Hiroshima Math. J.
44 (2014), 157–172

An infinite product associated to a hyperbolic three-holed sphere

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(Received April 8, 2013)
(Revised November 28, 2013)

Abstract. One of the generalizations of McShane's identities by Tan, Wong and Zhang is an identity concerning lengths of simple closed geodesics which pass through two Weierstrass points on a hyperbolic one-holed torus. The Fuchsian groups which uniformize the surface are purely hyperbolic and free of rank two. Another type of Fuchsian groups of the same property is of type $(0,3)$ corresponding to hyperbolic three-holed spheres. In this paper we show a McShane-type identity which holds for all Fuchsian groups of type $(0,3)$.

1. Introduction

The group $PSL(2,\mathbf{R})$ acts on the hyperbolic plane $\mathbf{H} = \{z = x + iy : y > 0\}$ by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto A(z) = \frac{az+b}{cz+d}.$$

It is identified with the group of conformal isometries of $\mathbf{H}$. An element g of $PSL(2,\mathbf{R})$ is *hyperbolic* if its trace satisfies $|\mathrm{tr}\, g| > 2$. A Fuchsian group G of type $(0,3)$ is a purely hyperbolic discrete subgroup of $PSL(2,\mathbf{R})$ such that the factor surface $\mathbf{H}/G$ is a sphere with three holes. The group G is a free group of rank two generated by elements a and b which correspond to two boundary components of $\mathbf{H}/G$ (see Section 3.2.) Each element g is written as a word in the letters of $\Gamma = \{a, a^{-1}, b, b^{-1}\}$:

$$g = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \dots a^{\alpha_r} b^{\beta_r},$$

where α_j and β_j are integers. Since G is free, the exponent sums $n_a(g) = \sum_{j=1}^{r} \alpha_j$, $n_b(g) = \sum_{j=1}^{r} \beta_j$ are well defined.

For an element g of G, let $\mathrm{Cl}(g)$ denote the set of elements of G conjugate to either g or g^{-1}. An element of G is called *primitive* if, along with another group element, it generates G. Let $\mathscr{P}Cl$ denote the set of all classes $\mathrm{Cl}(g)$ of

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This work was supported by JSPS KAKENHI Grant Number 22540191.
2010 *Mathematics Subject Classification*. Primary 30F35; Secondary 20H10, 57M50.
Key words and phrases. Fuchsian groups, McShane's identity.

primitive elements g of G. We denote by $\overline{\mathbf{Q}}$ the set of all rationals to which we include $\infty = 1/0$. Unless otherwise specified we express a rational by p/q in lowest terms with $q > 0$. Let $\varepsilon : \mathscr{P}Cl \to \overline{\mathbf{Q}}$ be the map defined by $\varepsilon(\mathrm{Cl}(g)) = n_b(g)/n_a(g)$. By using the representative system $\{E_{p/q}\}_{p/q \in \overline{\mathbf{Q}}}$ of $\mathscr{P}Cl$ produced by the Gilman-Keen enumeration scheme [3, Theorem 2.1] (see Section 1), we see that ε is bijective. Note that the absolute trace $|\mathrm{tr}\, g|$ depends only on the class $\mathrm{Cl}(g)$ of g. Our main result is to prove a McShane-type identity (see [5] for the original McShane identity):

THEOREM 1. *Let* (a,b) *be a canonical generating pair of a Fuchsian group* G *of type* $(0,3)$. *Let* $D = \mathrm{tr}(aba^{-1}b^{-1}) - 2$. *Then it holds*

$$\prod_{\mathrm{Cl}(g) \in \mathscr{E}_+} \left(\frac{\sqrt{(\mathrm{tr}\, g)^2 - 4} + \sqrt{D}}{\sqrt{(\mathrm{tr}\, g)^2 - 4} - \sqrt{D}} \right)^2 = \prod_{\mathrm{Cl}(g) \in \mathscr{E}_-} \left(\frac{\sqrt{(\mathrm{tr}\, g)^2 - 4} + \sqrt{D}}{\sqrt{(\mathrm{tr}\, g)^2 - 4} - \sqrt{D}} \right)^2$$

$$= \frac{|\mathrm{tr}(ab)| + |\mathrm{tr}(ab^{-1})| + 2\sqrt{D}}{|\mathrm{tr}(ab)| + |\mathrm{tr}(ab^{-1})| - 2\sqrt{D}}, \tag{1}$$

where $\mathscr{E}_\pm$ *is the set of* $\mathrm{Cl}(g) \in \mathscr{P}Cl$ *with* $n_a(g)n_b(g)$ *odd and positive for the plus sign and negative for the minus sign.*

As in the case of other McShane-type identities, (1) is obtained by the following manner. Let $J(a,b)$ denote the shortest segment between the axes $ax(a)$ and $ax(b)$ of a and b. Except for a subset E of linear measure zero, $J(a,b)$ is covered by disjoint subsegments called *gaps*. To each $g \in \mathscr{E}_+$ one gap is associated. Then the length $|J(a,b)|$ of the segment $J(a,b)$ equals the sum of the lengths of gaps, and this equation can be described as (1). The case for $\mathscr{E}_-$ is similar. We include an elementary proof of that the exceptional set E has linear measure zero. The identity (1) is different from the one given in [12, (17)] which also holds for hyperbolic three holed sphere. But there may be some relation between them.

The author thanks Hirotaka Akiyoshi and Makoto Sakuma for many valuable suggestions. He also thanks the referee for many comments and corrections.

2. Rank two free group

2.1. An enumeration scheme by Keen and Gilman. Let $F = \langle a, b \rangle$ be an abstract free group of rank two. Each non-trivial element g of F can be expressed by a unique reduced word in letters of $\Gamma = \{a, a^{-1}, b, b^{-1}\}$:

$$g = x_1 x_2 \dots x_n, \tag{2}$$

where $x_j \in \Gamma$ and $x_j x_{j+1} \neq 1$ for $j = 1, \dots, n-1$. Let $\ell(g) = n$ be the *word length* of g. Let $\overline{\mathbf{R}}$ be the boundary of the hyperbolic plane $\mathbf{H}$ in the Riemann sphere. For two distinct points x and y of $\overline{\mathbf{R}}$, let $L(x, y)$ denote the hyperbolic line between x and y. The decomposition of $\mathbf{H}$ by all $L(p/q, r/s)$ with rationals p/q and r/s satisfying $|ps - rq| = 1$ is called the *Farey tessellation*. It is an ideal triangulation of $\mathbf{H}$. We introduce the Gilman-Keen enumeration scheme for a representative system of $\mathscr{P}Cl$ given in [3]. We set

$$E_{0/1} = a, \qquad E_{1/0} = b.$$

For other positive rationals p/q, let m/n and r/s be the positive rationals such that

$$\frac{m}{n} < \frac{p}{q} = \frac{m+r}{n+s}\left(= \frac{m}{n} \oplus \frac{r}{s}\right) < \frac{r}{s} \qquad \text{and} \qquad ms - rn = -1,$$

where $\oplus$ means the Farey sum. We define

$$E_{p/q} = \begin{cases} E_{r/s}E_{m/n} & \text{if } pq \text{ is odd,} \\ E_{m/n}E_{r/s} & \text{if } pq \text{ is even.} \end{cases} \tag{3}$$

We see by induction on $p + q$ that $E_{p/q}$ is a primitive word in $\{a, b\}$, $\ell(E_{p/q}) = p + q$, $n_b(E_{p/q}) = p$ and $n_a(E_{p/q}) = q$, and m/n, p/q and r/s are vertices of a Farey triangle. We call m/n and r/s the *parents* of p/q. For negative p/q, define $E_{p/q}$ by replacing b in the word for $E_{-p/q}$ by b^{-1}. Theorems 2.1 and 2.2 in [3] state

THEOREM 2. *If g is a primitive element in G, then either g or g^{-1} is conjugate to a unique element in $\{E_{p/q}\}_{p/q \in \overline{\mathbf{Q}}}$. If p/q and r/s are such that $|ps - rq| = 1$, then $(E_{p/q}, E_{r/s})$ is a generating pair of F.*

From this theorem, each class $\mathrm{Cl}(g)$ in $\mathscr{P}Cl$ contains a unique $E_{p/q}$. We remark that the enumeration scheme is slightly modified in this paper; The word for $E_{p/q}$ is different from the one in [3], but only in that b is replaced by b^{-1}.

2.2. A family tree of rationals. Let (g, h) be a pair of elements in F. For $n = 1, 2, \dots$ we define

$$\sigma_n(g, h) = (g(hg)^{n-1}, g(hg)^n), \qquad \sigma_{-n}(g, h) = ((hg)^n h, (hg)^{n-1} h).$$

Let $\mathbf{Z}^*$ be the set of non-zero integers. Let $\mathscr{P}_0 = \{(a, b)\}$ and define inductively $\mathscr{P}_k$ to be the collection of $\{\sigma_n(g, h)\}_{n \in \mathbf{Z}^*}$ for all $(g, h) \in \mathscr{P}_{k-1}$, and let

$$\mathscr{P} = \bigcup_{k=0}^{\infty} \mathscr{P}_k.$$

If $(g_1, h_1) = \sigma_n(g, h)$ for some $n \in \mathbf{Z}^*$, then the commutator $g_1 h_1 g_1^{-1} h_1^{-1}$ is conjugate to $ghg^{-1}h^{-1}$. Hence

$$\mathrm{tr}(ghg^{-1}h^{-1}) = \mathrm{tr}(aba^{-1}b^{-1}) \qquad \text{for all } (g,h) \in \mathscr{P}. \tag{4}$$

If p/q and r/s are positive rationals, then $ps - qr = -1$ means $p/q < r/s$. Let $\mathscr{Q} = \{(E_{p/q}, E_{r/s}) : ps - rq = -1, pq \equiv rs \equiv 0 \bmod 2\}$. We will show that $\mathscr{P} = \mathscr{Q}$. If $(E_{p/q}, E_{r/s}) \in \mathscr{Q}$, then $(p+r)(q+s)$ must be odd. The Gilman-Keen scheme yields

$$E_{(p+r)/(q+s)} = E_{r/s} E_{p/q}.$$

Let

$$A = \begin{pmatrix} -pq - rs - 2ps & (p+r)^2 \\ -(q+s)^2 & pq + rs + 2qr \end{pmatrix} \in SL(2, \mathbf{R}). \tag{5}$$

As a Möbius transformation, A fixes $(p+r)/(q+s)$, sends r/s to p/q, and

$$A^n(p/q) = A^{n+1}(r/s) = \frac{(n+1)p + nr}{(n+1)q + ns}, \qquad A^{-n}(r/s) = A^{-n-1}(p/q) = \frac{np + (n+1)r}{nq + (n+1)s}$$

for $n = 1, 2, \dots$. Since $((n+1)p + nr)((n+1)q + ns) \equiv n(n+1)(ps + qr) \equiv 0 \bmod 2$ for all integers n, the Gilman-Keen scheme and induction on $n = 1, 2, \dots$ yield

$$\begin{aligned} E_{((n+1)p+nr)/((n+1)q+ns)} &= E_{(np+(n-1)r)/(nq+(n-1)s)} E_{(p+q)/(r+s)} \\ &= E_{(np+(n-1)r)/(nq+(n-1)s)} (E_{r/s} E_{p/q}) = E_{p/q} (E_{r/s} E_{p/q})^n, \\ E_{(np+(n+1)r)/(nq+(n+1)s)} &= E_{(p+r)/(q+s)} E_{((n-1)p+nr)/((n-1)q+ns)} \\ &= (E_{r/s} E_{p/q}) E_{((n-1)p+nr)/((n-1)q+ns)} = (E_{r/s} E_{p/q})^n E_{r/s}, \end{aligned}$$

and hence

$$\sigma_n(E_{p/q}, E_{r/s}) = (E_{A^n(r/s)}, E_{A^n(p/q)}), \tag{6}$$

for $n \in \mathbf{Z}^*$. Since

$$\begin{vmatrix} np + (n-1)r & (n+1)p + nr \\ nq + (n-1)s & (n+1)q + ns \end{vmatrix} = \begin{vmatrix} np + (n+1)r & (n-1)p + nr \\ nq + (n+1)s & (n-1)q + ns \end{vmatrix}$$

equals $ps - qr = -1$, $\sigma_n(E_{p/q}, E_{r/s}) \in \mathscr{Q}$ for all $n \in \mathbf{Z}^*$. Starting with $(a, b) = (E_{0/1}, E_{1/0}) \in \mathscr{P} \cap \mathscr{Q}$ we can show that $\mathscr{P} \subset \mathscr{Q}$.

For each $(E_{p/q}, E_{r/s}) \in \mathscr{Q} - \{(a, b)\}$, choose the positive rational t/u outside the interval $(p/q, r/s)$ so that t/u, p/q and r/s are vertices of a Farey triangle. We assume, say, that $t/u < p/q$. Then $t/u = (p-r)/(q-s)$. Let $A \in SL(2, \mathbf{R})$ be the parabolic transformation which fixes t/u and sends r/s

to p/q:

$$A = \begin{pmatrix} pq + rs - 2ps & -(p-r)^2 \\ (q-s)^2 & -pq - rs + 2qr \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2. \quad (7)$$

Let n be the integer such that $p_1/q_1 = A^n(r/s) < t/u < r_1/s_1 = A^n(p/q)$. Since $A \in SL(2, \mathbf{Z})$, A preserves the Farey tessellation. Hence $t/u = (p_1 + r_1)/(q_1 + s_1)$ and $p_1 s_1 - q_1 r_1 = -1$. (Therefore A is also of the form (5) with p, q, r and s replaced by p_1, q_1, r_1 and s_1, respectively.) From (7) $p_1 q_1$ and $r_1 s_1$ are even. Thus $(E_{p_1/q_1}, E_{r_1/s_1}) \in \mathcal{Q}$. By using A as in (6) we see that $(E_{p/q}.E_{r/s}) = \sigma_{-n}(E_{p_1/q_1}, E_{r_1/s_1})$. We define an operation τ on $\{(p/q, r/s) : ps - rq = -1, pq \equiv rs \equiv 0 \mod 2\}$ by $(p_1/q_1, r_1/s_1) = \tau(p/q, r/s)$ if $(p/q, r/s) \neq (0/1, 1/0)$ and $\tau(0/1, 1/0) = (0/1, 1/0)$. When $r/s < t/u$ we can find in the same way $(p_1/q_1, r_1/s_1) = \tau(p/q, r/s)$ such that $(E_{p/q}, E_{r/s}) = \sigma_n(E_{p_1/q_1}, E_{r_1/s_1})$ for some $n \in \mathbf{Z}^*$. By a finite number of operations τ, we reach $(a, b) = (E_{0/1}, E_{1/0})$ from $(E_{p/q}, E_{r/s})$. Thus $(E_{p/q}, E_{r/s}) \in \mathcal{P}$. Now we conclude $\mathcal{P} = \mathcal{Q}$.

Lemma 1. *The two sets* $\{E_{m/n} : m/n > 0$ *and* mn *is odd*$\}$ *and* $\{hg : (g, h) \in \mathcal{P}\}$ *are the same.*

Proof. If $m/n > 0$ is such that mn is odd, let p/q and r/s be the parents of m/n with $p/q < m/n < r/s$. Then $(E_{p/q}, E_{r/s}) \in \mathcal{Q} = \mathcal{P}$ and the Gilman-Keen scheme yields $E_{m/n} = E_{r/s} E_{p/q}$. This concludes the lemma. □

3. Fuchsian groups of type $(0, 3)$

3.1. A pair of hyperbolic elements in $SL(2, \mathbf{R})$. For a hyperbolic element A, let p_A and q_A denote the repelling and attracting fixed points of A, respectively. The *axis* $ax(A)$ of A is the hyperbolic line $L(p_A, q_A)$ and the *extended axis* is $ax(A) \cup \{p_A, q_A\}$. We will find hyperbolic X and $Y \in SL(2, \mathbf{R})$ with the following properties:

(1) The extended axes of X and Y are disjoint in $\overline{\mathbf{H}}$.
(2) The common orthogonal L_1 of $ax(X)$ and $ax(Y)$ separates the pair $\{q_X, q_Y\}$ from $\{p_X, p_Y\}$.
(3) $x = \operatorname{tr} X > 2$, $y = \operatorname{tr} Y > 2$ and $z = \operatorname{tr} XY$.

If L_1 is the imaginary axis and $q_X = -p_X = 1 < q_Y$, then there is a unique pair $\{X, Y\}$ of such matrices. This pair consists of

$$X = \begin{pmatrix} \dfrac{x}{2} & \dfrac{\sqrt{x^2-4}}{2} \\ \dfrac{\sqrt{x^2-4}}{2} & \dfrac{x}{2} \end{pmatrix}, \quad Y = \begin{pmatrix} \dfrac{y}{2} & \dfrac{2z - xy + 2\sqrt{D}}{2\sqrt{x^2-4}} \\ \dfrac{2z - xy - 2\sqrt{D}}{2\sqrt{x^2-4}} & \dfrac{y}{2} \end{pmatrix}, \quad (8)$$

where

$$D = x^2 + y^2 + z^2 - xyz - 4 = \mathrm{tr}\, XYX^{-1}Y^{-1} - 2.$$

Since $0 < 1/q_Y < 1 < q_Y$ we must have $2z - xy - 2\sqrt{D} > 0$ and hence

$$q_Y = \frac{\sqrt{(x^2-4)(y^2-4)}}{2z - xy - 2\sqrt{D}} = \frac{2z - xy + 2\sqrt{D}}{\sqrt{(x^2-4)(y^2-4)}} = \sqrt{\frac{2z - xy + 2\sqrt{D}}{2z - xy - 2\sqrt{D}}}. \tag{9}$$

Therefore, if X and Y are matrices as above, then they necessarily satisfy $D > 0$ and

$$2z - xy + 2\sqrt{D} > \sqrt{(x^2-4)(y^2-4)} > 2z - xy - 2\sqrt{D} > 0.$$

By adding the first term and the third term, we have $z > (2z - xy)/2 > 0$ and hence from the definition of D,

$$z = \frac{xy + \sqrt{(x^2-4)(y^2-4) + 4D}}{2}. \tag{10}$$

By using the matrices (8) we have

$$q_{XY} = \frac{\sqrt{x^2-4}(\sqrt{z^2-4} + \sqrt{D})}{2y - xz + x\sqrt{D}}, \qquad q_{YX} = \frac{\sqrt{x^2-4}(\sqrt{z^2-4} - \sqrt{D})}{2y - xz + x\sqrt{D}}. \tag{11}$$

3.2. Canonical generators. Let p_1, q_1, p_2, q_2, p_3 and q_3 be distinct points lying on $\bar{\mathbf{R}}$ in this order. Then $L_1 = L(p_1, q_1)$, $L_2 = L(p_2, q_2)$ and $L_3 = L(p_3, q_3)$ are disjoint hyperbolic lines none of which separates the other two. Let r_k denote the refection in the line L_k for $k = 1, 2, 3$. Then $a = r_2 r_1$ and $b = r_1 r_3$ give a canonical generating pair of a Fuchsian group G of type $(3, 0)$. We identify G with the free group $F = \langle a, b \rangle$. If we assume that $p_1 = \infty$, $q_1 = 0 < p_2$ and $p_2 q_2 = 1$, then $a = X$ and $b = Y^{-1}$ (see (8)) for some $x > 2$, $y > 2$ and z. The configuration of L_1, L_2 and L_3 implies that the axes of a and b are orthogonal to L_1 and

$$q_b < q_{ba} < p_{ba} < p_a = -1 < 0 < q_a = 1 < q_{ab} < p_{ab} < p_b = -q_b.$$

Let $w = \mathrm{tr}(ab) = \mathrm{tr}\, XY^{-1}$. Then $w = xy - z$ and $\sqrt{x^2-4}(\sqrt{w^2-4} + \sqrt{D})/(-2y + xw + x\sqrt{D})$ is a fixed point of $ab = XY^{-1}$. Thus we have $-2y + xw + x\sqrt{D} > 0$ and since $0 < q_{ab} < p_{ab}$,

$$q_{ab} = \frac{\sqrt{x^2-4}(\sqrt{D} - \sqrt{w^2-4})}{-2y + xw + x\sqrt{D}}, \qquad p_{ab} = \frac{\sqrt{x^2-4}(\sqrt{D} + \sqrt{w^2-4})}{-2y + xw + x\sqrt{D}}.$$

Since $x = \mathrm{tr}\, a > 0$ and $y = \mathrm{tr}\, b > 0$, $w = xy - z = \mathrm{tr}(ab) < -2$ (see [13, Lemma 33.4]).

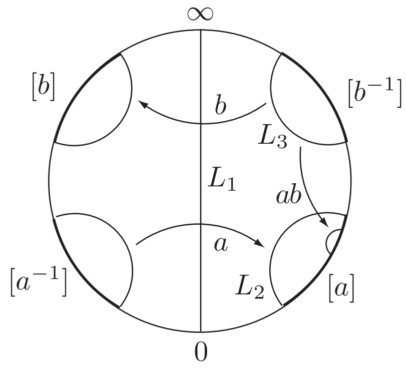

Fig. 1. Canonial generators

3.3. Palindrome words in the group. Let g be an element of G and $g = e_1 e_2 \dots e_r$ be the expression by a reduced word in $\{a, a^{-1}, b, b^{-1}\}$. Then the axis $ax(g)$ is orthogonal to L_1 if and only if g is a palindrome, that is, $e_i = e_{r+1-i}$, $i = 1, 2, \dots, r$. To see this, we assume that L_1 is the imaginary axis. Then a hyperbolic element $A \in SL(2, \mathbf{R})$ has an axis orthogonal to L_1 if and only if $A^* = A$, where

$$A^* = \begin{pmatrix} s & q \\ r & p \end{pmatrix} \qquad \text{for } A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

If $e \in \{a, a^{-1}, b, b^{-1}\}$, then $e^* = e$. Now $ax(g)$ is orthogonal to L_1 if and only if $g = e_1 e_2 \dots e_r$ is a palindrome, because

$$(e_1 e_2 \dots e_r)^* = e_r^* \dots e_2^* e_1^* = e_r \dots e_2 e_1.$$

This fact has interesting applications. See, for example, [4] and [10].

If (g, h) is a pair in $G = F$, then $g(hg)^n$ and $(hg)^n h$ are palindrome in $\{g, h\}$ for $n = 1, 2, \dots$. By induction on the index k of $\mathscr{P}_k$ (see Section 2.2), we see that if g belongs to a pair in $\mathscr{P}$, then g is palindrome as a word in a and b.

4. Proof of the identity

In what follows, we consider $G = \langle a, b \rangle$ as in Figure 1. We define the intervals $[a] = (p_2, q_2)$, $[b^{-1}] = (p_3, q_3)$, $[a^{-1}] = r_1[a]$ and $[b] = r_1[b^{-1}]$, where r_1 is the reflection in the imaginary axis L_1. Then a sends $\overline{\mathbf{R}} \backslash [a^{-1}]$ onto $[a]$ and b sends $\overline{\mathbf{R}} \backslash [b^{-1}]$ onto $[b]$. For a reduced word $W = e_1 e_2 \dots e_r e_{r+1}$ of letters of $\Gamma = \{a, a^{-1}, b, b^{-1}\}$, let $[W] = e_1 \dots e_r [e_{r+1}]$. If $e_{r+1} = a$, since W is orientation-preserving, the subintervals $[Wa]$, $[Wb]$ and $[Wb^{-1}]$ of $[W]$ are located in this order. Similar results hold when e_{r+1} is one of other letters (see (ii) in Lemma 2 below). If $W \in G$ is a non-trivial cyclically reduced word, then $\{[W^n]\}_{n=1}^{\infty}$ is a decreasing sequence of intervals and their limit is the attracting fixed point q_W of W. This observation leads to the following lemma. (See also [2].)

Lemma 2. *Let W_1 and W_2 be cyclically reduced words for non-trivial elements in G with disjoint axes. Let n and m be positive integers such that $\ell(W_1^m) \geq \ell(W_2)$ and $\ell(W_2^n) \geq \ell(W_1)$. Let $W_1^m = e_1 e_2 \dots$ and $W_2^n = f_1 f_2 \dots$. Then $q_{W_1} < q_{W_2}$ if and only if either*

(i) *(e_1, f_1) equals (b, a^{-1}), (b, a), (b, b^{-1}), (a^{-1}, a), (a^{-1}, b^{-1}) or (a, b^{-1}), or*

(ii) *There is a positive integer r such that $e_i = f_i$ for each $i = 1, 2, \dots, r$ and either*

  (a) *$e_r = a$ and $(e_{r+1}, f_{r+1}) = (a, b^{-1})$, (a, b) or (b^{-1}, b), or*

  (b) *$e_r = b^{-1}$ and $(e_{r+1}, f_{r+1}) = (a^{-1}, a)$, (a^{-1}, b^{-1}) or (a, b^{-1}), or*

  (c) *$e_r = a^{-1}$ and $(e_{r+1}, f_{r+1}) = (b^{-1}, b)$, (b^{-1}, a^{-1}) or (b, a^{-1}), or*

  (d) *$e_r = b$ and $(e_{r+1}, f_{r+1}) = (b, a^{-1})$, (b, a) or (a^{-1}, a).*

We prove (1) for the product over $\mathscr{E}_+$. If $g \in G$ is palindrome, then the axis $ax(g)$ is orthogonal to the imaginary axis L_1 and hence $p_g = -q_g$.

Lemma 3. *If $(g, h) \in \mathscr{P}(a, b)$, then for $n = 1, 2, \dots,$*

$$|q_{(gh)^{n-1}g}| < |q_{(gh)^n g}| < |q_{(hg)^n h}| < |q_{(hg)^{n-1}h}|. \tag{12}$$

Proof. For the pair (a, b), (12) follows from Lemma 2 and that $(ab)^n a$ is palindrome:

$$q_a < q_{aba} < q_{(ab)^{n-1}b} < q_{(ab)^n b} < p_{(ba)^n a} < p_{(ba)^{n-1}b} < p_b.$$

for $n = 1, 2, \dots$ and $p_{(ba)^n b} = -q_{(ba)^n b}$. Let $(g, h) \in \mathscr{P}(a, b) - \{(a, b)\}$. Note that the first letters in the words of g and h are the same. We consider the case where a is the first letter. In this case q_g and q_h belong to the interval $[a]$. We assume that $q_g < q_h$. We remark that the inequality $q_g < q_h$ is true for $(g, h) = ((ab)^{n-1}a, (ab)^n a)$ for $n \geq 1$. If $\ell(g) \leq \ell(h)$, then let the reduced word expressions for g and h be $g = e_1 \dots e_p e_{p+1} \dots e_q$ and $h = g^{\nu-1} e_1 \dots e_p f_{p+1} \dots f_r$ for some $\nu \geq 1$ where $e_{p+1} \neq f_{p+1}$. From Lemma 2 (e_p, e_{p+1}, f_{p+1}) is either (a, a, b) or (b, b, a). If $n = 1$, then

$$g^{\nu+1} = g^{\nu} e_1 \dots e_p e_{p+1} W_1, \qquad ghg = g^{\nu} e_1 \dots e_p f_{p+1} W_2,$$

$$hgh = hg^{\nu-1} e_1 \dots e_p e_{p+1} W_3, \qquad h^2 = hg^{\nu-1} e_1 \dots e_p f_{p+1} W_4,$$

for some words W_j, $j = 1, 2, 3, 4$, and hence by Lemma 2 $q_g < q_{ghg}$ and $q_{hgh} < g_h$. If $n \geq 2$, then

$$((gh)^{n-1}g)^2 = (gh)^{n-1}g^v e_1 \dots e_p e_{p+1} W_1, \quad (gh)^n g = (gh)^{n-1} g^v e_1 \dots e_p f_{p+1} W_2,$$

$$(hg)^n h = (hg)^{n-1} h e_1 \dots e_p e_{p+1} W_3, \qquad ((hg)^{n-1}h)^2 = (hg)^{n-1} h e_1 \dots e_p f_{p+1} W_4,$$

with some words W_j for $j = 1, 2, 3, 4$. Then Lemma 2 yields

$$0 < q_{(gh)^{n-1}g} < q_{(gh)^n g} < q_{(hg)^n h} < q_{(hg)^{n-1}h}.$$

The case of $\ell(g) > \ell(h)$ can be treated in a similar way. We also proved that the property $q_g < q_h$ holds if the pair (g, h) is replaced by $\sigma_n(g, h)$ for all $n \in \mathbf{Z}^*$. By induction on the index k of $\mathscr{P}_k$ we conclude (12) when the reduced words for g and h start with a. The other case proceeds in a similar way. □

For $(g, h) \in \mathscr{P}(a, b)$, let $J(g, h)$ be the interval on the imaginary axis L_1 between the axes of g and h. We compute the hyperbolic length $|J(g, h)|$ of $J(g, h)$. To this end, by taking their conjugates in $SL(2, \mathbf{R})$ we may assume that g and h are the matrices X and Y in (8), respectively, if $(g, h) \neq (a, b)$. For (a, b), q_a and $q_{b^{-1}}$ are in the same side of L_1. So we let $a = X$ and $b = Y^{-1}$. Since $ax(g)$ and $ax(h)$ are orthogonal to L_1 and $q_g = 1$, $|J(g, h)| = \log(|q_h|/|q_g|) = \log|q_h|$. We may assume that $x = \operatorname{tr} g$, $y = \operatorname{tr} h$ are positive. If $(g, h) \neq (a, b)$, then $z = \operatorname{tr} gh > 0$ and $xy - z = \operatorname{tr} gh^{-1} < 0$, and if $(g, h) = (a, b)$, $z = \operatorname{tr} ab^{-1} > 0$, and $xy - z = \operatorname{tr} ab < 0$. From (9) and (4)

$$|J(g, h)| = \frac{1}{2} \log\left(\frac{2z - xy + 2\sqrt{D}}{2z - xy - 2\sqrt{D}}\right) = \frac{1}{2} \log\left(\frac{|\operatorname{tr} gh| + |\operatorname{tr} gh^{-1}| + 2\sqrt{D}}{|\operatorname{tr} gh| + |\operatorname{tr} gh^{-1}| - 2\sqrt{D}}\right). \tag{13}$$

We remark that $\exp 2|J(a, b)|$ is the right hand side of (1). Since $(gh)^n g$ and $(hg)^n h$ are palindrome when expressed by words in $\{a, b\}$, their axes cut L_1 orhogonally. By Lemma 3, the axes $ax((gh)^{n-1}g)$ and $ax((gh)^n g)$ bound the interval $J(\sigma_n(g, h))$, and $ax((hg)^n h)$ and $ax((hg)^{n-1}h)$ bound $J(\sigma_{-n}(g, h))$. Therefore these axes decompose $J(g, h)$ into the subintervals $J(\sigma_n(g, h))$, $n \in \mathbf{Z}^*$, and an interval $I(g, h)$ called a *gap* for (g, h). Since

$$\lim_{n \to \infty} q_{(gh)^n g} = q_{gh}, \qquad \lim_{n \to \infty} q_{(hg)^n h} = q_{hg},$$

the gap $I(g, h)$ is the interval between $|q_{gh}|\sqrt{-1}$ and $|q_{hg}|\sqrt{-1}$. Again we use the matrices in (8) and set $(g, h) = (X, Y)$ for $(g, h) \neq (a, b)$ and $(a, b) = (X, Y^{-1})$. Then from (11) (for (a, b) we use also (8) to compute q_{ab} and q_{ba}) its hyperbolic length is

$$|I(g,h)| = \begin{cases} \log\left(\dfrac{\sqrt{(\operatorname{tr} gh)^2 - 4} + \sqrt{D}}{\sqrt{(\operatorname{tr} gh)^2 - 4} - \sqrt{D}}\right) & \text{if } (g,h) \neq (a,b) \\ \log\left(\dfrac{\sqrt{D} + \sqrt{(\operatorname{tr} ab)^2 - 4}}{\sqrt{D} - \sqrt{(\operatorname{tr} ab)^2 - 4}}\right) & \text{if } (g,h) = (a,b). \end{cases} \tag{14}$$

We define $E_1 = J(a,b) - I(a,b)$ and define inductively

$$\begin{aligned} E_{n+1} = E_n - \bigcup_{(g,h)\in\mathscr{P}_n(a,b)} I(g,h) &= \bigcup_{(g,h)\in\mathscr{P}_n(a,b)} \left(\bigcup_{k\in\mathbf{Z}^*} J(\sigma_k(g,h))\right) \\ &= \bigcup_{(g,h)\in\mathscr{P}_{n+1}(a,b)} J(g,h). \end{aligned}$$

Then $E_{n+1} \subset E_n$ and

$$J(a,b) - \bigcup_{(g,h)\in\mathscr{P}(a,b)} I(g,h) = \bigcap_{n=1}^{\infty} E_n.$$

Let $|\cdot|$ denote also the linear measure on L_1. If we show that

$$\lim_{n\to\infty} |E_n| = \left|\bigcap_{n=1}^{\infty} E_n\right| = 0, \tag{15}$$

then, since two distinct gaps are disjoint,

$$|J(a,b)| = \sum_{(g,h)\in\mathscr{P}(a,b)} |I(g,h)|,$$

or

$$\begin{aligned} &\frac{1}{2}\log\left(\frac{|\operatorname{tr}(ab)| + |\operatorname{tr}(ab^{-1})| + 2\sqrt{D}}{|\operatorname{tr}(ab)| + |\operatorname{tr}(ab^{-1})| - 2\sqrt{D}}\right) \\ &\quad = \log\left(\frac{\sqrt{D} + \sqrt{(\operatorname{tr} ab)^2 - 4}}{\sqrt{D} - \sqrt{(\operatorname{tr} ab)^2 - 4}}\right) + \sum_{(g,h)\in\mathscr{P}(a,b)-\{(a,b)\}} \log\left(\frac{\sqrt{(\operatorname{tr} gh)^2 - 4} + \sqrt{D}}{\sqrt{(\operatorname{tr} gh)^2 - 4} - \sqrt{D}}\right) \end{aligned}$$

and from this and Lemma 1 follows the desired product (1) for the plus sign. Although (15) follows from a general theorem, we will give an elementary proof. We need

Lemma 4. *There exists a positive constant $c < 1$ satisfying for all $(g,h) \in \mathscr{P}(a,b)$*

$$|I(g,h)| > c|J(g,h)|. \tag{16}$$

If this lemma is true, then (15) follows, because

$$\begin{aligned} |E_{n+1}| &= \sum_{(g,n)\in\mathscr{P}_n} (|J(g,h)| - |I(g,h)|) \\ &< (1-c) \sum_{(g,n)\in\mathscr{P}_n} |J(g,h)| \\ &= (1-c)|E_n| < (1-c)^n |E_1|. \end{aligned}$$

Proof of Lemma 4. Since G is discrete, the sequence $\{|\mathrm{tr}\, g| : g \in G\}$ has no accumulation points in $\mathbf{R}$ (see, for example, [1, Section 2.2]). Moreover G is purely hyperbolic group. Therefore there exists a positive constant c_0 such that $|(\mathrm{tr}\, g)|^2 > c_0 > 4$ for all $g \in G - \{1\}$, and for any positive constant $m > 2$

$$\max\{|\mathrm{tr}\, g|, |\mathrm{tr}\, h|\} > m \tag{17}$$

for all $(g,h) \in \mathscr{P}(a,b)$ but a finite number of pairs. Let $(g,h) \in \mathscr{P}(a,b) - \{(a,b)\}$. Let $x = \mathrm{tr}\, g$, $y = \mathrm{tr}\, h$. We may assume that $x, y > 2$ and since $(g,h) \neq (a,b)$, by taking conjugates in $SL(2,\mathbf{R})$, assume also that $g = X$ and $h = Y$ with X and Y as in (8). If $z = \mathrm{tr}\, gh$, then (10) shows $z > m$ if (17) holds. Since

$$\lim_{x\to+0} \frac{\log\left(\frac{1+x}{1-x}\right)}{2x} = 1,$$

if a constant c_1 with $0 < c_1 < 1$ is fixed, then from (13), (14) and (17)

$$\begin{aligned} \frac{|I(g,h)|}{|J(g,h)|} &> c_1 \frac{2z - xy}{\sqrt{z^2-4}} = c_1 \frac{\sqrt{(x^2-4)(y^2-4)+4D}}{\sqrt{z^2-4}} \\ &> c_1(1 - 4/c_0)\frac{xy}{z} \end{aligned}$$

except for a finite number of pairs (g,h) in $\mathscr{P}(a,b)$. Again except for a finite number of pairs (g,h), $4D < x^2y^2$ and hence

$$z = xy \frac{\sqrt{(1-4/x^2)(1-4/y^2) + 4D/(x^2y^2)}}{2} < xy.$$

Therefore, except for pairs in a finite subset $\mathscr{E}$ of $\mathscr{P}(a,b)$

$$\frac{|I(g,h)|}{|J(g,h)|} > c_1(1 - 4/c_0).$$

Any constant c will do if c satisfies $c < c_1(1 - 4/c_0)$ and

$$0 < c < \min\{|I(g,h)|/|J(g,h)| : (g,h) \in \mathscr{E}\}. \qquad \square$$

The proof of (1) for the negative sign is similar.

5. A characterization of generating pairs

Characterizations of primitive elements in the free group F of rank two are found in many literatures, see, for example, [3], [8] and [9]. In this section we introduce a simple method of finding generating pairs of F. This method is given in [8], but we consider it with its relation to palindrome words. Let $w = e_1e_2\dots e_n$ be a word of $\Gamma = \{a, a^{-1}, b, b^{-1}\}$. We denote by $M(w)$ the central subword of length one if n is odd or of length 2 if n is even. Thus $M(w) = e_{(n+1)/2}$ if n is odd and $M(w) = e_{n/2}e_{(n+2)/2}$ if n is even. Let $T(w)$ be the word w with $M(w)$ removed. Thus

$$T(w) = \begin{cases} e_1\dots e_{(n-1)/2}e_{(n+3)/2}\dots e_n & \text{if } n \text{ is odd,} \\ e_1\dots e_{n/2-1}e_{n/2+2}\dots e_n & \text{if } n \text{ is even.} \end{cases}$$

We consider the following conditions for a pair (V, W) of elements in F: If V and W are represented by reduced words, then 1) Their lengths $p = \ell(V)$ and $q = \ell(W)$ are relatively prime. 2) No cancellation occur in VW and hence $\ell(VW) = p + q$. 3) $T(V)$, $T(W)$ and $T(VW)$ are palindrome. 4) From (1) and (2) there is a unique element of even length among V, W and VW. If U is such an element, $M(U)$ consists of two different letters.

THEOREM 3. *If (V, W) satisfies (1)–(4) above, then it is a generating pair of F.*

The theorem is clear if $p = q = 1$. We prove the theorem by induction on $p + q$. Let

$$V = e_1e_2\dots e_p, \qquad W = e_{p+1}e_{p+2}\dots e_{p+q},$$

be the reduced words for V and W, where $e_j \in \Gamma$, $j = 1, \dots, p+q$. We define order-reversing substitution of indices of the first p letters and that of the last q letters:

$$\varphi_1(j) = \begin{cases} -j + p + 1 & \text{for } j = 1, \dots, p \\ -j + 2p + q + 1 & \text{for } j = p+1, \dots, p+q \end{cases}$$

and the order-reversing substitution on all indices:

$$\varphi_2(j) = -j + p + q + 1.$$

Case 1*: pq is odd.* In this case V and W are palindrome. By conditions (3) and (4) $e_j = e_{\varphi_1(j)}$ for all j and $e_j = e_{\varphi_2(j)}$ if and only if $j < (p+q)/2$ or $j > (p+q)/2+1$, and $M(VW)$ is a product of two different letters. Without loss of generality we assume that $M(VW) = e_{(p+q)/2}e_{(p+q)/2+1} = ab$. Let

$$\varphi(j) = \varphi_2 \circ \varphi_1(j) \equiv j+q \equiv j-p \qquad \text{mod } p+q.$$

We consider the indices modulo $p+q$. Then

$$\begin{aligned} e_{\varphi(j)} \neq e_j \qquad & \text{if and only if } \varphi(j) = \frac{p+q}{2} \quad \text{or} \quad \varphi(j) = \frac{p+q}{2}+1 \\ & \text{if and only if } \varphi_1(j) = \frac{p+q}{2} \quad \text{or} \quad \varphi_1(j) = \frac{p+q}{2}+1 \\ & \text{if and only if } j = \frac{3p+q}{2} \quad \text{or} \quad j = \frac{3p+q}{2}+1 \end{aligned}$$

Since p and q are relatively prime, φ acts transitively on $\{1,2,\dots,p+q\}$. Hence V and W are words containing no a^{-1}'s and b^{-1}'s. Let $x = n_a(V)$ and $y = n_a(W)$. The φ-orbit of indices shows

$$e_{(p+q)/2}, e_{(p+q)/2+q}, \dots, e_{(p+q)/2+(x+y-1)q} \text{ are the letter } a$$

$$e_{(p+q)/2+1}, e_{(p+q)/2+1+q}, \dots, e_{(p+q)/2+(p+q-1)q} \text{ are the letter } b$$

Here $e_{(p+q)/2+1} = e_{(p+q)/2+(x+y)q}$, and hence $(x+y)q \equiv 1 \mod p+q$. Since

$$\frac{p+q}{2} + (x+y-1)q \equiv \frac{3p+q}{2}+1 \qquad \text{mod } p+q,$$

we have

$$(*) \qquad -1 \equiv p+(1-x-y)q \equiv yp-qx \qquad \text{mod } p+q.$$

We consider only the case $p<q$, because the other case is treated in a similar way just by reversing the order of the indices. We shall show

(i) $e_1 \dots e_p = e_{p+1} \dots e_{2p}$.

(ii) $T(e_{2p+1} \dots e_{p+q})$ is palindrome.

(iii) $e_{2p+(q-p)/2}e_{2p+(q-p)/2+1} = ba$.

Suppose that these are true. Let $U = e_{2p+1} \dots e_{p+q}$. By (i) $W = VU$ and (V,U) satisfies conditions (1)–(4). By induction, (V,U) and hence (V,W) are generating pairs of F.

Proof of (i). Since $2p < (3p+q)/2$, $e_{p+j} = e_{\varphi(p+j)} = e_j$ for $j = 1,\dots,p$.

Proof of (ii). Let $\psi(j) = -j+3p+q+1$ for $j = 2p+1,\dots,p+q$, which is the order-reversing substitution of the indices of U. If $j \neq (3p+q)/2$ and

$j \neq (3p+q)/2+1$,

$$e_j = e_{\varphi_1(j)} = e_{-j+2p+q+1} = e_{\varphi(-j+3p+q+1)} = e_{-j+3p+q+1} = e_{\psi(j)}.$$

Hence $T(e_{2p+1}\dots e_{p+q})$ is palindrome.

PROOF OF (iii).

$$e_{2p+(q-p)/2} \neq e_{\varphi((3p+q)/2)} = e_{(p+q)/2} = a \neq b = e_{(p+q)/2+1}$$
$$= e_{\varphi((3p+q)/2+1)} \neq e_{2p+(q-p)/2+1}.$$

Hence $e_{2p+(q-p)/2}e_{2p+(q-p)/2+1} = ba$.

EXAMPLE. If $p=5$ and $q=7$, then e_6, e_1, e_8, e_3, e_{10} e_5 and e_{12} are a, and e_7, e_2, e_9, e_4 and e_{11} are b. Hence $V = ababa$ and $W = abababa$.

Case 2*: pq is even.* We consider the case where p is even and q is odd and $e_{p/2}e_{p/2+1} = ab$. From condition (3), $T(V)$, W and VW are palindrome (since $\ell(V)$ and $\ell(VW)$ are odd). By condition (4)

$$e_{\varphi_1(j)} = e_j \qquad \text{if and only if } j < p/2 \quad \text{or} \quad j > p/2+1$$

and

$$e_{\varphi_2(j)} = e_j \qquad \text{for all } j = 1,2,\dots,p+q.$$

In this case let $\varphi(j) = \varphi_1 \circ \varphi_2(j) \equiv j+p \equiv j-q \mod p+q$. Again φ acts transitively on $\{1,2,\dots,p+q\}$ and

$$\begin{aligned} e_{\varphi(j)} \neq e_j \qquad & \text{if and only if } \varphi(j) = \frac{p}{2} \quad \text{or} \quad \varphi(j) = \frac{p}{2}+1 \\ & \text{if and only if } \varphi_2(j) = \frac{p}{2} \quad \text{or} \quad \varphi_2(j) = \frac{p}{2}+1 \\ & \text{if and only if } j = \frac{p+2q}{2} \quad \text{or} \quad j = \frac{p+2q}{2}+1. \end{aligned}$$

Let $x = n_a(V)$, $y = n_a(W)$ again. The φ-orbit of indices shows the following:

$$e_{p/2}, e_{p/2+p}, \dots, e_{p/2+(x+y-1)p} \text{ are the letter } a;$$

$$e_{p/2+1}, e_{p/2+1+p}, \dots, e_{p/2+(p+q-1)p} \text{ are the letter } b.$$

Here $e_{p/2+1} = e_{p/2+(x+y)p}$, and hence $(x+y)p \equiv 1 \mod p+q$. Since

$$\frac{p}{2} + (x+y-1)p \equiv \frac{p}{2}+q+1 \qquad \mod p+q,$$

we have

$$(**) \qquad 1 \equiv -q + (x+y-1)p \equiv yp - qx \qquad \mod p+q.$$

Note that $(**)$ means x is odd, because p is even. We first show that if $p < q$, then

(i) $e_1 \dots e_p = e_{p+1} \dots e_{2p}$.

(ii) $e_{2p+1} \dots e_{p+q}$ and hence $T(e_{2p+1} \dots e_{p+q})$ are palindrome.

If these are true, we can write W as VU and, as in Case 1, conclude that (V, U) and hence (V, W) are generating pairs of F. We show also

(iii) $M(e_{p+1} \dots e_{p+q}) = e_{p+(q+1)/2} \neq e_{(3p+q+1)/2} = M(e_{2p+1} \dots e_{p+q})$.

Proof of (i). Since $p < p/2 + q$, $e_j = e_{\varphi(j)} = e_{p+j}$ for $j = 1, \dots, p$.

Proof of (ii). Let $\psi(j) = -j + 3p + q + 1$ for $j = 2p+1, \dots, p+q$. If $j \geq 2p+1$, then

$$e_j = e_{\varphi_1(j)} = e_{-j+2p+q+1} = e_{\varphi(-j+2p+q+1)} = e_{-j+3p+q+1} = e_{\psi(j)}.$$

Hence $e_{2p+1} \dots e_{p+q}$ is palindrome.

Proof of (iii). If $x + y$ is even, then $p + q - x - y$ is odd.

$$\frac{2p+q+1}{2} \equiv \frac{p}{2} + \frac{(x+y)p}{2} \quad \bmod p+q \qquad \text{hence } e_{(2p+q+1)/2} = a.$$

$$\frac{3p+q+1}{2} \equiv \frac{p}{2} + 1 + \frac{(p+q-x-y+1)p}{2} \quad \bmod p+q \qquad \text{hence } e_{(3p+q+1)/2} = b.$$

If $x + y$ is odd, then $p + q - x - y$ is even.

$$\frac{2p+q+1}{2} \equiv \frac{p}{2} + 1 + \frac{(p+q-x-y)p}{2} \quad \bmod p+q \qquad \text{hence } e_{(2p+q+1)/2} = b.$$

$$\frac{3p+q+1}{2} \equiv \frac{p}{2} + \frac{(x+y+1)p}{2} \quad \bmod p+q \qquad \text{hence } e_{(3p+q+1)/2} = a.$$

Thus (iii) holds. If $p > q$, we can show

(i) $e_1 \dots e_q = e_{p+1} \dots e_{p+q}$.

(ii) $e_{q+1} \dots e_p$ and hence $T(e_{q+1} \dots e_p)$ are palindrome.

(iii) $M(e_{q+1} \dots e_{p+q}) = e_{q+p/2} e_{q+p/2+1} = ba$.

Proof of (i). Since $q < p/2 + q$, $e_j = e_{\varphi(j)} = e_{p+j}$ for $j = 1, \dots, q$.

Proof of (ii). Since VW is palindrome, (ii) is obvious.

Proof of (iii). $e_{q+p/2} \neq e_{\varphi(q+p/2)} = e_{p/2} = a$, $e_{q+p/2+1} \neq e_{\varphi(q+p/2+1)} = e_{p/2+1} = b$. Thus (iii) follows. Again we can write V as WU and conclude by induction that (V, W) is a generating pair of F.

Example. If $p = 6$ and $q = 5$, then e_3 and e_9 are a, and e_4, e_{10}, e_5, e_{11}, e_6, e_1, e_7, e_2 and e_8 are b. Hence $V = bbabbb$ and $W = bbabb$.

REMARK. From $W = VU$ in the above argument, $n_a(W) = n_a(V) + n_a(U)$ and $n_b(W) = n_b(V) + n_b(U)$. By induction we can show that $yp - qx = -1$ (or $yp - qx = 1$) in $(*)$ (or in $(**)$).

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