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Asymptotic analysis of positive solutions of third order nonlinear differential equations

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ABSTRACT. It is shown that an application of the theory of regular variation (in the sense of Karamata) gives the possibility of determining the existence and precise asymptotic behavior of positive solutions of the third-order nonlinear differential equation $(|x''|^{\alpha-1}x'')' + q(t)|x|^{\beta}x = 0$, where $\alpha > \beta > 0$ are constants and $q : [a, \infty) \to (0, \infty)$ is a continuous regularly varying function.

1. Introduction

We consider the third order nonlinear differential equation

$$(|x''|^{\alpha-1}x'')' + q(t)|x|^{\beta-1}x = 0, \tag{A}$$

where α and β are positive constants such that $\alpha > \beta$ and $q : [a, \infty) \to (0, \infty)$ is a continuous function, $a > 0$.

By a solution of (A) we mean a function $x : [T_x, \infty) \to \mathbf{R}$, $T_x \geq a$, which satisfies (A) (so that $|x''|^{\alpha-1}x''$ is continuously differentiable) for all sufficiently large t and is nontrivial (proper) in the sense that

$$\sup\{|x(t)| : t \geq T\} > 0 \qquad \text{for any } T \geq T_x.$$

Such a solution is called *oscillatory* if it has an infinite sequence of zeros clustering at infinity, and *nonoscillatory* otherwise.

Our first goal in this paper is to obtain necessary and sufficient conditions for all proper solutions of (A) to be oscillatory or satisfying

$$|x^{(i)}(t)| \downarrow 0 \qquad \text{as } t \uparrow \infty, \ i = 0, 1, 2, \tag{1}$$

(the so-called Property A of equation (A)). It will be shown in Section 2, that the above property of equation (A) generalizing the known result for sublinear Emden-Fowler equation of the third order with $\alpha = 1$ is characterized by the

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condition

$$\int_a^\infty t^{2\beta} q(t)dt = \infty. \tag{2}$$

With regard to this result it is natural to ask the following two questions:

(i) If (2) holds, does equation (A) really possess nonoscillatory solutions satisfying (1)? And if the answer is "Yes", is it possible to describe asymptotic behavior of such solutions at infinity explicitly and precisely?

(ii) If (2) does not hold, is it possible to characterize the existence of nonoscillatory solutions of (A) which do not satisfy (1) and obtain accurate asymptotic formulas governing their behavior at infinity?

In looking for the answers to the above questions, a combination of methods of the theory of regular variation with a fixed point technique has been utilized. Such an approach has shown to be very effective and powerful, and produced a series of new interesting results recently (see [3], [4] and [5]).

To obtain the desired detailed information, we begin with classifying the set of all possible nonoscillatory (or equivalently positive) solutions of (A) into five disjoint subclasses. It suffices to restrict our consideration to positive solutions of (A), since if $x(t)$ is a solution of (A), then so is $-x(t)$.

Let $x(t)$ be an eventually positive solution of equation (A). Then, there are two possibilities for $x'(t)$ and $x''(t)$: either

$$x'(t) > 0 \quad \text{and} \quad x''(t) > 0 \qquad \text{for all large } t \tag{3}$$

or

$$x'(t) < 0 \quad \text{and} \quad x''(t) > 0 \qquad \text{for all large } t. \tag{4}$$

If (3) holds, then the limit $x''(\infty) = \lim_{t\to\infty} x''(t) = 2\lim_{t\to\infty} x(t)/t^2$ exists and is either zero or a finite positive number. If $x''(\infty) = 0$, then $x'(t)$ increases to a positive limit $x'(\infty)$, finite or infinite, as $t \to \infty$, implying that $\lim_{t\to\infty} x(t)/t = x'(\infty)$. If (4) holds, then $x(t)$ is an eventually decreasing function and $x(\infty) = \lim_{t\to\infty} x(t)$ is either zero or a strict positive finite value. In both cases $x''(\infty) = 0$. Summarizing the above observations, we see that eventually positive solutions of (A) fall into the following five types:

$$\lim_{t\to\infty} \frac{x(t)}{t^2} = \text{const} > 0 \tag{I}$$

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{t^2} = 0 \tag{II}$$

$$\lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0 \tag{III}$$

$$\lim_{t\to\infty} x(t) = \text{const} > 0 \tag{IV}$$

$$\lim_{t\to\infty} x(t) = 0. \tag{V}$$

Note that the functions $\{t^2, t, 1\}$ are particular solutions of the unperturbed differential equation

$$(|x''|^{\alpha-1}x'')' = 0.$$

The solutions of (A) which are asymptotic to constant multiples of t^2, t or 1 as $t \to \infty$, i.e., the solutions satisfying (I), (III) or (IV), respectively, are referred to as *primitive solutions* of equation (A). If we use the symbol $\sim$ to denote the asymptotic equivalence of two positive functions $f(t)$ and $g(t)$, i.e.

$$f(t) \sim g(t) \quad \text{as } t \to \infty \qquad \Leftrightarrow \qquad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1 \tag{5}$$

then a primitive solution $x(t)$ of (A) satisfies $x(t) \sim ct^2$, $x(t) \sim ct$ or $x(t) \sim c$ as $t \to \infty$ for some constant $c > 0$.

In Section 2 we show via the Schauder-Tychonoff fixed point theorem that the existence of primitive solutions of all three types for (A) can be completely characterized. Our efforts in the subsequent sections will be focused on proving the existence of *non-primitive solutions* of equation (A), that is, positive solutions satisfying either (II) or (V) and analyzing their asymptotic behavior at infinity as accurately as possible.

If (A) has a type (II)-solution $x(t)$ defined on $[T, \infty)$, then integrating (A) once on $[t, \infty)$, using that $x''(\infty) = 0$, raising to the power $1/\alpha$ and then integrating twice from T to t, we obtain

$$\begin{aligned} x(t) &= c_0 + c_1(t - T) + \int_T^t \int_T^s \left[\int_r^\infty q(u)x(u)^\beta du \right]^{1/\alpha} dr ds \\ &= c_0 + c_1(t - T) + \int_T^t (t - s) \left[\int_s^\infty q(r)x(r)^\beta dr \right]^{1/\alpha} ds, \qquad t \geq T, \end{aligned} \tag{6}$$

where $c_0 = x(T) > 0$ and $c_1 = x'(T) \geq 0$. In what follows we will often make use of the integral asymptotic relation

$$x(t) \sim \int_T^t \int_T^s \left[\int_r^\infty q(u)x(u)^\beta du \right]^{1/\alpha} dr ds, \qquad t \to \infty, \tag{AR$_1$}$$

which can be considered as an "approximation" of (6). If $x(t)$ is a non-primitive solution of type (V) for (A), then $x''(\infty) = x'(\infty) = x(\infty) = 0$ and the triple integration of (A) on $[t, \infty)$ leads to

$$x(t) = \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t \geq T, \tag{7}$$

which can be approximated by the integral asymptotic relation

$$x(t) \sim \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t \to \infty. \tag{AR$_2$}$$

If the coefficient $q(t)$ is a general continuous positive function, then it is a very difficult task to extract the information about the existence and precise asymptotic behavior of non-primitive solutions directly from (A) or from the corresponding integral equations (6) and (7). But if we restrict ourselves to the case of $q(t)$ which is a *regularly varying function* (in the sense of definition given below) and consider only the *regularly varying solutions*, then the asymptotic analysis of (A) can be made quite easily in two subsequent steps. First, in Section 3, we establish the existence of regularly varying solutions of the integral asymptotic relations $(\mathrm{AR})_1$ (resp. $(\mathrm{AR})_2$) and next, in Section 4, we show that these solutions of relations $(\mathrm{AR})_1$ (resp. $(\mathrm{AR})_2$) can be used to define suitable subsets of the locally convex space $C[T, \infty)$ so that the Schauder-Tychonoff fixed point theorem is effectively applicable to certain integral operators generated by (6) (resp. (7)) defined on these subsets.

For the reader's benefit we recall here the definition and some properties of regularly varying functions. A measurable function $f : (0, \infty) \to (0, \infty)$ is called *regularly varying of index* $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \qquad \text{for } \forall\lambda > 0,$$

or, equivalently, it is expressed in the form

$$f(t) = c(t) \exp\left\{\int_{t_0}^t \frac{\delta(s)}{s}\, ds\right\}, \qquad t \geq t_0,$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \qquad \text{and} \qquad \lim_{t\to\infty} \delta(t) = \rho.$$

If $c(t) \equiv c_0$, then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ.

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV instead of RV(0) and call members of SV *slowly varying functions.* By definition any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^{\rho} g(t)$ with $g(t) \in \mathrm{SV}$. So, the class SV of slowly varying functions is of fundamental importance in theory of regular variation. Typical examples of slowly varying functions are: all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathrm{R}, \qquad \text{and} \qquad \exp\left\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\right\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\{(\log t)^{1/3} \cos(\log t)^{1/3}\}$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \qquad \text{and} \qquad \liminf_{t\to\infty} L(t) = 0.$$

A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^{\rho} L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t\to\infty} L(t) = \text{const} > 0$. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) is used to denote the set of all trivial RV(ρ)-functions (or the set of all nontrivial RV(ρ)-functions). According to this definition a primitive solution $x(t)$ of (A) such that $x(t) \sim ct^j$, $t \to \infty$, for some $c > 0$ and $j \in \{0,1,2\}$, is a trivial regularly varying function of index j, i.e. $x(t) \in$ tr-RV(j).

The following proposition known as Karamata's integration theorem, is particularly useful in handling slowly and regularly varying functions analytically and is often used throughout the paper.

Proposition 1. *Let* $L(t) \in \mathrm{SV}$. *Then*
(i) *if* $\alpha > -1$,

$$\int_a^t s^{\alpha} L(s)\,ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \qquad t \to \infty;$$

(ii) *if* $\alpha < -1$,

$$\int_t^{\infty} s^{\alpha} L(s)\,ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \qquad t \to \infty;$$

(iii) *if* $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s}\,ds \in \mathrm{SV} \qquad \textit{and} \qquad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^\infty \frac{L(s)}{s}\,ds \in \mathrm{SV} \qquad \text{and} \qquad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

The reader is referred to Bingham et al. [1] for the most complete exposition of theory of regular variation and its applications and to Marić [6] for the comprehensive survey of results up to 2000 on the asymptotic analysis of second order linear and nonlinear ordinary differential equations in the framework of regular variation.

2. Existence of primitive solutions of (A)

In this section we establish necessary and sufficient conditions for the existence of trivial regularly varying solutions of indices 2, 1 and 0 of equation (A), that is, positive solutions of types (I), (III) and (IV), respectively.

Theorem 1. *Equation* (A) *has positive solutions* $x(t)$ *such that*

$$\lim_{t\to\infty} \frac{x(t)}{t^2} = \text{const} > 0 \tag{8}$$

if and only if

$$\int_a^\infty t^{2\beta} q(t)dt < \infty. \tag{9}$$

Proof. (The "only if" part.) Let $x(t)$ be an eventually positive solution of (A) satisfying (8). Then, there exist positive constants c_1, c_2 and $t_0 \geq a$ such that

$$c_1 t^2 \leq x(t) \leq c_2 t^2 \tag{10}$$

for $t \geq t_0$. An integration of (A) yields

$$\int_{t_0}^\infty q(t)x(t)^\beta dt < \infty, \tag{11}$$

which combined with (10) implies (9).

(The "if" part.) Let (9) hold and $c > 0$ be any given constant. Choose $t_0 \geq a$ large enough so that

$$\int_{t_0}^\infty t^{2\beta} q(t)dt \leq (2^\alpha - 1)c^{\alpha-\beta}. \tag{12}$$

Let $X \subset C[t_0, \infty)$ and $F : X \to C[t_0, \infty)$ be defined as follows:

$$X = \left\{ x \in C[t_0, \infty) : \frac{c}{2}(t - t_0)^2 \le x(t) \le c(t - t_0)^2, t \ge t_0 \right\},$$

$$Fx(t) = \int_{t_0}^{t} \int_{t_0}^{s} \left[c^{\alpha} + \int_{r}^{\infty} q(u)x(u)^{\beta} du \right]^{1/\alpha} drds, \qquad t \ge t_0.$$

Clearly, X is a closed convex subset of the Fréchet space $C[t_0, \infty)$ with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. It can be shown routinely that the integral operator F is a continuous self-map on X and sends X into a relatively compact subset of $C[t_0, \infty)$. Hence by the Schauder-Tychonoff fixed point theorem there exists a function $x(t) \in X$ such that $x(t) = Fx(t)$, $t \ge t_0$, that is,

$$x(t) = \int_{t_0}^{t} \int_{t_0}^{s} \left[c^{\alpha} + \int_{r}^{\infty} q(u)x(u)^{\beta} du \right]^{1/\alpha} drds, \qquad t \ge t_0. \tag{13}$$

Differentiation of (13) shows that $x(t)$ is a solution of (A) that satisfies (8).

Theorem 2. *Equation* (A) *has positive solutions* $x(t)$ *such that*

$$\lim_{t \to \infty} \frac{x(t)}{t} = \text{const} > 0 \tag{14}$$

if and only if

$$\int_{a}^{\infty} \left[\int_{t}^{\infty} s^{\beta} q(s) ds \right]^{1/\alpha} dt < \infty. \tag{15}$$

Proof. (The "only if" part.) Suppose that (A) has a positive solution $x(t)$ which satisfies (14). Integrating (A) from t to ∞, we have

$$(x''(t))^{\alpha} = \int_{t}^{\infty} q(s)x(s)^{\beta} ds, \qquad t \ge t_0,$$

or, equivalently,

$$x''(t) = \left[\int_{t}^{\infty} q(s)x(s)^{\beta} ds \right]^{1/\alpha}, \qquad t \ge t_0.$$

Integrating this equation again and using the inequality $x(t) \ge c_1 t$ holding for $t \ge t_0$ and some constant $c_1 > 0$ (a consequence of (14)), we conclude that

$$\int_{t_0}^{\infty} \left[\int_{t}^{\infty} s^{\beta} q(s) ds \right]^{1/\alpha} dt < \infty.$$

(The "if" part.) If (15) holds, then for any given constant $c > 0$ we can choose $t_0 > a$ so that

$$\int_{t_0}^{\infty}\left[\int_t^{\infty} s^{\beta} q(s)ds\right]^{1/\alpha} dt \leq \frac{1}{2} c^{1-\beta/\alpha}.$$

Then, as in the proof of Theorem 1, we can show that the integral operator G defined by

$$Gx(t) = \int_{t_0}^{t}\left(c - \int_s^{\infty}\left[\int_r^{\infty} q(u)x(u)^{\beta} du\right]^{1/\alpha} dr\right) ds, \qquad t \geq t_0,$$

has a fixed point $x(t)$ in the set

$$X = \left\{x(t) \in C[t_0, \infty) : \frac{c}{2}(t - t_0) \leq x(t) \leq c(t - t_0), t \geq t_0\right\},$$

which gives birth to a solutions of (A) satisfying (14).

Theorem 3. *Equation* (A) *has positive solutions* $x(t)$ *such that*

$$\lim_{t\to\infty} x(t) = \text{const} > 0 \tag{16}$$

if and only if

$$\int_a^{\infty} t\left[\int_t^{\infty} q(s)ds\right]^{1/\alpha} dt < \infty. \tag{17}$$

Proof. (The "only if" part.) Suppose that (A) has a solution $x(t)$ which is positive for $t \geq t_0$ and such that (16) holds. Repeated integration of (A) shows that

$$\int_{t_0}^{\infty} t\left[\int_t^{\infty} q(s)x(s)^{\beta} ds\right]^{1/\alpha} dt < \infty$$

which together with the inequality $x(t) \geq c_1$ holding for some constant $c_1 > 0$ and all sufficiently large t implies (17).

(The "if" part.) Let $c > 0$ be an arbitrary constant and choose $t_0 \geq a$ large enough so that

$$\int_{t_0}^{\infty} t\left[\int_t^{\infty} q(s)ds\right]^{1/\alpha} dt \leq 2^{-1/\alpha} c^{1-\beta/\alpha}.$$

This is possible because of (17). Define

$$X = \{x(t) \in C[t_0, \infty) : c \leq x(t) \leq 2c, t \geq t_0\}$$

and

$$Hx(t) = c + \int_t^{\infty} \int_s^{\infty} \left[\int_r^{\infty} q(u)x(u)^{\beta} du \right]^{1/\alpha} drds, \qquad t \geq t_0.$$

It is easy to verify that H is continuous and maps X into a compact subset of X, and hence the operator H has a fixed element x in X, which gives the desired solution of equation (A).

Lemma 1. *Let $x(t)$ be an eventually positive solution of* (A) *which satisfies* (3). *Then there exist $L > 0$ and $T > a$ such that*

$$x(t) \geq Lt^2 x''(t), \qquad t \geq T. \tag{18}$$

Proof. Since $x''(t)$ is positive and nonincreasing on $[t_0, \infty)$ for some $t_0 > a$, it follows that

$$x'(t) = x'(t_0) + \int_{t_0}^{t} x''(s)ds \geq (t - t_0)x''(t) \qquad \text{for } t \geq t_0. \tag{19}$$

Inequality (19) yields

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds \geq \int_{t_0}^{t} (s - t_0)x''(s)ds \geq \frac{(t - t_0)^2}{2} x''(t), \qquad t \geq t_0,$$

and so (18) holds for some constant $L > 0$ and $T > t_0$.

Lemma 2. *Assume that* (A) *has a positive solution which satisfies* (3). *Then the integral condition* (9) *holds.*

Proof. An integration of (A) gives

$$(x''(t))^{\alpha} \geq \int_t^{\infty} q(s)x(s)^{\beta} ds, \qquad t \geq T. \tag{20}$$

Denote the right-hand side of (20) by $y(t)$. Then, by (18) from Lemma 1,

$$x(t) \geq Lt^2 y(t)^{1/\alpha}, \qquad t \geq T, \tag{21}$$

where L is a positive constant. The inequality (21) implies

$$y'(t) = -q(t)x(t)^{\beta} \leq -L^{\beta} q(t) t^{2\beta} y(t)^{\beta/\alpha}, \qquad t \geq T,$$

from which it follows that

$$\int_T^t \frac{y'(s)}{y(s)^{\beta/\alpha}} ds \leq -L^{\beta} \int_T^t s^{2\beta} q(s) ds$$

and hence

$$\frac{\alpha}{\alpha-\beta}\{y(t)^{(\alpha-\beta)/\alpha} - y(T)^{(\alpha-\beta)/\alpha}\} \le -L^{\beta}\int_T^t s^{2\beta}q(s)ds, \qquad t \ge T.$$

Since $\alpha > \beta$, we obtain

$$\frac{\alpha}{\alpha-\beta}y(T)^{(\alpha-\beta)/\alpha} \ge L^{\beta}\int_T^t s^{2\beta}q(s)ds, \qquad t \ge T,$$

which implies (9).

The proof of the following lemma is patterned after the proof of Naito et al. [7, Lemma 8].

Lemma 3. *If $\alpha > \beta$, then the integral condition* (17) *implies* (9).

Proof. From (17) it follows that there exists an $M > 0$ such that

$$\int_a^t s\left[\int_s^{\infty} q(r)dr\right]^{1/\alpha} ds \le M$$

for $t \ge a$. Then

$$\int_a^t s\,ds \cdot \left[\int_t^{\infty} q(r)dr\right]^{1/\alpha} \le M, \qquad t \ge a,$$

and so there exists an M_1 such that

$$t^2\left[\int_t^{\infty} q(r)dr\right]^{1/\alpha} \le M_1, \qquad t \ge a,$$

or, equivalently,

$$\int_t^{\infty} q(r)dr \le M_1^{\alpha}t^{-2\alpha} \tag{22}$$

for $t \ge a$. Multiplying (22) by $t^{-1+2\beta}$ and integrating from a to t, we find that

$$\frac{t^{2\beta}}{2\beta}\int_t^{\infty} q(r)dr - \frac{a^{2\beta}}{2\beta}\int_a^{\infty} q(r)dr + \frac{1}{2\beta}\int_a^t s^{2\beta}q(s)ds \le M_1^{\alpha}\int_a^t s^{-1-2\alpha+2\beta}\,ds.$$

This gives

$$\int_a^t s^{2\beta}q(s)ds \le a^{2\beta}\int_a^{\infty} q(r)dr + 2\beta M_1^{\alpha}\int_a^t s^{-1-2\alpha+2\beta}\,ds. \tag{23}$$

Since the assumption $\alpha > \beta$ implies $-1-2\alpha+2\beta < -1$, the last integral in (23) (and consequently also the integral on the left-hand side) converges as $t \to \infty$. Thus, we get (9) and the proof is complete.

As a consequence of Lemmas 1–3 we obtain the following result.

Theorem 4. *Any proper solution* $x(t)$ *of* (A) *is oscillatory or satisfies*

$$|x^{(i)}(t)| \downarrow 0 \qquad \text{as } t \uparrow \infty,\ i = 0, 1, 2, \tag{24}$$

if and only if

$$\int_a^\infty t^{2\beta} q(t)dt = \infty. \tag{25}$$

Proof. The necessity of the condition (25) follows from Theorem 1. To prove the sufficiency part, note that the existence of eventually positive solutions satisfying (3) is impossible due to Lemma 2. On the other hand, by Lemma 3 the only possible positive solutions which satisfy (4) are those of type (V).

Nonexistence of eventually negative solutions other than those satisfying (24) follows from the fact that if $x(t)$ is a solution of (A), then so is $-x(t)$.

3. Integral asymptotic relations for non-primitive solutions of (A)

3.1. Asymptotic relations for moderately growing solutions. We begin by considering positive solutions of the integral asymptotic relation $(\mathrm{AR})_1$ with regularly varying $q(t)$ which satisfy

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{t^2} = 0. \tag{26}$$

All such solutions tend to infinity as $t \to \infty$ and are often referred to as *moderately growing*. The set of all moderately growing positive solutions of $(\mathrm{AR})_1$ consists of three disjoint subclasses which follows from the observation that a regularly varying function $x(t) = t^\rho \xi(t)$, where $\xi(t) \in \mathrm{SV}$, can satisfy $(\mathrm{AR})_1$ and (26) only if $\rho = 2$ and $\xi(t) \to 0$ as $t \to \infty$ (i.e. $x(t) \in \text{ntr-RV}(2)$), or $\rho \in (1, 2)$ (i.e. $x(t) \in \mathrm{RV}(\rho)$), or $\rho = 1$ and $\xi(t) \to \infty$ as $t \to \infty$ (i.e. $x(t) \in \text{ntr-RV}(1)$).

Theorem 5. *Let* $q(t)$ *be regularly varying of index* σ. *Relation* $(\mathrm{AR})_1$ *possesses nontrivial regularly varying solutions of index* 2 *if and only if* $\sigma = -2\beta - 1$ *and* (9) *holds, in which case any such solution* $x(t)$ *enjoys one and the same asymptotic behavior*

$$x(t) \sim t^2 \left[\frac{\alpha - \beta}{2^\alpha \alpha} \int_t^\infty s^{2\beta} q(s)ds\right]^{1/(\alpha-\beta)}, \qquad t \to \infty. \tag{27}$$

Theorem 6. *Let* $q(t)$ *be regularly varying of index* σ. *Relation* $(\mathrm{AR})_1$ *possesses regularly varying solutions of index* $\rho \in (1, 2)$ *if and only if* $\sigma \in$

$(-\alpha-\beta-1,-2\beta-1)$, *in which case* ρ *is given by*

$$\rho=\frac{\sigma+2\alpha+1}{\alpha-\beta}, \tag{28}$$

and any such solution $x(t)$ *enjoys one and the same asymptotic behavior*

$$x(t)\sim\left[\frac{t^{2\alpha+1}q(t)}{\alpha(2-\rho)(\rho-1)^{\alpha}\rho^{\alpha}}\right]^{1/(\alpha-\beta)}, \qquad t\to\infty. \tag{29}$$

THEOREM 7. *Let* $q(t)$ *be regularly varying of index* σ. *Relation* $(\mathrm{AR})_1$ *possesses nontrivial regularly varying solutions of index* 1 *if and only if* $\sigma=-\alpha-\beta-1$ *and*

$$\int_a^{\infty}(t^{\beta+1}q(t))^{1/\alpha}dt=\infty, \tag{30}$$

in which case any such solution $x(t)$ *enjoys one and the same asymptotic behavior*

$$x(t)\sim t\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t(s^{\beta+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)}, \qquad t\to\infty. \tag{31}$$

PROOF OF THEOREMS 5, 6 AND 7. (The "only if" part.) Suppose that $(\mathrm{AR})_1$ has a solution $x(t)\in\mathrm{RV}(\rho)$ on $[t_0,\infty)$ satisfying (26) and express it as $x(t)=t^{\rho}\xi(t)$, $\xi(t)\in\mathrm{SV}$. Because of (26), we must have $\rho\in[1,2]$ and $\xi(t)\to\infty$ or $\xi(t)\to 0$ as $t\to\infty$ according as $\rho=1$ or $\rho=2$, respectively. Using the expression $q(t)=t^{\sigma}l(t)$, $l(t)\in\mathrm{SV}$, we obtain

$$\int_t^{\infty}q(s)x(s)^{\beta}ds=\int_s^{\infty}s^{\sigma+\rho\beta}l(s)\xi(s)^{\beta}ds, \qquad t\ge t_0. \tag{32}$$

The convergence of the integral on the right-hand side of (32) implies that $\sigma+\rho\beta\le -1$. First consider the case where $\sigma+\rho\beta=-1$. Then (32) reduces to

$$\int_t^{\infty}q(s)x(s)^{\beta}ds=\int_t^{\infty}s^{-1}l(s)\xi(s)^{\beta}ds\in\mathrm{SV}.$$

Raising the above to $1/\alpha$ and integrating twice on $[t_0,\infty)$, we see from $(\mathrm{AR})_1$ that

$$x(t)\sim\frac{t^2}{2}\left[\int_t^{\infty}s^{-1}l(s)\xi(s)^{\beta}ds\right]^{1/\alpha}\in\mathrm{RV}(2). \tag{33}$$

This shows that the regularity index of $x(t)$ is $\rho=2$ and hence $\sigma=-2\beta-1$. Note that (33) is equivalent to

$$\xi(t)^{\alpha}\sim\frac{1}{2^{\alpha}}\int_t^{\infty}s^{-1}l(s)\xi(s)^{\beta}ds, \qquad t\to\infty. \tag{34}$$

Denoting the right-hand side of (34) by $\eta(t)$, from (34) we obtain the following differential asymptotic relation for $\eta(t)$:

$$-\eta(t)^{-\beta/\alpha}\eta'(t) \sim \frac{1}{2^\alpha} t^{-1} l(t) = \frac{1}{2^\alpha} t^{2\beta} q(t), \qquad t \to \infty. \tag{35}$$

Since $\eta(t) \to 0$ as $t \to \infty$, the left-hand side of (35) is integrable on $[t_0, \infty)$, which shows that (9) is satisfied. Integration of (35) from t to ∞ gives

$$\eta(t) \sim \left[\frac{\alpha - \beta}{2^\alpha \alpha} \int_t^\infty s^{2\beta} q(s) ds\right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty,$$

and hence

$$x(t) = t^2 \xi(t) \sim t^2 \eta(t)^{1/\alpha} \sim t^2 \left[\frac{\alpha - \beta}{2^\alpha \alpha} \int_t^\infty s^{2\beta} q(s)\right]^{1/(\alpha-\beta)}, \qquad t \to \infty.$$

Next consider the case where $\sigma + \rho\beta < -1$. In this case from (32) we have

$$\int_t^\infty q(s) x(s)^\beta ds \sim -\frac{1}{\sigma + \rho\beta + 1} t^{\sigma+\rho\beta+1} l(t) \xi(t)^\beta, \qquad t \to \infty, \tag{36}$$

or, equivalently,

$$\left[\int_t^\infty q(s) x(s)^\beta ds\right]^{1/\alpha} \sim \left[-\frac{1}{\sigma + \rho\beta + 1}\right]^{1/\alpha} t^{(\sigma+\rho\beta+1)\alpha} l(t)^{1/\alpha} \xi(t)^{\beta/\alpha}, \qquad t \to \infty. \tag{37}$$

Observe that (37) is not integrable over $[t_0, \infty)$, which means that

$$\frac{\sigma + \rho\beta + 1}{\alpha} \geq -1, \qquad \text{i.e., } \sigma + \rho\beta \geq -\alpha - 1.$$

We distinguish the two cases:

(a) $\sigma + \rho\beta > -\alpha - 1$, (b) $\sigma + \rho\beta = -\alpha - 1$.

If (a) holds, then integrating (37) twice from t_0 to t and using Karamata's integration theorem ((i) of Proposition 1), we obtain

$$x(t) \sim \frac{\alpha^2 t^{(\sigma+\rho\beta+2\alpha+1)/\alpha} l(t)^{1/\alpha} \xi(t)^{\beta/\alpha}}{[-(\sigma + \rho\beta + 1)]^{1/\alpha} (\sigma + \rho\beta + 1 + \alpha)(\sigma + \rho\beta + 1 + 2\alpha)}, \qquad t \to \infty, \tag{38}$$

which shows that $x(t) \in \mathrm{RV}((\sigma + \rho\beta + 2\alpha + 1)/\alpha)$ with $(\sigma + \rho\beta + 2\alpha + 1)/\alpha \in (1, 2)$. Therefore,

$$\rho = \frac{\sigma + \rho\beta + 2\alpha + 1}{\alpha} \quad \Rightarrow \quad \rho = \frac{\sigma + 2\alpha + 1}{\alpha - \beta}.$$

Notice that $\rho \in (1,2)$ determines the range of σ to be $\sigma \in (-\alpha-\beta-1, -2\beta-1)$. Using the fact that the numerator and the denominator of the right-hand side of (38) can be rewritten, respectively, as

$$t^{(\sigma+\rho\beta+2\alpha+1)/\alpha} l(t)^{1/\alpha} \xi(t)^{\beta/\alpha} = t^{(2\alpha+1)/\alpha} q(t)^{1/\alpha} x(t)^{\beta/\alpha}$$

and

$$[-(\sigma+\rho\beta+1)]^{1/\alpha}(\sigma+\rho\beta+1+\alpha)(\sigma+\rho\beta+1+2\alpha)\alpha^{-2}$$
$$= \rho(\rho-1)(2-\rho)^{1/\alpha}\alpha^{1/\alpha},$$

we obtain from (38) the following asymptotic expression for $x(t)$:

$$x(t) \sim \left[\frac{t^{2\alpha+1} q(t)}{\alpha(2-\rho)(\rho-1)^{\alpha}\rho^{\alpha}}\right]^{1/(\alpha-\beta)}, \qquad t \to \infty.$$

If (b) holds, then integrating (37) twice from t_0 to t, we get

$$\int_{t_0}^{t}\left[\int_{s}^{\infty} q(r)x(r)^{\beta}dr\right]^{1/\alpha} ds \sim \alpha^{-1/\alpha}\int_{t_0}^{t} s^{-1} l(s)^{1/\alpha}\xi(s)^{\beta/\alpha}ds, \qquad t \to \infty,$$

$$\int_{t_0}^{t}\int_{t_0}^{s}\left[\int_{r}^{\infty} q(u)x(u)^{\beta}du\right]^{1/\alpha} drds \sim \alpha^{-1/\alpha} t\int_{t_0}^{t} s^{-1} l(s)^{1/\alpha}\xi(s)^{\beta/\alpha}ds, \qquad t \to \infty, \tag{39}$$

which, in view of $(\mathrm{AR})_1$, gives

$$x(t) \sim \alpha^{-1/\alpha} t\int_{t_0}^{t} s^{-1} l(s)^{1/\alpha}\xi(s)^{\beta/\alpha}ds \in \mathrm{RV}(1), \qquad t \to \infty, \tag{40}$$

(cf. (iii) of Proposition 1). This implies that $x(t) \in \mathrm{RV}(1)$, so that $\rho = 1$ and $\sigma = -\alpha-\beta-1$. Relation (40) is equivalent to

$$\xi(t) \sim \alpha^{-1/\alpha}\int_{t_0}^{t} s^{-1} l(s)^{1/\alpha}\xi(s)^{\beta/\alpha}ds, \qquad t \to \infty. \tag{41}$$

Let $\eta(t)$ denote the right-hand side of (41). Then, we can convert (41) into the differential asymptotic relation

$$\eta(t)^{-\beta/\alpha}\eta'(t) \sim \alpha^{-1/\alpha} t^{-1} l(t)^{1/\alpha} = \alpha^{-1/\alpha} t^{(\beta+1)/\alpha} q(t)^{1/\alpha}, \qquad t \to \infty. \tag{42}$$

Integrating (42) from t_0 to t, we see that (30) must hold and obtain the asymptotic formula

$$\eta(t) \sim \left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_{t_0}^{t}(s^{\beta+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)}$$
$$\sim \left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_{a}^{t}(s^{\beta+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty,$$

which combined with $(\mathrm{AR})_1$ gives

$$x(t) = t\xi(t) \sim t\eta(t) \sim t\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t (s^{\beta+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty.$$

This completes the proof of the "only if" parts of Theorems 5, 6 and 7.

(The "if" parts.) We show that the function $X(t)$ defined by

$$X(t) = \begin{cases} t^2\left[\frac{\alpha-\beta}{2^\alpha\alpha}\int_t^\infty s^{2\beta}q(s)ds\right]^{1/(\alpha-\beta)} & \text{if } \sigma = -2\beta-1 \text{ and (9) holds;} \\ \left[\frac{t^{2\alpha+1}q(t)}{\alpha(2-\rho)(\rho-1)^\alpha\rho^\alpha}\right]^{1/(\alpha-\beta)} & \text{if } \sigma \in (-\alpha-\beta-1, -2\beta-1), \\ & \quad \text{where } \rho = \frac{\sigma+2\alpha+1}{\alpha-\beta}; \\ t\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t (s^{\beta+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)} & \text{if } \sigma = -\alpha-\beta-1 \text{ and (30) holds} \end{cases} \tag{43}$$

satisfies for any $b \geq a$ the integral asymptotic relation

$$\int_b^t \int_b^s \left[\int_r^\infty q(u)X(u)^\beta du\right]^{1/\alpha} drds \sim X(t), \qquad t \to \infty. \tag{44}$$

Let $\sigma = -2\beta - 1$ and (9) hold. Then, we have

$$\begin{aligned}\int_t^\infty q(s)X(s)^\beta ds &= \int_t^\infty s^{2\beta}q(s)\left[\frac{\alpha-\beta}{2^\alpha\alpha}\int_s^\infty r^{2\beta}q(r)dr\right]^{\beta/(\alpha-\beta)} ds \\ &= 2^\alpha\left[\frac{\alpha-\beta}{2^\alpha\alpha}\int_t^\infty s^{2\beta}q(s)ds\right]^{\alpha/(\alpha-\beta)} = 2^\alpha t^{-2\alpha}X(t)^\alpha, \end{aligned} \tag{45}$$

which, raised to the power $1/\alpha$ and integrated twice on $[b, t]$, gives via application of Karamata's integration theorem

$$\int_b^t \int_b^s \left[\int_r^\infty q(u)X(u)^\beta du\right]^{1/\alpha} drds \sim t^2\left[\frac{\alpha-\beta}{2^\alpha\alpha}\int_t^\infty s^{2\beta}q(s)ds\right]^{1/(\alpha-\beta)} = X(t), \qquad t \to \infty.$$

Next, let $\sigma \in (-\alpha-\beta-1, -2\beta-1)$ and define ρ by (28). Expressing $X(t)$ as

$$X(t) = \frac{t^\rho l(t)^{1/(\alpha-\beta)}}{[\alpha(2-\rho)(\rho-1)^\alpha\rho^\alpha]^{1/(\alpha-\beta)}}, \qquad t \to \infty,$$

and using Karamata's integration theorem, we get

$$\begin{aligned}\int_t^\infty q(s)X(s)^\beta ds &= \frac{\int_t^\infty s^{\alpha\rho-2\alpha-1}l(s)^{\alpha/(\alpha-\beta)}ds}{[\alpha(2-\rho)(\rho-1)^\alpha\rho^\alpha]^{\beta/(\alpha-\beta)}} \\ &\sim \frac{t^{\alpha(\rho-2)}l(t)^{\alpha/(\alpha-\beta)}}{\alpha(2-\rho)[\alpha(2-\rho)(\rho-1)^\alpha\rho^\alpha]^{\beta/(\alpha-\beta)}},\end{aligned}$$

and

$$\int_b^t \int_b^s \left[\int_r^\infty q(u)X(u)^\beta du\right]^{1/\alpha} drds \sim \frac{t^\rho l(t)^{1/(\alpha-\beta)}}{[\alpha(2-\rho)(\rho-1)^\alpha \rho^\alpha]^{1/(\alpha-\beta)}} = X(t)$$

for $t \to \infty$.

Finally, let $\sigma = -\alpha - \beta - 1$ and (30) hold. Then, we have

$$\begin{aligned}\int_t^\infty q(s)X(s)^\beta ds &= \int_t^\infty s^\beta q(s)\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^s (r^{\beta+1}q(r))^{1/\alpha} dr\right]^{\alpha\beta/(\alpha-\beta)} ds \\ &= \int_t^\infty s^{-\alpha-1} l(s)\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^s (r^{\beta+1}q(r))^{1/\alpha} dr\right]^{\alpha\beta/(\alpha-\beta)} ds \\ &\sim \frac{1}{\alpha} t^{-\alpha} l(t)\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t (s^{\beta+1}q(s))^{1/\alpha} ds\right]^{\alpha\beta/(\alpha-\beta)} \\ &= \frac{1}{\alpha} tq(t)X(t)^\beta, \qquad t \to \infty. \end{aligned} \tag{46}$$

Integrating the above (raised to the power $1/\alpha$) twice on $[b, t]$ we conclude that

$$\begin{aligned}\int_b^t \left[\int_s^\infty q(r)X(r)^\beta dr\right]^{1/\alpha} ds &\sim \left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t (s^{\beta+1}q(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \\ &= t^{-1}X(t)\end{aligned}$$

and

$$\begin{aligned}\int_b^t \int_b^s \left[\int_r^\infty q(u)X(u)^\beta du\right]^{1/\alpha} drds &\sim t\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t (s^{\beta+1}q(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \\ &= X(t).\end{aligned} \tag{47}$$

This completes the proof of Theorems 5, 6 and 7.

3.2. Asymptotic relations for strongly decaying solutions. We now turn to studying positive solutions of the asymptotic integral relation $(AR)_2$ with regularly varying coefficient $q(t)$. Clearly, all solutions of $(AR)_2$ tend to 0 as $t \to \infty$ and are often referred to as *strongly decaying.* There are only two possible types of strongly decaying solutions of $(AR)_2$. In fact, a regularly varying function $x(t)$, which is expressed as $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$, can satisfy $(AR)_2$ if $\rho < 0$, in which case $x(t) \in \mathrm{RV}(\rho)$, or if $\rho = 0$ and $\xi(t) \to 0$ as $t \to \infty$, in which case $x(t) \in \text{ntr-SV} = \text{ntr-RV}(0)$.

Theorem 8. *Let $q(t)$ be regularly varying of index σ. Relation* $(\mathrm{AR})_2$ *possesses nontrivial slowly varying solutions if and only if $\sigma = -2\alpha - 1$ and*

$$\int_a^\infty (t^{\alpha+1}q(t))^{1/\alpha}dt < \infty, \tag{48}$$

in which case any such solution $x(t)$ has the unique asymptotic behavior

$$x(t) \sim \left[\frac{\alpha-\beta}{\alpha(2\alpha)^{1/\alpha}}\int_t^\infty (s^{\alpha+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty. \tag{49}$$

Theorem 9. *Let $q(t)$ be regularly varying of index σ. Relation* $(\mathrm{AR})_2$ *possesses regularly varying solutions of index $\rho < 0$ if and only if $\sigma < -2\alpha - 1$, in which case ρ is given by* (28) *and any such solution $x(t)$ has the unique asymptotic behavior*

$$x(t) \sim \left[\frac{t^{2\alpha+1}q(t)}{\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha}\right]^{1/(\alpha-\beta)}, \qquad t \to \infty. \tag{50}$$

Proof of theorems 8 and 9. (The "only if" part.) Let $x(t)$ be a regularly varying solution of index ρ of $(\mathrm{AR})_2$ defined on $[t_0, \infty)$ which is strongly decaying. Clearly, $\rho \le 0$. Using the expressions $q(t) = t^\sigma l(t)$, $x(t) = t^\rho \xi(t)$, $l(t), \xi(t) \in \mathrm{SV}$, we obtain

$$\int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds = \int_t^\infty \int_s^\infty \left[\int_r^\infty u^{\sigma+\rho\beta} l(u)\xi(u)^\beta du\right]^{1/\alpha} drds$$

for $t \ge t_0$. The convergence of the integral on the right-hand side implies that $\sigma + \rho\beta \le -2\alpha - 1$. First consider the case where $\sigma + \rho\beta = -2\alpha - 1$. Then, in view of (iii) of Proposition 1, we have

$$\int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds \sim \left(\frac{1}{2\alpha}\right)^{1/\alpha} \int_t^\infty s^{-1} l(s)^{1/\alpha}\xi(s)^{\beta/\alpha}ds \in \mathrm{SV}, \qquad t \to \infty,$$

which means that $\rho = 0$ (i.e., $x(t) = \xi(t)$) and $\sigma = -2\alpha - 1$. Then, from $(\mathrm{AR})_2$ we obtain

$$x(t) \sim \left(\frac{1}{2\alpha}\right)^{1/\alpha} \int_t^\infty s^{-1} l(s)^{1/\alpha}\xi(s)^{\beta/\alpha}ds, \qquad t \to \infty. \tag{51}$$

Denoting the right-hand side of (51) by $y(t)$, from (51) we get the following differential asymptotic relation for $y(t)$:

$$-y(t)^{-\beta/\alpha}y'(t) \sim \left(\frac{1}{2\alpha}\right)^{1/\alpha} t^{-1}l(t)^{1/\alpha} = \left(\frac{1}{2\alpha}\right)^{1/\alpha} t^{(\alpha+1)/\alpha}q(t)^{1/\alpha}, \qquad t \to \infty. \tag{52}$$

The left-hand side of (52) is integrable on $[t_0, \infty)$ (note that $y(t) \to 0$ as $t \to \infty$), and so is $(t^{\alpha+1} q(t))^{1/\alpha}$, that is, (48) must hold. An integration of (52) on $[t, \infty)$ yields

$$x(t) \sim y(t) \sim \left[\frac{\alpha-\beta}{\alpha}\left(\frac{1}{2\alpha}\right)^{1/\alpha}\int_t^\infty (s^{\alpha+1}q(s))^{1/\alpha}ds\right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty.$$

Next consider the case where $\sigma + \rho\beta < -2\alpha - 1$. Repeated application of Karamata's integration theorem ((ii) of Proposition 1) yields

$$\int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds$$
$$\sim \frac{\alpha^2 t^{(\sigma+\rho\beta+2\alpha+1)/\alpha} l(t)^{1/\alpha} \xi(t)^{\beta/\alpha}}{[-(\sigma+\rho\beta+1)]^{1/\alpha}[-(\sigma+\rho\beta+1+\alpha)][-(\sigma+\rho\beta+1+2\alpha)]}, \qquad t \to \infty$$

which, combined with $(\mathrm{AR})_2$, gives

$$x(t) \sim \frac{\alpha^2 t^{(\sigma+\rho\beta+2\alpha+1)/\alpha} l(t)^{1/\alpha} \xi(t)^{\beta/\alpha}}{[-(\sigma+\rho\beta+1)]^{1/\alpha}[-(\sigma+\rho\beta+1+\alpha)][-(\sigma+\rho\beta+1+2\alpha)]},$$
$$t \to \infty. \tag{53}$$

This means that $x(t) \in \mathrm{RV}((\sigma + \rho\beta + 2\alpha + 1)/\alpha)$ with $\sigma + \rho\beta + 2\alpha + 1 < 0$, and hence

$$\rho = \frac{\sigma+\rho\beta+2\alpha+1}{\alpha} \quad \Rightarrow \quad \rho = \frac{\sigma+2\alpha+1}{\alpha-\beta}.$$

The requirement $\rho < 0$ implies $\sigma < -2\alpha - 1$. Taking into account the fact that

$$t^{(\sigma+\rho\beta+2\alpha+1)/\alpha} l(t)^{1/\alpha} \xi(t)^{\beta/\alpha} = t^{(2\alpha+1)/\alpha} q(t)^{1/\alpha} x(t)^{\beta/\alpha}$$

and

$$[-(\sigma+\rho\beta+1)]^{1/\alpha}[-(\sigma+\rho\beta+1+\alpha)][-(\sigma+\rho\beta+1+2\alpha)]$$
$$= (-\rho)(1-\rho)(2-\rho)^{1/\alpha}\alpha^{(2\alpha+1)/\alpha},$$

we can rewrite (53) as

$$x(t) \sim \left[\frac{t^{2\alpha+1}q(t)}{\alpha(-\rho)^\alpha(1-\rho)^\alpha(2-\rho)}\right]^{1/(\alpha-\beta)}, \qquad t \to \infty.$$

(The "if" part.) Define the function $Y(t)$ by

$$Y(t) = \begin{cases} \left[\frac{\alpha-\beta}{\alpha(2\alpha)^{1/\alpha}} \int_t^\infty (s^{\alpha+1} q(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} & \text{if } \sigma = -2\alpha - 1 \text{ and (48) holds;} \\ \left[\frac{t^{2\alpha+1} q(t)}{\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha}\right]^{1/(\alpha-\beta)} & \text{if } \sigma < -2\alpha - 1, \text{ where } \rho = \frac{\sigma+2\alpha+1}{\alpha-\beta}, \end{cases} \tag{54}$$

and verify that it satisfies the integral asymptotic relation

$$\int_t^\infty \int_s^\infty \left[\int_r^\infty q(u) Y(u)^\beta du\right]^{1/\alpha} \sim Y(t), \qquad t \to \infty. \tag{55}$$

If $\sigma = -2\alpha - 1$ and (48) holds, then repeated use of Karamata's integration theorem gives

$$\int_t^\infty q(s) Y(s)^\beta ds = \int_t^\infty s^{-2\alpha-1} l(s) Y(s)^\beta ds \sim \frac{1}{2\alpha} t^{-2\alpha} l(t) Y(t)^\beta,$$

$$\begin{aligned} \int_t^\infty \left[\int_s^\infty q(r) Y(r)^\beta dr\right]^{1/\alpha} ds &\sim \frac{1}{(2\alpha)^{1/\alpha}} \int_t^\infty s^{-2} l(s)^{1/\alpha} Y(s)^{\beta/\alpha} ds \\ &\sim \frac{1}{(2\alpha)^{1/\alpha}} t^{-1} l(t)^{1/\alpha} Y(t)^{\beta/\alpha}, \end{aligned} \tag{56}$$

and hence

$$\begin{aligned} \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u) Y(u)^\beta du\right]^{1/\alpha} dr ds &\sim \frac{1}{(2\alpha)^{1/\alpha}} \int_t^\infty s^{-1} l(s)^{1/\alpha} Y(s)^{\beta/\alpha} ds \\ &= \left[\frac{\alpha-\beta}{\alpha(2\alpha)^{1/\alpha}} \int_t^\infty (s^{\alpha+1} q(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \\ &= Y(t), \qquad t \to \infty. \end{aligned}$$

If $\sigma < -2\alpha - 1$, then we use the expression

$$Y(t) = \frac{t^\rho l(t)^{1/(\alpha-\beta)}}{[\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha]^{1/(\alpha-\beta)}}, \qquad \rho = \frac{\sigma + 2\alpha + 1}{\alpha - \beta},$$

and compute

$$\begin{aligned} \int_t^\infty q(s) Y(s)^\beta ds &= \frac{\int_t^\infty s^{\sigma+\rho\beta} l(s)^{\alpha/(\alpha-\beta)} ds}{[\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha]^{\beta/(\alpha-\beta)}} \\ &\sim \frac{t^{\alpha(\rho-2)} l(t)^{\alpha/(\alpha-\beta)}}{\alpha(2-\rho)[\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha]^{\beta/(\alpha-\beta)}} \end{aligned}$$

as $t \to \infty$. Raising to the power $1/\alpha$ and continuing to integrate the above twice on $[t, \infty)$, we obtain

$$\int_t^\infty \int_s^\infty \left[\int_r^\infty q(u) Y(u)^\beta du\right]^{1/\alpha} drds$$

$$\sim \frac{t^\rho l(t)^{1/(\alpha-\beta)}}{[\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha]^{1/(\alpha-\beta)}} = Y(t), \qquad t \to \infty.$$

This completes the proof of Theorems 8 and 9.

4. Existence of non-primitive positive solutions for equations (A)

We now turn our attention to the existence of moderately growing and strongly decaying positive solutions of the differential equation (A). In what follows, the following notation will be used extensively.

Let $f(t)$ and $g(t)$ be two positive continuous functions defined in a neighborhood of infinity, say for $t \ge T$. We use the notation $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exists positive constants m and M such that

$$mg(t) \le f(t) \le Mg(t) \qquad \text{for } t \ge T.$$

If $f(t)$ satisfies $f(t) \asymp g(t)$, $t \to \infty$, for some $g(t)$ which is regularly varying of index ρ, then $f(t)$ is called a *nearly regularly varying function of index* ρ.

Our purpose in this section is to show that equation (A) with nearly regularly varying coefficient $q(t)$ can have nearly regularly varying positive solutions of types (II) and (V), which behave for $t \to \infty$ like the regularly varying solutions of the asymptotic relations $(\mathrm{AR})_1$ and $(\mathrm{AR})_2$ whose existence was established in Theorems 5–9.

4.1. Existence of moderately growing non-primitive solutions of (A).

Theorem 10. *Let $q(t)$ be nearly regularly varying of index σ, that is, $q(t) \asymp q_\sigma(t)$, $t \to \infty$, for some $q_\sigma(t) \in \mathrm{RV}(\sigma)$. Suppose that $\sigma = -2\beta - 1$ and* (9) *holds. Then, equation* (A) *possesses a nearly regularly varying solution $x(t)$ of index $\rho = 2$ such that*

$$x(t) \asymp t^2 \left[\frac{\alpha-\beta}{\alpha 2^\alpha} \int_t^\infty s^{2\beta} q_\sigma(s) ds\right]^{1/(\alpha-\beta)}, \qquad t \to \infty. \tag{57}$$

Theorem 11. *Let $q(t)$ be nearly regularly varying of index σ, that is, $q(t) \asymp q_\sigma(t)$, $t \to \infty$, for some $q_\sigma(t) \in \mathrm{RV}(\sigma)$. Suppose that $\sigma \in (-\alpha - \beta - 1, -2\beta - 1)$. Then, equation* (A) *possesses a nearly regularly varying solution $x(t)$*

of index $\rho = (\sigma + 2\alpha + 1)/(\alpha - \beta) \in (1, 2)$ *such that*

$$x(t) \asymp \left[\frac{t^{2\alpha+1} q_\sigma(t)}{\alpha(2-\rho)(\rho-1)^\alpha \rho^\alpha}\right]^{1/(\alpha-\beta)}, \qquad t \to \infty. \tag{58}$$

Theorem 12. *Let* $q(t)$ *be nearly regularly varying of index* σ, *that is,* $q(t) \asymp q_\sigma(t)$, $t \to \infty$, *for some* $q_\sigma(t) \in \mathrm{RV}(\sigma)$. *Suppose that* $\sigma = -\alpha - \beta - 1$ *and* (30) *holds. Then, equation* (A) *possesses a nearly regularly varying solution* $x(t)$ *of index* $\rho = 1$ *such that*

$$x(t) \asymp t\left[\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \int_a^t (s^{\beta+1} q_\sigma(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty. \tag{59}$$

Proof of theorems 10, 11 and 12. We give a simultaneous proof of all three theorems on the basis of Theorems 5–7 concerning moderately growing regularly varying solutions of the integral asymptotic relation $(\mathrm{AR})_1$.

By hypothesis, there are positive constants k and K such that

$$k q_\sigma(t) \le q(t) \le K q_\sigma(t), \qquad t \ge a. \tag{60}$$

Define $X(t)$ by

$$X(t) = \begin{cases} t^2\left[\frac{\alpha-\beta}{2^\alpha \alpha} \int_t^\infty s^{2\beta} q_\sigma(s) ds\right]^{1/(\alpha-\beta)} & \text{if } \sigma = -2\beta - 1 \text{ and (9) holds;} \\ \left[\frac{t^{2\alpha+1} q_\sigma(t)}{\alpha(2-\rho)(\rho-1)^\alpha \rho^\alpha}\right]^{1/(\alpha-\beta)} & \text{if } \sigma \in (-\alpha-\beta-1, -2\beta-1), \\ & \quad \text{where } \rho = \frac{\sigma+2\alpha+1}{\alpha-\beta}; \\ t\left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}} \int_a^t (s^{\beta+1} q_\sigma(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} & \text{if } \sigma = -\alpha - \beta - 1 \text{ and (30) holds} \end{cases} \tag{61}$$

Then, $X(t)$ satisfies for any $b \ge a$ the asymptotic relation

$$\int_b^t \int_b^s \left[\int_r^\infty q_\sigma(u) X(u)^\beta du\right]^{1/\alpha} dr ds \sim X(t), \qquad t \to \infty. \tag{62}$$

Choose $T_0 > a$ so that

$$\int_{T_0}^t \int_{T_0}^s \left[\int_r^\infty q_\sigma(u) X(u)^\beta du\right]^{1/\alpha} dr ds \le 2X(t), \qquad t \ge T_0. \tag{63}$$

We may assume that $X(t)$ is increasing for $t \ge T_0$. Since by (62) with $b = T_0$

$$\int_{T_0}^t \int_{T_0}^s \left[\int_r^\infty q_\sigma(u) X(u)^\beta du\right]^{1/\alpha} dr ds \sim X(t), \qquad t \to \infty,$$

there exists $T_1 > T_0$ such that

$$\int_{T_0}^{t}\int_{T_0}^{s}\left[\int_{r}^{\infty} q_\sigma(u)X(u)^\beta du\right]^{1/\alpha} drds \ge \frac{X(t)}{2}, \qquad t \ge T_1. \tag{64}$$

One may choose positive constants m and M so that

$$m^{\alpha-\beta} \le \frac{k}{2^\alpha}, \qquad M^{\alpha-\beta} \ge 4^\alpha K, \qquad \text{and} \qquad mX(T_1) \le \frac{1}{2}MX(T_0). \tag{65}$$

Let the integral operator F be defined by

$$Fx(t) = x_0 + \int_{T_0}^{t}\int_{T_0}^{s}\left[\int_{r}^{\infty} q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t \ge T_0, \tag{66}$$

where x_0 is a positive constant such that

$$mX(T_1) \le x_0 \le \frac{1}{2}MX(T_0), \tag{67}$$

and let it act on the set

$$X_0 = \{x(t) \in C[T_0,\infty) : mX(t) \le x(t) \le MX(t), t \ge T_0\}$$

which is a closed convex subset of $C[T_0,\infty)$.

(i) $F(X_0) \subset X_0$. Let $x(t) \in X_0$. Then, we obtain

$$Fx(t) \ge x_0 \ge mX(T_1) \ge mX(t) \qquad \text{for } T_0 \le t \le T_1,$$

$$\begin{aligned} Fx(t) &\ge \int_{T_0}^{t}\int_{T_0}^{s}\left[\int_{r}^{\infty} kq_\sigma(u)(mX(u))^\beta du\right]^{1/\alpha} drds \\ &\ge \frac{1}{2}k^{1/\alpha}m^{\beta/\alpha}X(t) \ge mX(t) \qquad \text{for } t \ge T_1, \end{aligned}$$

and

$$\begin{aligned} Fx(t) &\le \frac{1}{2}MX(T_0) + K^{1/\alpha}M^{\beta/\alpha}\int_{T_0}^{t}\int_{T_0}^{s}\left[\int_{r}^{\infty} q_\sigma(u)X(u)^\beta du\right]^{1/\alpha} drds \\ &\le \frac{1}{2}MX(t) + 2K^{1/\alpha}M^{\beta/\alpha}X(t) \le \frac{1}{2}MX(t) + \frac{1}{2}MX(t) \\ &= MX(t) \qquad \text{for } t \ge T_0. \end{aligned}$$

This implies that $Fx(t) \in X_0$.

(ii) $F(X_0)$ is relatively compact. The local uniform boundedness of $F(X_0)$ is a consequence of the inclusion $F(X_0) \subset X_0$. The local equicontinuity of $F(X_0)$ follows from the inequality

$$0 \le (Fx)'(t) \le K^{1/\alpha} M^{\beta/\alpha} \int_{T_0}^{t} \left[\int_{s}^{\infty} q_\sigma(r) X(r)^\beta dr \right]^{1/\alpha} ds, \qquad t \ge T_0$$

which holds for all $x(t) \in X_0$. The Arzela-Ascoli lemma then ensures the relative compactness of $F(X_0)$.

(iii) F is continuous. Let $\{x_n(t)\}$ be a sequence in X_0 converging to $x(t) \in X_0$ uniformly on compact subintervals of $[T_0, \infty)$. Then, by (66) we have

$$|Fx_n(t) - Fx(t)| \le \int_{T_0}^{t} \int_{T_0}^{s} F_n(r) dr ds, \qquad t \ge T_0, \tag{68}$$

where

$$F_n(r) = \left| \left[\int_r^\infty q(u) x_n(u)^\beta du \right]^{1/\alpha} - \left[\int_r^\infty q(u) x(u)^\beta du \right]^{1/\alpha} \right|.$$

To evaluate $F_n(r)$ the two cases $\alpha \ge 1$ and $\alpha < 1$ must be distinguished.

If $\alpha \ge 1$, then applying the inequality $|A^\gamma - B^\gamma| \le |A - B|^\gamma$ ($A > 0$, $B > 0$, $0 < \gamma < 1$), we see that

$$F_n(r) \le \left[\int_r^\infty q(u) |x_n(u)^\beta - x(u)^\beta| du \right]^{1/\alpha},$$

which combined with (68) gives

$$|Fx_n(t) - Fx(t)| \le \int_{T_0}^{t} \int_{T_0}^{s} \left[\int_r^\infty q(u) |x_n(u)^\beta - x(u)^\beta| du \right]^{1/\alpha} dr ds.$$

This implies

$$|Fx_n(t) - Fx(t)| \le \frac{(t - T_0)^2}{2} \left[\int_{T_0}^\infty q(s) |x_n(s)^\beta - x(s)^\beta| ds \right]^{1/\alpha}, \qquad t \ge T_0,$$

and so the Lebesgue dominated convergence theorem ensures that $Fx_n(t) \to Fx(t)$, $n \to \infty$, uniformly on compact subintervals of $[t_0, \infty)$.

If $\alpha < 1$, then using the mean value theorem, we find

$$F_n(r) \le \frac{1}{\alpha} \left(\int_r^\infty q(u) (MX(u))^\beta du \right)^{(1-\alpha)/\alpha} \int_r^\infty q(u) |x_n(u)^\beta - x(u)^\beta| du,$$

which implies that

$$|Fx_n(t) - Fx(t)| \le \frac{(t - T_0)^2}{2\alpha} \left(\int_{T_0}^\infty q(s) (MX(s))^\beta ds \right)^{(1-\alpha)/\alpha} \int_{T_0}^\infty q(s) |x_n(s)^\beta - x(s)^\beta| ds.$$

From this it follows via the Lebesgue dominated convergence theorem that $Fx_n(t) \to Fx(t)$ as $n \to \infty$ uniformly on compact subintervals of $[T_0, \infty)$.

Thus, by the Schauder-Tychonoff fixed point theorem F has a fixed element $x(t) \in X_0$ which satisfies the integral equation

$$x(t) = x_0 + \int_{T_0}^{t} \int_{T_0}^{s} \left[\int_r^\infty q(u) x(u)^\beta du \right]^{1/\alpha} dr ds, \qquad t \geq T_0. \tag{69}$$

Differentiating (69), we conclude that $x(t)$ is a positive solutions of equation (A) such that $mX(t) \leq x(t) \leq MX(t)$ for $t \geq T_0$, which means that $x(t)$ is a nearly regularly varying function of index 2, $\rho = (\sigma + 2\alpha + 1)/(\alpha - \beta)$ $\in (1, 2)$ or 1 according to whether $\sigma = -2\beta - 1$, $\sigma \in (-\alpha - \beta - 1, -2\beta - 1)$ or $\sigma = -\alpha - \beta - 1$. This completes the proof of Theorems 10, 11 and 12.

REMARK 1. If $\sigma = -\alpha - \beta - 1$, (30) is equivalent to the negation of (15), i.e.,

$$\int_a^\infty (t^{\beta+1} q(t))^{1/\alpha} dt = \infty \quad \Leftrightarrow \quad \int_a^\infty \left[\int_t^\infty s^\beta q(s) ds \right]^{1/\alpha} dt = \infty.$$

4.2. Existence of strongly decaying non-primitive solutions of (A).

THEOREM 13. *Let $q(t)$ be nearly regularly varying of index σ, that is, $q(t) \asymp q_\sigma(t)$, $t \to \infty$, for some $q_\sigma(t) \in \mathrm{RV}(\sigma)$. Suppose that $\sigma = -2\alpha - 1$ and* (48) *holds. Then, equation* (A) *possesses a nearly slowly varying solution $x(t)$ such that*

$$x(t) \asymp \left[\frac{\alpha - \beta}{\alpha (2\alpha)^{1/\alpha}} \int_t^\infty (s^{\alpha+1} q_\sigma(s))^{1/\alpha} ds \right]^{\alpha/(\alpha-\beta)}, \qquad t \to \infty. \tag{70}$$

THEOREM 14. *Let $q(t)$ be nearly regularly varying of index σ, that is, $q(t) \asymp q_\sigma(t)$, $t \to \infty$, for some $q_\sigma(t) \in \mathrm{RV}(\sigma)$. Suppose that $\sigma < -2\alpha - 1$. Then, equation* (A) *possesses a nearly regularly varying solution $x(t)$ of index $\rho = (\sigma + 2\alpha + 1)/(\alpha - \beta) < 0$ such that*

$$x(t) \asymp \left[\frac{t^{2\alpha+1} q_\sigma(t)}{\alpha (2 - \rho)(1 - \rho)^\alpha (-\rho)^\alpha} \right]^{1/(\alpha-\beta)}, \qquad t \to \infty. \tag{71}$$

PROOF OF THEOREMS 13 AND 14. Define the function $Y(t)$ by

$$Y(t) = \begin{cases} \left[\frac{\alpha-\beta}{\alpha(2\alpha)^{1/\alpha}} \int_t^\infty (s^{\alpha+1} q_\sigma(s))^{1/\alpha} ds \right]^{\alpha/(\alpha-\beta)} & \text{if } \sigma = -2\alpha - 1 \text{ and (48) holds;} \\ \left[\frac{t^{2\alpha+1} q_\sigma(t)}{\alpha(2-\rho)(1-\rho)^\alpha(-\rho)^\alpha} \right]^{1/(\alpha-\beta)} & \text{if } \sigma < -2\alpha - 1, \text{ where } \rho = \frac{\sigma+2\alpha+1}{\alpha-\beta}. \end{cases} \tag{72}$$

Since $Y(t)$ satisfies the relation

$$Y(t) \sim \int_t^\infty \int_s^\infty \left[\int_r^\infty q_\sigma(u) Y(u)^\beta du\right]^{1/\alpha} drds \tag{73}$$

as $t \to \infty$, there exists $T > a$ such that

$$\frac{Y(t)}{2} \leq \int_t^\infty \int_s^\infty \left[\int_r^\infty q_\sigma(u) Y(u)^\beta du\right]^{1/\alpha} drds \leq 2Y(t), \qquad t \geq T. \tag{74}$$

Choose positive constants m and M so that

$$m^{\alpha-\beta} \leq \frac{k}{2^\alpha}, \qquad M^{\alpha-\beta} \geq 2^\alpha K, \tag{75}$$

which is possible because of $\alpha > \beta$, and consider the set X_2 and the integral operator G defined, respectively, by

$$X_2 = \{x(t) \in C[T,\infty) : mY(t) \leq x(t) \leq MY(t), t \geq T\} \tag{76}$$

and

$$Gx(t) = \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t \geq T. \tag{77}$$

It is clear that X_2 is a closed convex subset of the locally convex space $C[T,\infty)$. It can be shown that G is a continuous self-map on X_2 and sends X_2 into a relatively compact subset of $C[T,\infty)$.

(i) $G(X_2) \subset X_2$. If $x(t) \in X_2$, then using (74)–(77), we see that

$$\begin{aligned} Gx(t) &\geq k^{1/\alpha} \int_t^\infty \int_s^\infty \left[\int_r^\infty q_\sigma(u)(mY(u))^\beta du\right]^{1/\alpha} drds \\ &\geq \frac{k^{1/\alpha}}{2} m^{\beta/\alpha} Y(t) \geq mY(t), \end{aligned}$$

and

$$\begin{aligned} Gx(t) &\leq K^{1/\alpha} \int_t^\infty \int_s^\infty \left[\int_r^\infty q_\sigma(u)(mY(u))^\beta du\right]^{1/\alpha} drds \\ &\leq 2K^{1/\alpha} M^{\beta/\alpha} Y(t) \leq MY(t), \end{aligned}$$

for $t \geq T$. This implies that $Gx(t) \in X_2$.

(ii) $G(X_2)$ is relatively compact. The inclusion $G(X_2) \subset X_2$ shows that $G(X_2)$ is uniformly bounded on $[T,\infty)$. The inequality

$$0 \geq (Gx)'(t) \geq -M^{\beta/\alpha} \int_t^\infty \left[\int_s^\infty q(r) Y(r)^\beta dr\right]^{1/\alpha} ds, \qquad t \geq T,$$

holding for all $x(t) \in X_2$ implies that $G(X_2)$ is equicontinuous on $[T, \infty)$. The relative compactness of $G(X_2)$ then follows from the Arzela-Ascoli lemma.

(iii) G is continuous. Letting $\{x_n(t)\}$ be a sequence in X_2 converging as $n \to \infty$ to $x(t) \in X_2$ uniformly on any compact subset of $[T, \infty)$, we have to verify that $Gx_n(t) \to Gx(t)$ as $n \to \infty$ uniformly on compact subintervals of $[T, \infty)$. To this end we need to distinguish the two cases $\alpha \geq 1$ and $\alpha < 1$ in the following manner.

Let $\alpha \geq 1$. Then, we have

$$\begin{aligned} |Gx_n(t) - Gx(t)| &\leq \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)|x_n(u)^\beta - x(u)^\beta| du\right]^{1/\alpha} drds \\ &\leq \int_T^\infty s\left[\int_s^\infty q(r)|x_n(r)^\beta - x(r)^\beta| dr\right]^{1/\alpha} ds, \qquad t \geq T. \end{aligned} \tag{78}$$

Since the function

$$g_n(s) = s\left[\int_s^\infty q(r)|x_n(r)^\beta - x(r)^\beta| dr\right]^{1/\alpha}$$

satisfies

$$g_n(s) \leq s\left(\int_s^\infty q(r)(MY(r))^\beta dr\right)^{1/\alpha}$$

which is integrable over $[T, \infty)$, and $g_n(s) \to 0$ as $n \to \infty$ for each $s \geq T$, we are able to apply the Lebesgue convergence theorem to (78), concluding that $Gx_n(t) \to Gx(t)$, $n \to \infty$, uniformly on $[T, \infty)$.

Let $\alpha < 1$. The, we have

$$\begin{aligned} &|Gx_n(t) - Gx(t)| \\ &\quad \leq \frac{1}{\alpha}\int_t^\infty \int_s^\infty \left(\int_r^\infty q(u)(MY(u))^\beta du\right)^{(1-\alpha)/\alpha} \int_r^\infty q(u)|x_n(u)^\beta - x(u)^\beta| dudrds \\ &\quad \leq \frac{1}{\alpha}\int_T^\infty s\left(\int_s^\infty q(r)(MY(r))^\beta dr\right)^{(1-\alpha)/\alpha} \\ &\qquad \times \int_s^\infty q(r)|x_n(r)^\beta - x(r)^\beta| drds, \qquad t \geq T. \end{aligned} \tag{79}$$

The function

$$h_n(t) = s\left(\int_s^\infty q(r)(MY(r))^\beta dr\right)^{(1-\alpha)/\alpha} \int_s^\infty q(r)|x_n(r)^\beta - x(r)^\beta| dr$$

is bounded from above by

$$s\left(\int_s^\infty q(r)(MY(r))^\beta dr\right)^{1/\alpha}$$

which is integrable on $[T,\infty)$, and tends to 0 as $n\to\infty$ at each $s\ge T$, we conclude via the Lebesgue convergence theorem that $Gx_n(t)\to Gx(t)$ uniformly on any compact subinterval of $[T,\infty)$.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and there exists $x(t)\in X_2$ such that $x(t)=Gx(t)$ for $t\ge T$, that is,

$$x(t)=\int_t^\infty\int_s^\infty\left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t\ge T. \tag{80}$$

Differentiating (80) three times, we conclude that $x(t)$ is a solution of equation (A) satisfying $mY(t)\le x(t)\le MY(t)$ for $t\ge T$. From (72) it follows that the solution $x(t)$ is nearly regularly varying function of index 0 or of index $\rho=(\sigma+2\alpha+1)/(\alpha-\beta)<0$ according to whether $\sigma=-2\alpha-1$ or $\sigma<-2\alpha-1$. This completes the proof.

Remark 2. If $\sigma=-2\alpha-1$, (48) is equivalent to (17), i.e.,

$$\int_a^\infty (t^{\alpha+1}q(t))^{1/\alpha}dt<\infty \quad\Leftrightarrow\quad \int_a^\infty t\left[\int_t^\infty q(s)ds\right]^{1/\alpha}dt<\infty.$$

5. Regularly varying solutions of (A)

Our purpose in this section is to demonstrate that in the case where the coefficient $q(t)$ in (A) is a regularly varying function, the existence of regularly varying solutions of index $\rho\in(-\infty,0]\cup[1,2]$ can be completely characterized and, moreover, the exact asymptotic behavior of these solutions can be described explicitly by the unique asymptotic formula. This can be done with the help of the following generalization of the L'Hospital rule (see Haupt and Aumann [2]). The use of this lemma was suggested by J. Manojlović.

Lemma 4. *Let $f,g\in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t)=\lim_{t\to\infty} g(t)=\infty \quad \text{and} \quad g'(t)>0 \qquad \text{for all large } t,$$

or

$$\lim_{t\to\infty} f(t)=\lim_{t\to\infty} g(t)=0 \quad \text{and} \quad g'(t)<0 \qquad \text{for all large } t.$$

Then

$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty}\frac{f(t)}{g(t)}, \qquad \limsup_{t\to\infty}\frac{f(t)}{g(t)} \le \limsup_{t\to\infty}\frac{f'(t)}{g'(t)}.$$

First we characterize the existence of regularly varying solutions which grow moderately at infinity.

THEOREM 15. *Let $q(t)$ be regularly varying of index σ. Then, equation* (A) *possesses nontrivial regularly varying solutions of index* 2 *if and only if $\sigma = -2\beta - 1$ and* (9) *holds, in which case the asymptotic behavior of any such solution $x(t)$ is governed by the formula* (27).

THEOREM 16. *Let $q(t)$ be regularly varying of index σ. Then, equation* (A) *possesses regularly varying solutions of index $\rho \in (1,2)$ if and only if $\sigma \in (-\alpha - \beta - 1, -2\beta - 1)$, in which case ρ is given by* (28) *and the asymptotic behavior of any such solution $x(t)$ is governed by* (29).

THEOREM 17. *Let $q(t)$ be regularly varying of index σ. Then, equation* (A) *possesses nontrivial regularly varying solutions of index* 1 *if and only if $\sigma = -\alpha - \beta - 1$ and* (30) *holds, in which case the asymptotic behavior of any such solution $x(t)$ is governed by the formula* (31).

PROOF OF THEOREMS 15, 16 AND 17. We give a simultaneous proof of these theorems.

The "only if" parts follow from the "only if" parts of Theorems 5, 6 and 7, respectively, because all moderately growing solutions of equation (A) satisfy the asymptotic relation $(\mathrm{AR})_1$.

To prove the "if" parts, suppose that σ and $q(t)$ satisfy the conditions specified in these theorems. We use the function $X(t)$ defined by (43). From Theorems 10, 11 and 12 applied to the special case of (A) where $q(t) \in \mathrm{RV}(\sigma)$ (i.e. $q(t) = q_\sigma(t)$) we see that equation (A) possesses nearly regularly varying solutions $x(t)$ which are obtained as solutions of the integral equation

$$x(t) = x_0 + \int_{T_0}^{t}\int_{T_0}^{s}\left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t \ge T_0, \tag{81}$$

(cf. (69)) satisfying the inequality

$$mX(t) \le x(t) \le MX(t), \qquad t \ge T_0, \tag{82}$$

for some suitably chosen positive constants T_0, x_0, m and M. Define $J(t)$ by

$$J(t) = \int_{T_0}^{t}\int_{T_0}^{s}\left[\int_r^\infty q(u)X(u)^\beta du\right]^{1/\alpha} drds, \qquad t \ge T_0, \tag{83}$$

which satisfies (cf. (44))

$$J(t) \sim X(t), \qquad t \to \infty. \tag{84}$$

Put

$$l = \liminf_{t\to\infty} \frac{x(t)}{J(t)}, \qquad L = \limsup_{t\to\infty} \frac{x(t)}{J(t)}. \tag{85}$$

From (81) and (82) it follows that $0 < l \le L < \infty$.

Repeated application of the generalized L'Hospital rule gives

$$\begin{aligned} l = \liminf_{t\to\infty} \frac{x(t)}{J(t)} &\ge \liminf_{t\to\infty} \frac{x'(t)}{J'(t)} = \liminf_{t\to\infty} \frac{\int_{T_0}^{t} [\int_s^\infty q(r)x(r)^\beta dr]^{1/\alpha} ds}{\int_{T_0}^{t} [\int_s^\infty q(r)X(r)^\beta dr]^{1/\alpha} ds} \\ &\ge \liminf_{t\to\infty} \left[\frac{\int_t^\infty q(s)x(s)^\beta ds}{\int_t^\infty q(s)X(s)^\beta ds}\right]^{1/\alpha} = \left[\liminf_{t\to\infty} \frac{\int_t^\infty q(s)x(s)^\beta ds}{\int_t^\infty q(s)X(s)^\beta ds}\right]^{1/\alpha} \\ &\ge \left[\liminf_{t\to\infty} \frac{q(t)x(t)^\beta}{q(t)X(t)^\beta}\right]^{1/\alpha} = \left[\liminf_{t\to\infty} \frac{x(t)}{X(t)}\right]^{\beta/\alpha} = \left[\liminf_{t\to\infty} \frac{x(t)}{J(t)}\right]^{\beta/\alpha} = l^{\beta/\alpha}. \end{aligned}$$

Note that in the last step we have used (84). Since $l > 0$ is finite and $\alpha > \beta$ the inequality $l \ge l^{\beta/\alpha}$ implies

$$1 \le l < \infty. \tag{86}$$

Similarly, it can be shown that

$$0 < L \le 1. \tag{87}$$

From (86) and (87) it follows that $l = L = 1$, that is, $\lim_{t\to\infty} x(t)/J(t) = 1$. Therefore, in view of (84) we conclude that

$$x(t) \sim J(t) \sim X(t), \qquad t \to \infty,$$

which establishes the regularity of $x(t)$ and the validity of the desired precise asymptotic formula for $x(t)$ simultaneously.

In the next two theorems we characterize the existence of regularly varying strongly decaying solutions of (A) where $q(t) \in \mathrm{RV}(\sigma)$.

Theorem 18. *Let $q(t)$ be regularly varying of index σ. Then, equation* (A) *possesses nontrivial slowly varying solutions if and only if $\sigma = -2\alpha - 1$ and* (48) *holds, in which case the asymptotic behavior of any such solution $x(t)$ is governed by the formula* (49).

Theorem 19. *Let $q(t)$ be regularly varying of index σ. Then, equation* (A) *possesses regularly varying solutions of index $\rho < 0$ if and only if $\sigma < -2\alpha - 1$,* *in which case* ρ *is given by* (28) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by* (50).

Proof of theorems 18 and 19. We give a simultaneous proof of both theorems.

(The "only if" parts) Notice that all strongly decaying solutions of equation (A) satisfy the asymptotic relation $(\mathrm{AR})_2$ and apply the "only if" parts of Theorems 8 and 9.

(The "if" parts) Suppose that σ and $q(t)$ satisfy the conditions specified in the theorems. We use the function $Y(t)$ defined by (72). From Theorems 13 and 14 applied to the special case of (A) where $q(t) \in \mathrm{RV}(\sigma)$ (i.e. $q(t) = q_\sigma(t)$) we see that equation (A) possesses nearly regularly varying solutions $x(t)$ which are obtained as solutions of the integral equation

$$x(t) = \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)x(u)^\beta du\right]^{1/\alpha} drds, \qquad t \geq T, \tag{88}$$

(cf. (80)) satisfying the inequality

$$mY(t) \leq x(t) \leq MY(t), \qquad t \geq T, \tag{89}$$

where T, m and M are suitably chosen positive constants. Define $K(t)$ by

$$K(t) = \int_t^\infty \int_s^\infty \left[\int_r^\infty q(u)Y(u)^\beta du\right]^{1/\alpha} drds, \qquad t \geq T, \tag{90}$$

which satisfies (cf. (73))

$$K(t) \sim Y(t), \qquad t \to \infty. \tag{91}$$

Put

$$\lambda = \liminf_{t\to\infty} \frac{x(t)}{K(t)}, \qquad \Lambda = \limsup_{t\to\infty} \frac{x(t)}{K(t)}. \tag{92}$$

From (88) and (89) it follows that $0 < \lambda \leq \Lambda < \infty$.

Repeated application of the generalized L'Hospital rule gives

$$\lambda = \liminf_{t\to\infty} \frac{x(t)}{K(t)} \geq \liminf_{t\to\infty} \frac{x'(t)}{K'(t)} = \liminf_{t\to\infty} \frac{\int_t^\infty [\int_s^\infty q(r)x(r)^\beta dr]^{1/\alpha} ds}{\int_t^\infty [\int_s^\infty q(r)Y(r)^\beta dr]^{1/\alpha} ds}$$

$$\geq \liminf_{t\to\infty} \left[\frac{\int_t^\infty q(s)x(s)^\beta ds}{\int_t^\infty q(s)Y(s)^\beta ds}\right]^{1/\alpha} = \left[\liminf_{t\to\infty} \frac{\int_t^\infty q(s)x(s)^\beta ds}{\int_t^\infty q(s)Y(s)^\beta ds}\right]^{1/\alpha}$$

$$\geq \left[\liminf_{t\to\infty} \frac{q(t)x(t)^\beta}{q(t)Y(t)^\beta}\right]^{1/\alpha} = \left[\liminf_{t\to\infty} \frac{x(t)}{Y(t)}\right]^{\beta/\alpha} = \left[\liminf_{t\to\infty} \frac{x(t)}{K(t)}\right]^{\beta/\alpha} = \lambda^{\beta/\alpha}.$$

Note that (91) has been used in the last step. Since $\lambda > 0$ is finite and $\alpha > \beta$ the inequality $\lambda \geq \lambda^{\beta/\alpha}$ implies

$$1 \leq \lambda < \infty. \tag{93}$$

Similarly, it can be shown that

$$0 < \Lambda \leq 1. \tag{94}$$

From (93) and (94) it follows that $\lambda = \Lambda = 1$, that is, $\lim_{t\to\infty} x(t)/K(t) = 1$. Therefore, in view of (91) we conclude that

$$x(t) \sim K(t) \sim Y(t), \qquad t \to \infty.$$

This shows that $x(t)$ is regularly varying and enjoys the precise asymptotic behavior as formulated in the theorems.

6. Concluding remarks and examples

Remark 3. We are now able to answer (at least partially) the questions (i) and (ii) raised in Section 1.

(i) Let $q(t)$ be a regularly varying function. According to the results in Section 3, equation (A) may possess regularly varying solutions which are strongly decaying only when (2) does not hold. Therefore, if (2) holds, (A) cannot have nonoscillatory solutions $x(t)$ such that $|x(t)|$ is regularly varying and tending to 0 as $t \to \infty$. We are tempted to conjecture that in this case all proper solutions of equation (A) are oscillatory if and only if (2) holds.

(ii) If (2) fails to hold (i.e., the condition (9) is satisfied), then the existence of trivial regularly varying solutions of indices 0, 1 and 2 of (A) is completely characterized by Theorems 1, 2 and 3, where the coefficient $q(t)$ is a general positive continuous function and does not need to be regularly varying. On the other hand, if we limit ourself to the special case where $q(t)$ is assumed to vary regularly at infinity, the in-depth analysis carried out in the preceding sections shows that we can obtain a series of new results on the existence and precise asymptotic behavior of nontrivial regularly varying solutions of equation (A). Summarizing and combining these results, we are able to draw fairly precise and clear picture of the overall structure of the set of positive regularly varying solutions of equation (A). In particular, we can determine whether or not the coexistence of trivial and nontrivial regularly varying solutions of index $j \in \{0, 1, 2\}$ takes place for (A).

Example 1. Let $0 < \beta < \alpha$ and consider the equation

$$(|x''|^{\alpha-1}x'')' + q_1(t)|x|^{\beta-1}x = 0, \tag{A$_1$}$$

where

$$q_1(t) \sim \frac{2\alpha}{t^{2\alpha+1}(\log t)^{\alpha}(\log\log t)^{2\alpha-\beta}}, \qquad t \to \infty.$$

It is easy to see that the function $q_1(t) \in \mathrm{RV}(-2\alpha-1)$ satisfies

$$\int_t^{\infty} (s^{\alpha+1}q_1(s))^{1/\alpha} ds \sim \frac{(2\alpha)^{1/\alpha}\alpha}{\alpha-\beta}(\log\log t)^{(\beta-\alpha)/\alpha}, \qquad t \to \infty.$$

Hence Theorem 18 ensures the existence of nontrivial slowly varying solutions $x(t)$ of (A_1) all of which have the unique asymptotic behavior

$$x(t) \sim \frac{1}{\log\log t}, \qquad t \to \infty.$$

Example 2. Let $0 < \beta < \alpha$ and consider the equation

$$\begin{aligned} &(|x''|^{\alpha-1}x'')' + q_2(t)|x|^{\beta-1}x = 0, \\ &q_2(t) \sim \frac{\alpha}{t^{\alpha+\beta+1}(\log t)^{\alpha}(\log\log t)^{\beta}}, \qquad t \to \infty. \end{aligned} \tag{A$_2$}$$

The function $q_2(t) \in \mathrm{RV}(-\alpha-\beta-1)$ satisfies

$$\begin{aligned} \left[\frac{\alpha-\beta}{\alpha^{1+1/\alpha}}\int_a^t (s^{\beta+1}q_2(s))^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} &\sim \left[\frac{\alpha-\beta}{\alpha}\int_a^t \frac{ds}{s\log s(\log\log s)^{\beta/\alpha}}\right]^{\alpha/(\alpha-\beta)} \\ &\sim \log\log t \end{aligned}$$

as $t \to \infty$, where $a = \exp(e)$, and so from Theorem 17 it follows that equation (A_2) possesses moderately growing solutions which are regularly varying of index 1. All such solutions $x(t)$ have the unique asymptotic behavior

$$x(t) \sim t\log\log t, \qquad t \to \infty.$$

Example 3. Let $0 < \beta < \alpha$ and consider the equation

$$(|x''|^{\alpha-1}x'')' + q_3(t)|x|^{\beta-1}x = 0, \qquad q_3(t) \sim \frac{2^{\alpha}\alpha}{t^{2\beta+1}(\log t)^{\alpha-\beta+1}}, \qquad t \to \infty. \tag{A$_3$}$$

As easily checked, the function $q_3(t) \in \mathrm{RV}(-2\beta-1)$ satisfies

$$\begin{aligned} \left[\frac{\alpha-\beta}{2^{\alpha}\alpha}\int_t^{\infty} s^{2\beta}q_3(s) ds\right]^{1/(\alpha-\beta)} &\sim \left[(\alpha-\beta)\int_t^{\infty} \frac{ds}{s(\log s)^{\alpha-\beta+1}}\right]^{1/(\alpha-\beta)} \\ &\sim \frac{1}{\log t}, \qquad t \to \infty, \end{aligned}$$

and so Theorems 15 ensures the existence of moderately growing solutions which are regularly varying of index 2. All such solutions $x(t)$ obey the unique asymptotic formula

$$x(t) \sim \frac{t^2}{\log t}, \qquad t \to \infty.$$

Example 4. Let $0 < \beta < \alpha$ and consider the equation

$$(|x''|^{\alpha-1}x'')' + q_4(t)|x|^{\beta-1}x = 0, \qquad q_4(t) = t^{\sigma} \exp(\delta(\log t)^{1/3} \cos(\log t)^{1/3}), \quad (A_4)$$

where σ and δ are constants.

(i) Let $\sigma = -2\alpha - 1 - \frac{\alpha-\beta}{2\alpha+1}$ which is the regularity index of $q_4(t)$. Note that $\sigma < -2\alpha - 1$ and the constant ρ defined by (28) is $\rho = -\frac{1}{2\alpha+1}$, and

$$(2-\rho)(1-\rho)^{\alpha}(-\rho)^{\alpha} = \frac{2^{\alpha}(4\alpha+3)(\alpha+1)^{\alpha}}{(2\alpha+1)^{2\alpha+1}}.$$

By Theorem 19 there exist strongly decaying solutions $x(t)$ of equation (A_4) which are regularly varying of index $-\frac{1}{2\alpha+1}$ and behave like

$$x(t) \sim \left[\frac{(2\alpha+1)^{2\alpha+1}}{2^{\alpha}\alpha(4\alpha+3)(\alpha+1)^{\alpha}}\right]^{1/(\alpha-\beta)}$$
$$\times\, t^{-1/(2\alpha+1)} \exp\left[\frac{\delta}{\alpha-\beta}(\log t)^{1/3} \cos(\log t)^{1/3}\right], \qquad t \to \infty.$$

(ii) Let $\sigma = -2\alpha - 1 + \frac{3}{2}(\alpha - \beta)$ which is the regularity index of $q_4(t)$. Note that $\sigma \in (-\alpha - \beta - 1, -2\beta - 1)$ and the constant ρ defined by (28) is $\rho = \frac{3}{2}$, and

$$\alpha(2-\rho)(\rho-1)^{\alpha}\rho^{\alpha} = \frac{\alpha}{2}\left(\frac{3}{4}\right)^{\alpha}.$$

Therefore, Theorem 16 shows that equation (A_4) has moderately growing solutions $x(t)$ which are regularly varying of index $\frac{3}{2}$ and behave like

$$x(t) \sim \left(\frac{2^{2\alpha+1}}{3^{\alpha}\alpha}\right)^{1/(\alpha-\beta)} t^{3/2} \exp\left[\frac{\delta}{\alpha-\beta}(\log t)^{1/3} \cos(\log t)^{1/3}\right], \qquad t \to \infty.$$

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