepsilon t|^p}{t^p} - \frac{|a + i\varepsilon s|^p}{ts^{p-1}} - \frac{p|a + i\varepsilon s|^{p-2}a(b-a)}{ts^{p-1}}$$
$$- \frac{p|a + i\varepsilon s|^{p-2}\varepsilon^2 s(t-s)}{ts^{p-1}} + \frac{(p-1)|a + i\varepsilon s|^p(t-s)}{ts^p}.$$

Proof. The result follows from (26) because by (10) we have

$$F_\varepsilon\left(\frac{a}{s}, \frac{b}{t}\right) = \operatorname{Re} F\left(\frac{a + i\varepsilon s}{s}, \frac{b + i\varepsilon t}{t}\right). \qquad \square$$

Lemma 15. *If u is singular α-harmonic on D, $U \subset\subset D$ and $p > 1$, then*

$$\mathbf{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0)|^p}{h(x_0)^{p-1}} + \int_U G_U(x_0, y) \int_{\mathbf{R}^d} F\left(\frac{u(y)}{h(y)}, \frac{u(z)}{h(z)}\right) \mathscr{A} \frac{h(z)dzdy}{|z-y|^{d+\alpha}}.$$

Proof. As in Lemma 8 we assume that $\mathbf{E}_{x_0}|u(X_{\tau_U})|^p/h(X_{\tau_U})^{p-1} < \infty$, equivalently $\int_D |u(y)|^p h(y)^{1-p}(1+|y|)^{-d-\alpha}dy < \infty$, else both sides of the equality in the statement are infinite. If $p \geq 2$, then $|u|^p/h^{p-1} \in C^2(D)$. By (7),

$$\mathbf{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0)|^p}{h(x_0)^{p-1}} + \int_U G_U(x_0, y) \Delta^{\alpha/2}\left(\frac{|u|^p}{h^{p-1}}\right)(y)dy.$$

By α-harmonicity of h and u,

$$\Delta^{\alpha/2}\left(\frac{|u|^p}{h^{p-1}}\right)(y)$$
$$= \Delta^{\alpha/2}\left(\frac{|u|^p}{h^{p-1}}\right)(y) - \frac{p|u(y)|^{p-2}u(y)}{h(y)^{p-1}} \Delta^{\alpha/2}u(y) + \frac{(p-1)|u(y)|^p}{h(y)^p} \Delta^{\alpha/2}h(y)$$
$$= \lim_{\eta \to 0^+} \int_{\{z \in \mathbf{R}^d : |z-y| > \eta\}} \left[\frac{|u(z)|^p}{h(z)^{p-1}} - \frac{|u(y)|^p}{h(y)^{p-1}} - \frac{p|u(y)|^{p-2}u(y)}{h(y)^{p-1}}(u(z) - u(y))\right.$$
$$\left. + \frac{(p-1)|u(y)|^p}{h(y)^p}(h(z) - h(y))\right] \mathscr{A}|z-y|^{-d-\alpha}dz.$$

By Lemma 13 with $a = u(y)$, $s = h(y)$, $b = u(z)$, $t = h(z)$, the above equals

$$\int_{\mathbf{R}^d} h(z) F(u(y)/h(y), u(z)/h(z)) \mathscr{A} |z-y|^{-d-\alpha} dz.$$

This gives the result for $p \geq 2$. If $1 < p < 2$ then we argue as follows. Let $\varepsilon > 0$. By α-harmonicity of u and h, $\Delta^{\alpha/2}(|u + i\varepsilon h|^p h^{1-p})(y)$ equals

$$\lim_{\eta \to 0^+} \int_{\{z \in \mathbf{R}^d : |z-y| > \eta\}} \left[\frac{|u(z) + i\varepsilon h(z)|^p}{h(z)^{p-1}} - \frac{|u(y) + i\varepsilon h(y)|^p}{h(y)^{p-1}} \right.$$
$$- \frac{p|u(y) + i\varepsilon h(y)|^{p-2} u(y)}{h(y)^{p-1}} (u(z) - u(y)) - \frac{p|u(y) + i\varepsilon h(y)|^{p-2} \varepsilon^2}{h(y)^{p-2}} (h(z) - h(y))$$
$$\left. + \frac{(p-1)|u(y) + i\varepsilon h(y)|^p}{h(y)^p} (h(z) - h(y)) \right] \mathscr{A} |z-y|^{-d-\alpha} dz.$$

By Corollary 14 with $a = u(y)$, $s = h(y)$, $b = u(z)$, $t = h(z)$, the above equals

$$\int_{\mathbf{R}^d} h(z) F_\varepsilon(u(y)/h(y), u(z)/h(z)) \mathscr{A} |z-y|^{-d-\alpha} dz.$$

By (7) we get

$$E_{x_0} \frac{|u(X_{\tau_U}) + i\varepsilon h(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0) + i\varepsilon h(x_0)|^p}{h(x_0)^{p-1}}$$
$$+ \int_U G_U(x_0, y) \int_{\mathbf{R}^d} F_\varepsilon\left(\frac{u(y)}{h(y)}, \frac{u(z)}{h(z)}\right) \mathscr{A} \frac{h(z) dz dy}{|z-y|^{d+\alpha}}.$$

We then proceed as in the proof of Lemma 8, letting $\varepsilon \to 0$, using (25), Remark 7, and the assumed finiteness of $\mathbf{E}_{x_0} |u(X_{\tau_U})|^p / h(X_{\tau_U})^{p-1}$ and $\mathbf{E}_{x_0} h(X_{\tau_U})$. □

THEOREM 16. *Let $1 < p < \infty$. For singular α-harmonic functions u on D, $\|u\|^p_{\mathscr{H}^p_h}$ is comparable with*

$$\frac{|u(x_0)|^p}{h(x_0)^p} + \int_D \frac{G_D(x_0, y)}{h(x_0) h(y)} \int_{\mathbf{R}^d} \left(\frac{|u(z)|}{h(z)} \vee \frac{|u(y)|}{h(y)} \right)^{p-2} \left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)} \right]^2 \frac{h(z) dz h^2(y) dy}{h(y) |z-y|^{d+\alpha}}.$$

PROOF. The result follows from Lemma 15 and (24). In fact,

$$\|u\|^p_{\mathscr{H}^p_h} = \frac{|u(x_0)|^p}{h(x_0)^p} + \int_D \frac{G_D(x_0, y)}{h(x_0) h(y)} \int_{\mathbf{R}^d} F\left(\frac{u(y)}{h(y)}, \frac{u(z)}{h(z)} \right) \mathscr{A} \frac{h(z) dz h^2(y) dy}{h(y) |z-y|^{d+\alpha}}. \quad (27)$$

□

We remark in passing that for $h \equiv 1$ we obtain $\mathscr{H}^p_h = \mathscr{H}^p$. To rigorously state this observation, one should discuss conditioning by functions h with nontrivial values on D^c. In this connection we note that [21] suggest that the stopped (rather than the killed) process should be used to this end (see also [14, Remark 11] and [23, Chapter 11]). We will not embark on this endeavor, instead in the next section we fully discuss the conditional Hardy spaces of a local operator, in which case the values of h on D^c are irrelevant.

4. Classical Hardy spaces

Here we describe the Hardy spaces and the conditional Hardy spaces of harmonic functions of the Laplacian $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$. The former case has been widely studied in the literature, mainly for the ball and the half-space, but also for smooth and Lipschitz domains, see [2, 36, 50, 51, 34]. The characterization of the Hardy spaces in terms of quadratic functions appeared in [49] and [58] for harmonic functions on the half-space in $\mathbf{R}^d$. The case of D being the unit ball was studied in detail in [54, 45]. For more general domains in $\mathbf{R}^d$ see [55, 51, 34].

Throughout this section we assume that $D \subset \mathbf{R}^d$ is open and *connected*, i.e. it is a domain, and $x_0 \in D$. For $0 < p < \infty$, the classical Hardy space $H^p(D)$ may be defined as the family of all those functions u on D which are harmonic on D (i.e. $u \in C^2(D)$ and $\Delta u(x) = 0$ for $x \in D$) and satisfy

$$\|u\|_{H^p} := \sup_{U \subset\subset D} (\mathbf{E}_{x_0}|u(W_{\tau_U})|^p)^{1/p} < \infty.$$

Here W is the Brownian motion on $\mathbf{R}^d$ and $\tau_U = \inf\{t \geq 0 : W_t \notin D\}$. For a positive harmonic function h on D and $0 < p < \infty$ we consider the space $H_h^p(D)$ of all those functions u harmonic on D which satisfy

$$\|u\|_{H_h^p}^p := \sup_{U \subset\subset D} \mathbf{E}_{x_0}^h \left|\frac{u(W_{\tau_U})}{h(W_{\tau_U})}\right|^p = \frac{1}{h(x_0)} \sup_{U \subset\subset D} \mathbf{E}_{x_0} \frac{|u(W_{\tau_U})|^p}{h(W_{\tau_U})^{p-1}} < \infty,$$

where $\mathbf{E}_x^h$ is the expectation for the conditional Brownian motion (compare Section 3 or see [27]). Let G_D be the classical Green function of D for Δ. If $1 < p < \infty$ and u is harmonic on D, then the following Hardy-Stein identity holds

$$\|u\|_{H^p}^p = |u(x_0)|^p + p(p-1) \int_D G_D(x_0, y)|u(y)|^{p-2}|\nabla u(y)|^2 dy. \tag{28}$$

The identity (28) obtains by taking $h \equiv 1$ in the next theorem. (28) generalizes [54, Lemma 1] and [45, Theorem 4.3], where the formula was given for the ball in $\mathbf{R}^d$, see also [52]. We note that (28) is implicit in [55, Lemma 6], but apparently the identity did not receive enough attention for general domains.

If sharp two-sided estimates of G_D are known, then we obtain explicit estimate for $\|u\|_{H^p}$. For instance, if D is a bounded $C^{1,1}$ domain in $\mathbf{R}^d$ and $d \geq 3$, then $G_D(x_0, y) \asymp \delta_D(y)|y - x_0|^{2-d}$, where $\delta_D(y) := \operatorname{dist}(y, D^c)$, see [57, 59] or [8]. For Lipschitz domains we also refer to [8].

THEOREM 17. *If $1 < p < \infty$ and u is harmonic on D, then*

$$\|u\|_{H_h^p}^p = \frac{|u(x_0)|^p}{h(x_0)^p} + p(p-1)\int_D \frac{G_D(x_0, y)}{h(x_0)h(y)} \left|\frac{u(y)}{h(y)}\right|^{p-2} \left|\nabla \frac{u}{h}(y)\right|^2 h^2(y)dy. \tag{29}$$

The remainder of this section is devoted to the proof of Theorem 17. The reader interested mostly in (28) is encouraged to carry out similar but simpler calculations for $h \equiv 1$ and $p > 2$. We note that (29) is quite more general than (28) because usually u/h is not harmonic. The same remark concerns (31, 32) for general h as opposed to (31, 32) for $h = 1$, which is a classical result ([50, VII.3]). We start with the following well-known Green-type equality. Consider an open set $U \subset\subset D$ and a real-valued function $\phi : \mathbf{R}^d \to \mathbf{R}$ which is C^2 in a neighborhood of $\bar{U}$. Then $\Delta\phi$ is bounded on $\bar{U}$, and for every $x \in D$,

$$\phi(x) = \mathbf{E}_x\phi(W_{\tau_U}) - \int_U G_U(x, y)\Delta\phi(y)dy, \tag{30}$$

see, e.g., [29, p. 133] for the proof.

LEMMA 18. *Let $\varepsilon \neq 0$ and $p > 1$, and let u be harmonic on D. We have*

$$\Delta\left[\left(\frac{u^2}{h^2} + \varepsilon^2\right)^{p/2} h\right] = p\left(\frac{u^2}{h^2} + \varepsilon^2\right)^{(p-4)/2} \left[(p-1)\frac{u^2}{h^2} + \varepsilon^2\right] \left|\nabla \frac{u}{h}\right|^2 h. \tag{31}$$

If $u \neq 0$ or $p \geq 2$, then

$$\Delta\left(\frac{|u|^p}{h^{p-1}}\right) = p(p-1)\left|\frac{u}{h}\right|^{p-2} \left|\nabla \frac{u}{h}\right|^2 h. \tag{32}$$

PROOF. Denote $u_i = \partial u/\partial x_i$, $h_i = \partial h/\partial x_i$, $u_{ii} = \partial^2 u/\partial x_i^2$ and $h_{ii} = \partial^2 h/\partial x_i^2$, $i = 1, \ldots, d$. The lemma results from straightforward calculations based on the following observations:

$$\nabla|u|^p = \nabla(u^2)^{p/2} = p|u|^{p-2}u\nabla u, \qquad \text{if } p \geq 2 \text{ or } u \neq 0,$$

$$\frac{\partial^2}{\partial x_i^2}|u|^p = p(p-1)|u|^{p-2}u_i^2 + p|u|^{p-2}uu_{ii}, \qquad \text{if } p \geq 2 \text{ or } u \neq 0,$$

$$\nabla h^{1-p} = (1-p)h^{-p}\nabla h,$$

$$\Delta(fg) = f\Delta g + 2\nabla f \circ \nabla g + g\Delta f.$$

This yields (32) if $p \geq 2$ or $u(x) \neq 0$ at the point x where the derivatives are calculated (and so $|u|^p h^{1-p}$ is of class C^2 there). To prove (31) we let $\varepsilon \neq 0$, denote $f(x) = u^2/h^2 + \varepsilon^2$, and use a few more identities:

$$\nabla \frac{u}{h} = \frac{\nabla u}{h} - \frac{u \nabla h}{h^2}, \qquad \nabla \left(\frac{u}{h}\right)^2 = 2 \frac{u}{h} \nabla \frac{u}{h},$$

$$\Delta \left(\frac{u}{h}\right)^2 = \frac{2|\nabla u|^2}{h^2} - \frac{8u \nabla u \circ \nabla h}{h^3} + \frac{6u^2 |\nabla h|^2}{h^4},$$

$$\nabla f^{p/2} = \frac{p}{2} f^{p/2-1} \nabla \left(\frac{u}{h}\right)^2,$$

$$\Delta f^{p/2} = \frac{p(p-2)}{4} f^{p/2-2} \left|\nabla \left(\frac{u}{h}\right)^2\right|^2 + \frac{p}{2} f^{p/2-1} \Delta \left(\frac{u}{h}\right)^2,$$

$$\Delta (f^{p/2} h) = \frac{p(p-2)}{4} f^{p/2-2} \left|\nabla \left(\frac{u}{h}\right)^2\right|^2 h + p f^{p/2-1} \left|\nabla \frac{u}{h}\right|^2 h. \qquad \square$$

Noteworthy, we obtained nonnegative expressions in (31) and (32). Also, if $\varepsilon \to 0$, then $\Delta[(u^2/h^2 + \varepsilon^2)^{p/2} h] \to \Delta(|u|^p h^{1-p})$ almost everywhere on D.

Lemma 19. *If u is harmonic on D, $U \subset\subset D$ and $p > 1$, then*

$$\mathbf{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0)|^p}{h(x_0)^{p-1}} + p(p-1) \int_U G_U(x_0, y) \left|\frac{u(y)}{h(y)}\right|^{p-2} \left|\nabla \frac{u}{h}(y)\right|^2 h(y) dy.$$

Proof. For $p \geq 2$ we have $|u|^p h^{1-p} \in C^2(D)$ and the result follows from (30) and Lemma 18. If $1 < p < 2$, then we consider $u + i\varepsilon h$ in place of u and we let $\varepsilon \to 0$. By (31), (30) and dominated convergence we obtain the result. $\square$

Proof (Proof of Theorem 17). The conclusion follows from Lemma 19 and monotone convergence, after dividing by $h(x_0)$ and rearranging the integrand. $\square$

We observe very close similarities between the Hardy-Stein identities and conditional Hardy-Stein identities discussed in this paper. Specifically, functions u and u/h undergo the same transformation under the integral sign. In each case we see the Green function (and jump kernels in the non-local case) appropriate for the given operator, and in the conditional case, $h^2(y)dy$ appears as a natural reference measure. We remark in passing that the framework of conditional semigroups (20) should be convenient for such calculations in more general settings.

5. Further results

We now discuss the structure of $\mathscr{H}^p$. We start with $p = 1$. The following is a counterpart of the theorem of Krickeberg for martingales ([25]), and an extension of [43, Theorem 1], where the result was proved for singular α-harmonic functions on bounded Lipschitz open sets.

Lemma 20. *Let $u \in \mathscr{H}^1$. There exist nonnegative functions f and g which are α-harmonic on D, satisfy $u = f - g$ and uniquely minimize $f(x_0) + g(x_0)$. In fact, $f(x_0) + g(x_0) = \|u\|_1$. If u is singular α-harmonic on D, then so are f and g. If $1 \le p < \infty$ and $u \in \mathscr{H}^p$, then $\|u\|_p^p = \|f\|_p^p + \|g\|_p^p$.*

Proof. Let U_n be open, $U_n \subset\subset U_{n+1}$ for $n = 1, 2\dots$ and $\bigcup_n U_n = D$. Let $\tau_n = \tau_{U_n}$. We have

$$\|u\|_1 = \lim_{n\to\infty} \mathbf{E}_{x_0}|u(X_{\tau_n})| < \infty.$$

Let $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. For $n = 1, 2, \dots,$ we define

$$f_n(x) = \mathbf{E}_x u^+(X_{\tau_n}), \qquad g_n(x) = \mathbf{E}_x u^-(X_{\tau_n}), \qquad x \in \mathbf{R}^d.$$

Obviously, functions f_n and g_n are nonnegative on $\mathbf{R}^d$, and finite and α-harmonic on U_n. We have $u = f_n - g_n$. Since $\tau_n \le \tau_{n+1}$, for every $x \in \mathbf{R}^d$,

$$f_n(x) = \mathbf{E}_x[\mathbf{E}_{X_{\tau_n}} u(X_{\tau_{n+1}}); u(X_{\tau_n}) > 0] \le \mathbf{E}_x[\mathbf{E}_{X_{\tau_n}} u^+(X_{\tau_{n+1}})] = f_{n+1}(x),$$

and $g_n(x) \le g_{n+1}(x)$. We let $f(x) = \lim f_n(x)$ and $g(x) = \lim g_n(x)$. By the monotone convergence theorem, the mean value property (2) holds for f and g. We obtain

$$f(x_0) + g(x_0) = \lim_{n\to\infty} \mathbf{E}_{x_0}|u(X_{\tau_n})| = \|u\|_1 < \infty.$$

In view of Harnack inequality we conclude that f and g are finite, hence α-harmonic on D. Also, $u = f - g$. If u vanishes on D, then so do f and g. For the uniqueness, we observe that if $\tilde{f}, \tilde{g} \ge 0$ are α-harmonic on D, and $u = \tilde{f} - \tilde{g}$, then $-\tilde{g} \le u \le \tilde{f}$, hence $f \le \tilde{f}$ and $g \le \tilde{g}$ by the construction of f and g. Therefore $f(x_0) + g(x_0) \le \tilde{f}(x_0) + \tilde{g}(x_0)$, and equality holds if and only if $f(x_0) = \tilde{f}(x_0)$ and $g(x_0) = \tilde{g}(x_0)$, henceforth $f = \tilde{f}$ and $g = \tilde{g}$.

Let $p > 1$ and suppose that $u \in \mathscr{H}^p \subset \mathscr{H}^1$. By Jensen's inequality,

$$f_n(x)^p \le \mathbf{E}_x u^+(X_{\tau_n})^p, \qquad g_n(x)^p \le \mathbf{E}_x u^-(X_{\tau_n})^p,$$

hence

$$f_n(x)^p + g_n(x)^p \le \mathbf{E}_x|u(X_{\tau_n})|^p.$$

For $m < n$ we have

$$\mathbf{E}_{x_0}(f_n(X_{\tau_m})^p + g_n(X_{\tau_m})^p) \le \mathbf{E}_{x_0}\mathbf{E}_{X_{\tau_m}}|u(X_{\tau_n})|^p = \mathbf{E}_{x_0}|u(X_{\tau_n})|^p.$$

Letting $n \to \infty$, we get

$$\mathbf{E}_{x_0}(f(X_{\tau_m})^p + g(X_{\tau_m})^p) \le \|u\|_p^p.$$

Hence $\|f\|_p^p + \|g\|_p^p \le \|u\|_p^p$. On the other hand, $f, g \ge 0$, hence

$$\begin{aligned}\|u\|_p^p &= \lim_{n\to\infty} \mathbf{E}_{x_0}|f(X_{\tau_n}) - g(X_{\tau_n})|^p \\ &\le \lim_{n\to\infty} \mathbf{E}_{x_0}(f(X_{\tau_n})^p + g(X_{\tau_n})^p) = \|f\|_p^p + \|g\|_p^p.\end{aligned}$$

The proof is complete. □

We note that $\|u\|_p^p = \|f\|_p^p + \|g\|_p^p$ has a trivial analogue for L^p spaces.

Lemma 21. *Let $u \in \mathscr{H}_h^1$. There are nonnegative functions $f, g \in \mathscr{H}_h^1$ which satisfy $u = f - g$ and uniquely minimize $f(x_0) + g(x_0)$. In fact, $f(x_0) + g(x_0) = \|u\|_{\mathscr{H}_h^1} h(x_0)$. If $1 \le p < \infty$ and $u \in \mathscr{H}_h^p$, then $\|u\|_{\mathscr{H}_h^p}^p = \|f\|_{\mathscr{H}_h^p}^p + \|g\|_{\mathscr{H}_h^p}^p$.*

Proof. If $u \in \mathscr{H}_h^1$, then u is singular α-harmonic on $D, u \in \mathscr{H}^1$ and $\|u\|_{\mathscr{H}_h^1} = h(x_0)^{-1}\|u\|_1$ (conditioning is trivial for $p = 1$). By Lemma 20, u has the Krickeberg decomposition $u = f - g$, and f, g are nonnegative and singular α-harmonic on D. In particular $\|f\|_{\mathscr{H}_h^1} = f(x_0)/h(x_0)$ and $\|g\|_{\mathscr{H}_h^1} = g(x_0)/h(x_0)$ are finite. The reader may easily verify the rest of the statement of the lemma, following the previous proof and using the conditional expectation $\mathbf{E}^h$. □

Remark 22. *Analogues of Lemma 20 and Lemma 21 are true for the classical Hardy spaces $H^p(D)$ and $H_h^p(D)$ for connected D.*

As an application of (28) we give a short proof of the following Littlewood-Paley type inequality (see [44], where the result was given for the ball in $\mathbf{R}^2$). Recall the notation $\delta_D(y) = \mathrm{dist}(y, D^c)$.

Proposition 23. *Consider a domain $D \subset \mathbf{R}^d$, and let $p \ge 2$. For every function u harmonic on D we have*

$$\|u\|_{H^p}^p - |u(x_0)|^p \ge p(p-1)d^{2-p}2^{1-p}\int_D G_D(x_0, y)\delta_D(y)^{p-2}|\nabla u(y)|^p dy.$$

Proof. We may assume that $\|u\|_{H^p} < \infty$. In view of Lemma 20 and Remark 22, $u = f - g$, where f, g are positive and harmonic on D and $\|u\|_{H^p}^p = \|f\|_{H^p}^p + \|g\|_{H^p}^p$. Clearly, $|u(x_0)|^p \le f(x_0)^p + g(x_0)^p$, hence

$\|u\|_{H^p}^p - |u(x_0)|^p \geq \|f\|_{H^p}^p - |f(x_0)|^p + \|g\|_{H^p}^p - |g(x_0)|^p$. Furthermore, by Jensen's inequality,

$$|\nabla u|^p \leq 2^{p-1}(|\nabla f|^p + |\nabla g|^p).$$

Recall the following gradient estimate for the nonnegative harmonic function f,

$$f(x) \geq |\nabla f(x)|\delta_D(x)/d, \qquad x \in D,$$

([31, Exercise 2.13], see also [3]). Here d is the dimension. By (28),

$$\begin{aligned} \|f\|_{H^p}^p - |f(x_0)|^p &= p(p-1)\int_D G_D(x_0, y)|f(y)|^{p-2}|\nabla f(y)|^2 dy \\ &\geq p(p-1)d^{2-p}\int_D G_D(x_0, y)\delta_D(y)^{p-2}|\nabla f(y)|^p dy, \end{aligned}$$

and a similar estimate holds for g. □

Acknowledgement

We thank Rodrigo Bañuelos and Jacek Zienkiewicz for comments on classical Hardy spaces and on the present paper. We thank the referee for remarks and suggestions.

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