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An algorithmic approach to Hurwitz equivalences

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ABSTRACT. For a group G with a certain positive presentation, we provide a criterion for two tuples of generators of G to be Hurwitz equivalent. Based on this criterion, we present an algorithmic approach to solve the Hurwitz equivalence and the Hurwitz search problems by using the word reversing method.

1. Introduction

Let B_n be the braid group of n-strands and $\sigma_1, \dots, \sigma_{n-1}$ be the standard generators. For a group G, we denote by G^n the n-fold direct product of G and we call an element of G^n a *G-system* of length n. For a fixed positive presentation $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ of G, we call an element of $\mathscr{S}^n$, a G-system consisting of positive generators $\mathscr{S}$, a *generator G-system*.

The *Hurwitz action* is a right action of B_n on G^n defined by

$$(g_1, g_2, \dots, g_n) \cdot \sigma_i = (g_1, g_2, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_n).$$

The Hurwitz action is diagrammatically represented as in Figure 1. Two G-systems $\mathbf{g}$ and $\mathbf{g}'$ are called *Hurwitz equivalent* if they belong to the same orbit of the Hurwitz action and denote by $\mathbf{g} \sim_H \mathbf{g}'$.

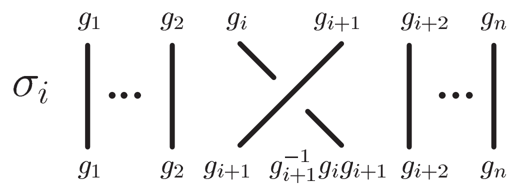

Fig. 1. Diagrammatic description of Hurwitz action

In this paper we study the following two problems.

Hurwitz equivalence problem: Given two G-systems, determine whether they are Hurwitz equivalent or not.

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Hurwitz search problem: Given two Hurwitz equivalent G-systems $\mathbf{g}$ and $\mathbf{g}'$, find a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$.

These problems are very hard compared to the well-known word or conjugacy problems. Lieberman-Teicher showed that these problems are undecidable even for the braid groups [6].

Although the Hurwitz equivalence/search problems are purely algebraic problems, they are closely related to geometry and topology. By considering certain monodromy representations [1] associated to singular points, many geometric objects in 4-dimensional topology and geometry such as braided surfaces or surface braids [5], Lefschetz fibrations [7], and complex surfaces or complex curves [8] are represented by a G-system for an appropriate group G. Such a G-system representative is not unique, and two G-system represent the same geometric object if and only if they are Hurwitz equivalent. Thus, the Hurwitz equivalence/search problems are directly related to the classification problems these topological or geometric objects.

The aim of this paper is to give an algorithmic approach to the Hurwitz equivalence/search problem using a method of *word-reversing* developed by Dehornoy [2], [3].

In Theorem 2 we give a criterion for two generator G-systems to be Hurwitz equivalent for a certain nice positive presentation which we call a Hurwitz-compatible presentation. This allows us to reduce the Hurwitz equivalence problem to much familiar problem, the word problem in monoids. Based on this observation, we give algorithmic approaches (Algorithm 2, Algorithm 4) to solve the Hurwitz equivalence/search problems for a generator G-system $\mathbf{g}$ and a general G-system $\mathbf{g}'$.

Unfortunately, our algorithms are not algorithms in the strict sense: they often fail to solve the Hurwitz equivalence/search problems. However in successful case our algorithm solves not only Hurwitz equivalence problems but also Hurwitz search problems. We can apply our algorithms to try to test the Hurwitz equivalences for *arbitrary* G-systems, as we will discuss in Section 4.3. For an application of geometry or topology, in many cases one can show two G-systems are *not* Hurwitz equivalent by using certain invariants of corresponding geometric objects. Thus, our algorithmic approach will provide a complementary method to studying Hurwitz equivalences.

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2. Word reversing and complete presentation

In this section we summarize the theory of word reversing and complete presentation due to Dehornoy. For details, see [2] and [3]. We only use the right word reversing and the right complete presentations, so we always drop the word "right".

Let $\mathscr{S} = \{a_1, \dots, a_m\}$ be a finite set and $\mathscr{S}^*$ be the free monoid generated by $\mathscr{S}$. For a word $V \in \mathscr{S}^*$ we denote the length of V with respect to $\mathscr{S}$ by $l(V)$. A *positive relation* is a pair of elements in $\mathscr{S}^*$, denoted by $W \equiv V$. A positive relation $W \equiv V$ is *homogeneous* if $l(V) = l(W)$. A positive relation of the form $aV \equiv aW$ or $Va \equiv Wa$ is called a *reducible relation*: As a group presentation, a reducible relation can be replaced by the simpler relation $V \equiv W$.

A *positive group presentation* is a group presentation of the form $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$, where $\mathscr{R}$ is a set of positive relations. In this paper we will always consider *finite* positive presentations, that is, we always assume both $\mathscr{S}$ and $\mathscr{R}$ are finite sets.

Each positive relation $V \equiv W$ is understood as a group relation $V^{-1}W$. We say $\mathscr{P}$ is *homogeneous* if all relations are homogeneous. The *associated monoid* $M_{\mathscr{P}}^+$ is a monoid $\mathscr{S}^*/\equiv$, where $\equiv$ is the smallest congruence on $\mathscr{S}^*$ that includes $\mathscr{R}$.

Definition 1 (Word reversing). Let W and W' be a word on $\mathscr{S} \cup \mathscr{S}^{-1}$. We say the word W' is obtained from W by performing one *word reversing* if one of the following holds.

(1) W' is obtained from W by replacing a subword of the form $u^{-1}v$ with a subword $u'v'^{-1}$, where u, v are nonempty words on $\mathscr{S}$ and u', v' are word on $\mathscr{S}$ possibly an empty word, such that the positive relation $uu' \equiv vv'$ is contained in $\mathscr{R}$.

(2) W' is obtained from W by deleting a subword of the form $u^{-1}u$ where u is a nonempty word on $\mathscr{S}$.

Diagrammatically, the word reversing is expressed as in Figure 2.

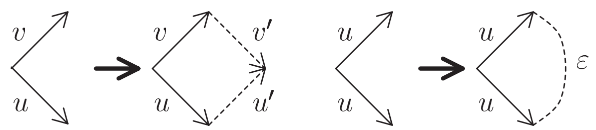

Fig. 2. Diagrammatic description of word reversing

We say a word W on $\mathscr{S} \cup \mathscr{S}^{-1}$ is *reversible* to a word W' on $\mathscr{S} \cup \mathscr{S}^{-1}$ if W' is obtained from W by iterated applications of word reversing. We denote by $W \curvearrowright W'$ if W is reversible to W'.

For $u, v \in \mathscr{S}^*$, $u^{-1}v \curvearrowright \varepsilon$ implies $u \equiv v$ [2, Proposition 1.9]. In fact, if $u^{-1}v \curvearrowright \varepsilon$ then the word reversing not only shows u and v are congruent but also gives a Van-Kampen diagram of (u, v), which describes a congruence of two words u and v.

Let $W, W' \in \mathscr{S}^*$ be words on $\mathscr{S}$ that are congruent. A *Van-Kampen diagram* of (W, W') is an oriented sub-graph D of the Cayley graph of $M^+_{\mathscr{P}}$ having the following properties.

(1) D has the unique source vertex which corresponds to an element 1, and the unique sink vertex which corresponds to an element $W = W' \in M^+_{\mathscr{P}}$.

(2) D is a planer graph, and bounded by two edge paths defined by the word W and W'. (In particular, D defines a cellular decomposition $\mathscr{T}_D$ of a 2-disc).

(3) The labeling of the boundary of each 2-cell in $\mathscr{T}_D$ is a relation in $\mathscr{R}$. That is, the labeling is of the form $u^{-1}v$ and the relation $u \equiv v$ lies in $\mathscr{R}$.

See Figure 3 for example. Once a Van-Kampen diagram of (W, W') is constructed, one can find how to change the word W into W' by using the relations in $\mathscr{R}$. That is, one can find a sequence of words on $\mathscr{S}$

$$W = W_0 \to W_1 \to \cdots \to W_{k-1} \to W_k = W'$$

where each W_{i+1} is obtained from W_i by performing a relation in $\mathscr{R}$.

Recall the diagrammatic expression of word-reversing in Figure 2. The word reversing is considered as an operation to glue a 2-cell along paths $u^{-1}v$, or an operation to identify two 1-cells having the same label. Thus, from a diagrammatic expression of word reversing, one obtains a Van-Kampen diagram for (u, v).

Example 1. Let us consider a positive presentation $\mathscr{P}_1$ of the braid group B_3,

$$\mathscr{P}_1 = \langle \mathscr{S} | \mathscr{R} \rangle = \langle x, y, z \mid xyx \equiv yxy, xy \equiv yz \equiv zx \rangle.$$

Here the relation $xy \equiv yz \equiv zx$ is understood as the three relations $xy \equiv yz$, $yz \equiv zx$ and $xy \equiv zx$. Let us reverse the word $(xxyx)^{-1}zxyz$.

$$x^{-1}y^{-1}x^{-1}\underline{x^{-1}z}xyz \curvearrowright^{(1)} x^{-1}y^{-1}x^{-1}yx^{-1}xyz$$
$$x^{-1}y^{-1}x^{-1}y\underline{x^{-1}x}yz \curvearrowright^{(2)} x^{-1}y^{-1}x^{-1}yyz$$
$$x^{-1}\underline{y^{-1}x^{-1}y}yz \curvearrowright^{(3)} x^{-1}xy^{-1}x^{-1}yz$$
$$\underline{x^{-1}x}y^{-1}x^{-1}yz \curvearrowright^{(4)} y^{-1}x^{-1}yz$$
$$\underline{y^{-1}x^{-1}yz} \curvearrowright^{(5)} \varepsilon.$$

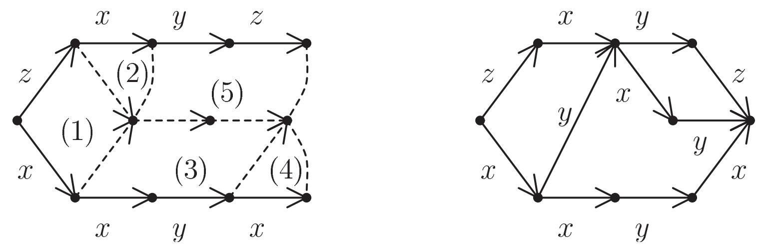

Fig. 3. Construction of Van-Kampen Diagram of $(xxyx, zxyz)$

According to this sequence of word reversing, we attach 2-cells or identify 1-cells to get a Van-Kampen diagram of $(xxyx, zxyz)$ shown in Figure 3. From this Van-Kampen diagram, we read a sequence of words

$$xxyx \to xyxy \to zxxy \to zxyz$$

which converts the word $xxyx$ to $zxyz$ by using the relations in $\mathscr{R}$.

In general a word reversing is not sufficient to detect congruence relations since $u \equiv v$ does not always imply $u^{-1}v \curvearrowright \varepsilon$. A *complete presentation* is a positive presentation such that the converse is true.

Definition 2 (Complete positive group presentation). A positive group presentation $\langle \mathscr{S} | \mathscr{R} \rangle$ is *complete* if $u^{-1}v \curvearrowright \varepsilon$ is equivalent to $u \equiv v$ for all $u, v \in \mathscr{S}^*$.

There is a nice characterization of complete presentations for a finite positive homogeneous presentation. This allows us to check whether a given homogeneous finite presentation $\mathscr{P}$ is complete or not.

Theorem 1 ([2], Proposition 4.4). *A finite positive homogeneous presentation $\langle \mathscr{S} | \mathscr{R} \rangle$ is complete if and only if the condition $SC(\mathscr{S})$ (called the strong cube condition on $\mathscr{S}$) holds.*

$SC(\mathscr{S})$: *For $s, r, t \in \mathscr{S}$ and $u, v \in \mathscr{S}^*$, if $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$ then $(su)^{-1}(tv) \curvearrowright \varepsilon$.*

Based on the strong cube condition, one can try to make a non-complete finite homogeneous positive presentation complete, without changing the associated monoid as follows. Assume that the strong cube condition fails for some s, r, t, u, v. That is, $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$ but $(su)^{-1}(tv) \not\curvearrowright \varepsilon$. Then we add a new relation $su \equiv tv$ so that the strong cube condition is satisfied for such s, r, t, u, v. Adding a new relation may produce a new word reversing sequence, so the new presentation is not necessarily complete. We may iterate this operation to try to get a complete presentation. The precise algorithm is given as Algorithm 1. This algorithm does not necessarily terminate, so this is not an algorithm in a strict sense.

Algorithm 1: Presentation Completion Algorithm

Input: A finite homogeneous positive presentation $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ of a group G.
Output: A complete presentation of G.

(1) Compute all pairs of words $u, v \in \mathscr{S}^*$ such that $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$ for some $s, r, t \in \mathscr{S}$.

(2) Check $(su)^{-1}(tv) \curvearrowright \varepsilon$ holds for all u, v obtained by Step (1). If $(su)^{-1}(tv) \not\curvearrowright \varepsilon$, then replace the presentation $\mathscr{P}$ with the new presentation

$$\langle \mathscr{S} | \mathscr{R} \cup \{su \equiv tv\} \rangle$$

and go back to Step (1).

(3) Stop.

Example 2. Let us consider a presentation $\mathscr{P}_0$ of the braid group B_3 given by

$$\mathscr{P}_0 = \langle x, y, z \,|\, xyx \equiv yxy, xy \equiv yz \rangle.$$

Observe that $y^{-1}xx^{-1}y \curvearrowright xyx^{-1}z^{-1}$, but $(yxy)^{-1}yzx \not\curvearrowright \varepsilon$. So we add a new relation $yxy \equiv yzx$ to $\mathscr{P}_0$ and obtain the new presentation

$$\mathscr{P}_0' = \langle x, y, z \,|\, xyx \equiv yxy, xy \equiv yz, yxy \equiv yzx \rangle.$$

In $\mathscr{P}_0'$, a new word reversing sequence $x^{-1}yy^{-1}x \curvearrowright (yxxy)(yxzx)^{-1}$ appears. The reversing of the word $(xyxxy)^{-1}(xyxzx)$ eventually arrives at the word of the form $\dots z^{-1}xzx$, $\dots x^{-1}zx$, or $\dots y^{-1}zx$. Since there are no relations of the form $z\cdots \equiv \cdots$, this shows $(xyxxy)^{-1}(xyxzx) \not\curvearrowright \varepsilon$.

Thus the presentation $\mathscr{P}_0'$ is not complete. We need to add a new relation $xyxzx \equiv xyxxy$. In this case, a similar argument reveals that the completion procedure never terminates.

On the other hand, let us consider another presentation of B_3

$$\mathscr{P}_1 = \langle x, y \,|\, xyx \equiv yxy, xy \equiv yz \equiv zx \rangle$$

used in Example 1. $\mathscr{P}_1$ satisfies the strong cube conditions, so it is complete.

3. Hurwitz equivalence criterion

In this section we provide a criterion for two generator G-systems to be Hurwitz equivalent. To state our results, we introduce the notion of Hurwitz-compatible relations.

Definition 3. Let $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ be a finite homogeneous positive presentation of a group G and $R : V \equiv W$ be a positive relation in $\mathscr{R}$. For words $V = a_1 a_2 \dots a_l$, $W = a_1' \dots a_l'$ on $\mathscr{S}$, let $\mathbf{g}_V$, $\mathbf{g}_W$ be generator G-systems defined by

$$\mathbf{g}_V = (a_1, \dots, a_l), \qquad \mathbf{g}_W = (a_1', \dots, a_l').$$

We say a homogeneous positive relation R is *Hurwitz compatible* if there exists an l-braid β_R such that $\mathbf{g}_V \cdot \beta_R = \mathbf{g}_W$.

We say a finite presentation $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ is *Hurwitz-compatible* if all relations of $\mathscr{R}$ are Hurwitz compatible.

By definition, Hurwitz compatible relations are homogeneous. Knowing a homogeneous relation R is Hurwitz-compatible is difficult, since it is equivalent to solve the Hurwitz search problem. However, there is a class of Hurwitz compatible relations which can be easily recognized.

Definition 4. A *word-conjugacy relation* is a positive relation of the form $R : aV \equiv Va'$, where $a, a' \in \mathscr{S}$ and $V \in \mathscr{S}^*$.

It is directly checked that a word-conjugacy relation is a Hurwitz compatible relation: $\mathbf{g}_{aV} \cdot (\sigma_1 \sigma_2 \dots \sigma_{l(V)}) = \mathbf{g}_{Va'}$.

Observe that there is an obvious and fundamental invariant of Hurwitz equivalence classes. The *Coxeter element* (or, *the global monodromy*) of a G-system $\mathbf{g} = (g_1, \dots, g_m)$ is an element $C(\mathbf{g}) = g_1 g_2 \dots g_m \in G$. It is easy to see if $\mathbf{g} \sim_H \mathbf{g}'$ then $C(\mathbf{g}) = C(\mathbf{g}')$. The Coxeter element serves as a fundamental invariant to study Hurwitz equivalence class. For example, in [4] the author studied Hurwitz equivalence classes for 3-strand braid groups whose Hurwitz orbits are finite by studying the centralizer of the Coxeter element.

For a generator G-system $\mathbf{g}$, we consider a refinement of the Coxeter element. We call the word $g_1 g_2 \dots g_m \in \mathscr{S}^*$ the *Coxeter word* of $\mathbf{g}$ and denote by $W(\mathbf{g})$. The Coxeter word contains more information than the Coxeter element itself, as the next lemma suggests.

Lemma 1. *Let $G = \langle \mathscr{S} | \mathscr{R} \rangle$ be a positively presented group and $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{a}' = (a_1', \dots, a_m')$ be generator G-systems of the same length. If $W(\mathbf{a}')$ is obtained from $W(\mathbf{a})$ by applying a Hurwitz-compatible relation $R : U \equiv V$ in $\mathscr{R}$, then $\mathbf{a}$ and $\mathbf{a}'$ are Hurwitz equivalent.*

Proof. Let us write $W(\mathbf{a}) = XUY$, $W(\mathbf{a}') = XVY$ and $sh : B_l \to B_{l+k}$ be the k-fold shift map defined by $\sigma_i \to \sigma_{i+k}$ where $k = l(X)$ and $l = l(U) = l(V)$. Let $\imath : B_{l+k} \hookrightarrow B_m$ be the natural embedding of B_{l+k}. Assume the relation $R : U \equiv V$ is Hurwitz-compatible, and let β_R be an l braid such that $\mathbf{g}_U \cdot \beta_R = \mathbf{g}_V$. Then, $\mathbf{a} \cdot \imath \circ sh(\beta_R) = \mathbf{a}'$, thus $\mathbf{a}$ and $\mathbf{a}'$ are Hurwitz equivalent.

Theorem 2 below shows the relationships between word reversing and Hurwitz equivalences. It reveals that under some conditions the Coxeter word or the Coxeter element completely determines the Hurwitz equivalence class.

Theorem 2. *Let $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ be a finite homogeneous positive presentation of a group G. Assume that $\mathscr{P}$ is Hurwitz compatible, and let* $\mathbf{a}$, $\mathbf{a}'$ *be generator G-systems of the same length.*

(1) *If $W(\mathbf{a}) \equiv W(\mathbf{a}')$, then $\mathbf{a} \sim_H \mathbf{a}'$.*
(2) *If $W(\mathbf{a})^{-1} W(\mathbf{a}') \curvearrowright \varepsilon$, then $\mathbf{a} \sim_H \mathbf{a}$. Moreover, in this case we can solve the Hurwitz search problem for $\mathbf{a}$ and $\mathbf{a}'$.*
(3) *If $M_{\mathscr{P}}^{+}$ injects in G, then $\mathbf{a} \sim_H \mathbf{a}'$ if and only if $C(\mathbf{a}) = C(\mathbf{a}')$.*
(4) *If $M_{\mathscr{P}}^{+}$ injects in G and the presentation $\mathscr{P}$ is complete, then $\mathbf{a} \sim_H \mathbf{a}'$ if and only if $W(\mathbf{a})^{-1} W(\mathbf{a}') \curvearrowright \varepsilon$. Moreover, in this case we can solve the Hurwitz search problem for $\mathbf{a}$ and $\mathbf{a}'$.*

Proof (Proof of Theorem 2). (1) directly follows from Lemma 1. To prove (2), recall that if $u^{-1} v \curvearrowright \varepsilon$ then word reversing gives a Van-Kampen diagram for (u, v). By using a Van-Kampen diagram we find a sequence of words on $\mathscr{S}$

$$W = W_0 \to W_1 \to \cdots \to W_{k-1} \to W_k = W'$$

where each W_{i+1} is obtained from W_i by applying the relations in $\mathscr{R}$. Let $\mathbf{a}_i$ be the generator G-system of length m whose Coxeter word is W_i. Then by Lemma 1, we can find a braid β_i such that $\mathbf{a}_i \cdot \beta_i = \mathbf{a}_{i+1}$. Thus, $\mathbf{a} \cdot (\beta_0 \beta_1 \dots \beta_{k-1}) = \mathbf{a}'$ so we solved the Hurwitz search problem. If the associated monoid $M_{\mathscr{P}}^{+}$ embeds in G, then $C(\mathbf{a}) = C(\mathbf{a}')$ is equivalent to $W(\mathbf{a}) \equiv W(\mathbf{a}')$, hence (3) follows from (1). Finally (4) follows from (2), (3) and the definition of the complete presentation.

As we have given as Algorithm 1, for a finite homogeneous positive presentation $\mathscr{P}$ one can try to check whether $\mathscr{P}$ is complete or not. Moreover, even if $\mathscr{P}$ is not complete one can try to make $\mathscr{P}$ complete. Thus, one can algorithmically try to show whether two generator G-systems are Hurwitz equivalent or not by using Theorem 2 (1) and (2). This point of view will be pursued in next section.

We remark that in a theory of word-reversing and complete presentation, there is a sufficient conditions for $M_{\mathscr{P}}^{+}$ to inject into G [2, Proposition 7.2]. Thus sometimes one can also apply Theorem 2 (3) (4) to solve Hurwitz equivalence/search problem.

Example 3 (Artin groups). Let $M = (m_{ij})_{1 \le i,j \le m}$ be a Coxeter matrix, which is a symmetric matrix such that $m_{ii} = 1$ and $m_{ij} \in \{2, 3, \dots, \infty\}$ for distinct i and j. The *Artin group G* corresponding to M is a group defined by the positive presentation

$$G = \langle a_1, \dots, a_m \mid R_{ij} \ \ (m_{i,j} \neq \infty) \rangle$$

where R_{ij} is a positive irreducible word conjugacy relation

$$R_{ij} : \underbrace{a_i a_j a_i \dots}_{m_{ij}} \equiv \underbrace{a_j a_i a_j \dots}_{m_{ij}}.$$

We call this presentation the *standard presentation* of an Artin group G. It is known that the associated monoid $M_{\mathscr{P}}^{+}$ of the standard presentation $\mathscr{P}$ of an Artin group G injects in G [9]. Thus, by Theorem 2 (3), a two generator G-system $\mathbf{a}$ and $\mathbf{a}'$, $\mathbf{a} \sim_H \mathbf{a}'$ if and only if $C(\mathbf{a}) = C(\mathbf{a}')$. This implies that for generator G-systems, the Hurwitz equivalence problem is equal to the classical word problem.

4. Algorithm to attack Hurwitz equivalence and Hurwitz search problems

In this section we present an algorithmic approach to solve the Hurwitz equivalence and Hurwitz search problems. In Sections 4.1 and 4.2 we will treat generator G-systems of Hurwitz-compatible finite positive presentation. We present a simple algorithm based on Hurwitz-compatible presentation and word reversing in Section 4.1, and give an improvement using complete presentation in Section 4.2. Finally in Section 4.3 we explain how to apply these algorithms for general case, namely, Hurwitz equivalences for non-generator G-systems of general groups.

4.1. Naive algorithm. First we provide a simple algorithm. This naive version of algorithm still has an advantage compared to the modified algorithm given in Section 4.2, since it requires less computations.

Let $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ be a Hurwitz-compatible finite positive presentation of a group G. Typically we consider the finite presentation such that all relations are word-conjugacy relations. We further assume that both the word and the conjugacy (search) problems of G are solvable.

Let $\mathbf{g} = (g_1, \dots, g_m)$ be a generator G-system and $\mathbf{g}' = (g_1', \dots, g_m')$ be an arbitrary G-system. We try to check whether $\mathbf{g} \sim_H \mathbf{g}'$ or not as follows.

We begin with two simple tests. First we compare the Coxeter elements of $\mathbf{g}$ and $\mathbf{g}'$. If $C(\mathbf{g}) \neq C(\mathbf{g}')$, then $\mathbf{g} \not\sim_H \mathbf{g}'$. Next for each i, we check whether there is a permutation τ of indices such that g_i' is conjugate to $g_{\tau(i)}$ for all $i = 1, 2, \dots, m$. If such a permutation does not exist, then $\mathbf{g} \not\sim_H \mathbf{g}'$. We will call these two tests the *primary test*.

Assume that $\mathbf{g}$ and $\mathbf{g}'$ pass the primary test. The next step is to construct a Hurwitz compatible presentation $\mathscr{P}' = \langle \mathscr{S}' | \mathscr{R}' \rangle$ of G so that both $\mathbf{g}$ and $\mathbf{g}'$ are generator G-systems with respect to the new presentation $\mathscr{P}'$.

Let τ be the permutation obtained from the primary tests. Let us denote by $g_i' = V_i^{-1} g_{\tau(i)} V_i$ where V_i are fixed words on $\mathscr{S} \cup \mathscr{S}^{-1}$, computed in the primary tests. Let $L(i) = l(V_i)$, and write V_i as

$$V_i = a_{n_1^{(i)}}^{\varepsilon_1^{(i)}} a_{n_2^{(i)}}^{\varepsilon_2^{(i)}} \dots a_{n_{L(i)}^{(i)}}^{\varepsilon_{L(i)}^{(i)}}$$

where we put $\mathscr{S} = \{a_1, \dots, a_M\}$ and $n_k^{(i)} \in \{1, 2, \dots, M\}$, $\varepsilon_k^{(i)} \in \{\pm 1\}$.

We introduce new generators $\{g_1', \dots, g_m'\} \cup \{g_{i,j}\}_{i=1,\dots,m, j=1,\dots,L(i)-1}$ and new word conjugacy relations $\{\mathscr{R}_{i,j}\}_{i=1,\dots,m, j=1,\dots,L(i)}$ as follows. For $j = 1$, we define the relation $\mathscr{R}_{i,1}$ as

$$\mathscr{R}_{i,1} : \begin{cases} g_{\tau(i)} a_{n_1^{(i)}} \equiv a_{n_1^{(i)}} g_{i,1} & (\varepsilon_1^{(i)} = +1) \\ a_{n_1^{(i)}} g_{\tau(i)} \equiv g_{i,1} a_{n_1^{(i)}} & (\varepsilon_1^{(i)} = -1). \end{cases}$$

For $1 < j < L(i)$, we define the relation $\mathscr{R}_{i,j}$ as

$$\mathscr{R}_{i,j} : \begin{cases} g_{i,j-1} a_{n_j^{(i)}} \equiv a_{n_j^{(i)}} g_{i,j} & (\varepsilon_j^{(i)} = +1) \\ a_{n_j^{(i)}} g_{i,j-1} \equiv g_{i,j} a_{n_j^{(i)}} & (\varepsilon_j^{(i)} = -1). \end{cases}$$

Finally, for $j = L(i)$, we define the relation $\mathscr{R}_{i,L(i)}$ as

$$\mathscr{R}_{i,j} : \begin{cases} g_{i,L(i)-1} a_{n_{L(i)}^{(i)}} \equiv a_{n_{L(i)}^{(i)}} g_i' & (\varepsilon_{L(i)}^{(i)} = +1) \\ a_{n_{L(i)}^{(i)}} g_{i,L(i)-1} \equiv g_i' a_{n_{L(i)}^{(i)}} & (\varepsilon_{L(i)}^{(i)} = -1). \end{cases}$$

Let us consider the new positive presentation of G,

$$\mathscr{P}' = \langle \mathscr{S} \cup \{g_{i,j}\} \cup \{g_1', \dots, g_m'\} \mid \mathscr{R} \cup \{\mathscr{R}_{i,j}\} \rangle.$$

We call this positive presentation $\mathscr{P}'$ the *expanded presentation*. All of the newly-added relations $\mathscr{R}_{i,j}$ are word-conjugacy relations, so $\mathscr{P}'$ is Hurwitz-compatible.

Now we reverse the word $W(\mathbf{g})^{-1} W(\mathbf{g}')$ by using the presentation $\mathscr{P}'$. The reversing procedure stops in finite time because the expanded presentation is finite and homogeneous. By Theorem 2 (2), if $W(\mathbf{g})^{-1} W(\mathbf{g}') \curvearrowright \varepsilon$ then we conclude $\mathbf{g} \sim_H \mathbf{g}'$, and we are able to read a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$ from a Van-Kampen diagram.

The precise algorithm is given as Algorithm 2. Algorithm 2 returns Undecidable if it fails to determine whether $\mathbf{g} \sim_H \mathbf{g}'$ or not.

Algorithm 2: Hurwitz equivalence and search—Naive algorithm

Input: A finite Hurwitz-compatible presentation $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ of G, a generator G-system $\mathbf{g} = (g_1, \dots, g_m)$, and a G-system $\mathbf{g}' = (g_1', \dots, g_m')$.
Output: The truth value of $\mathbf{g} \sim_H \mathbf{g}'$ or Undecidable. In case of $\mathbf{g} \sim_H \mathbf{g}'$, also return a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$.

(1) If $C(\mathbf{g}) \neq C(\mathbf{g}')$, then return false.
(2) Check whether there is a permutation τ of indices such that g_i' is conjugate to $g_{\tau(i)}$. If such a permutation does not exist, then return false.
(3) Compute an expanded presentation $\mathscr{P}'$ of G.
(4) Check whether $W(\mathbf{g})^{-1} W(\mathbf{g}') \curvearrowright \varepsilon$ or not. If not, then return Undecidable.
(5) If $W(\mathbf{g})^{-1} W(\mathbf{g}') \curvearrowright \varepsilon$, then construct a Van-Kampen diagram for $(W(\mathbf{g}), W(\mathbf{g}'))$ and compute a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$ from the Van-Kampen diagram.
(6) Return true and the braid β.

4.2. A better Algorithm. In Algorithm 2, the word reversing of $W(\mathbf{g})^{-1} W(\mathbf{g}')$ is not sufficient to show $\mathbf{g} \sim_H \mathbf{g}'$ because the word reversing might fail to detect the congruence of $W(\mathbf{g})$ and $W(\mathbf{g}')$. To improve Algorithm 2 we try to make the expanded presentation complete so that it is more likely to succeed in showing $\mathbf{g} \sim_H \mathbf{g}'$.

The modified algorithm goes as follows. The inputs $G = \mathscr{P}$, $\mathbf{g}$, $\mathbf{g}'$ and the first three steps are the same as in Algorithm 2: We do the primary test to exclude obviously non-Hurwitz-equivalent case, and get an expanded presentation $\mathscr{P}'$.

The next step is the core of the modified algorithm. We try to make the expanded presentation $\mathscr{P}'$ complete preserving the property that the presentation is Hurwitz compatible.

We slightly modify Algorithm 1 so that it is more effective for our purposes. Recall that in the completion procedure, we add a new relation $su \equiv tv$ if $s^{-1} r r^{-1} t \curvearrowright uv^{-1}$ but $(su)^{-1}(tv) \not\curvearrowright \varepsilon$.

Since we need Hurwitz-compatible presentation, we must check whether the new relation $su \equiv tv$ is Hurwitz-equivalent or not. Fortunately, adding the relation $su \equiv tv$ does not cause a problem.

Lemma 2. *Assume that $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ is a Hurwitz-compatible group presentation, and take s, r, t, u, v as above. Then the relation $su \equiv tv$ is also Hurwitz-compatible.*

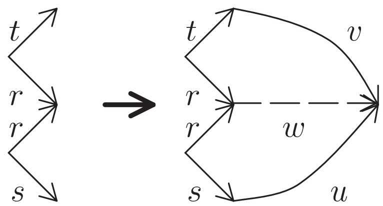

Fig. 4. Van-Kampen-like Diagram from word reversing $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$

Proof. From the reversing sequence $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$, one can construct a diagram which is similar to a Van-Kampen diagram. Indeed, one can find a word w such that this diagram is obtained from two Van-Kampen diagrams of (su, rw) and (rw, tv) by gluing along the path w as shown in Figure 4. Thus, one can read a sequence of words

$$su = W_0 \to W_1 \to \cdots \to W_i = rw \to W_{i+1} \to W_{k-1} \to W_k = tv$$

where each W_{j+1} is obtained from W_j by performing the relation in $\mathscr{R}$. Since all relations $\mathscr{R}$ are Hurwitz-compatible, we find a braid β such that $\mathbf{g}_{su}\beta = \mathbf{g}_{tv}$, where $\mathbf{g}_{su}$, $\mathbf{g}_{tv}$ are generator G-systems whose Coxeter words are su, tv. Thus the relation $su \equiv tv$ is Hurwitz-compatible.

Now we consider the case $s = t$, so the relation $su \equiv tv$ is reducible. As a group relation, $u \equiv v$ is equivalent to $su \equiv tv$. Since to detect the Hurwitz equivalences it is better to use finer congruence relations, it is better to add $u \equiv v$ instead of $su \equiv tv$. Adding the relation $u \equiv v$ also makes the strong cube condition for s, r, t, u, v is satisfied, because $u^{-1}s^{-1}tv \curvearrowright u^{-1}v \curvearrowright \varepsilon$.

However, one problem occurs. We cannot expect the relation $u \equiv v$ is Hurwitz-compatible unlike Lemma 2. So we will do as follows: We add the relation $u \equiv v$ instead of $su \equiv tv$ if we know the relation $u \equiv v$ is Hurwitz-compatible, such as, the case $u \equiv v$ is a word-conjugacy relation. Otherwise, we add the relation $su \equiv tv$.

Summarizing, we modify the completion procedure as follows. Assume that $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$ but $(su)^{-1}(tv) \not\curvearrowright \varepsilon$. Assume that $s = t$ and the relation $u \equiv v$ is a word-conjugacy relation, then we add a new relation $u \equiv v$. Otherwise, we add a new relation $su \equiv tv$. The precise description of the modified completion algorithm is given as Algorithm 3.

Suppose that Algorithm 3 terminates and we obtained a complete Hurwitz compatible presentation $\overline{\mathscr{P}'}$. (As we will explain in Remark 1, in our purpose we are able to modify Algorithm 3 so that it always terminates in finite time, without affecting the output of the algorithm.)

Algorithm 3: Modified Presentation Completion Algorithm

Input: A finite positive homogeneous presentation $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ of a group G such that all relations are Hurwitz-compatible.

Output: A complete, Hurwitz-compatible presentation of G.

(1) Compute all pair of words $u, v \in \mathscr{S}^*$ such that $s^{-1}rr^{-1}t \curvearrowright uv^{-1}$ for some $s, r, t \in \mathscr{S}$.

(2) Check $(su)^{-1}(tv) \curvearrowright \varepsilon$ holds for all u, v obtained by Step (1). Assume that $(su)^{-1}(tv) \not\curvearrowright \varepsilon$ for some u, v.

(a) If $s \neq t$, then replace the presentation $\mathscr{P}$ with the new presentation

$$\langle \mathscr{S} | \mathscr{R} \cup \{su \equiv tv\} \rangle$$

and go back to Step (1).

(b) If $s = t$, then replace the presentation $\mathscr{P}$ with the new presentation

$$\begin{cases} \langle \mathscr{S} | \mathscr{R} \cup \{u \equiv v\} \rangle, & \text{if } u \equiv v \text{ is a word-conjugacy relation,} \\ \langle \mathscr{S} | \mathscr{R} \cup \{su \equiv tv\} \rangle, & \text{if } u \equiv v \text{ is not a word-conjugacy relation} \end{cases}$$

and go back to Step (1). (See Remark 1 below)

(3) Stop.

It should be noted that the monoids $M^{+}_{\mathscr{P}'}$ and $M^{+}_{\overline{\mathscr{P}'}}$ might be different unlike the usual completion procedure described in Algorithm 1.

The rest are the same as the previous algorithm. We reverse the word $W(\mathbf{g})^{-1}W(\mathbf{g}')$ by using the complete presentation $\overline{\mathscr{P}'}$. If $W(\mathbf{g})^{-1}W(\mathbf{g}') \curvearrowright \varepsilon$, then we conclude $\mathbf{g} \sim_H \mathbf{g}'$ and compute a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$ from a Van-Kampen diagram.

The explicit description of the above algorithm is given as Algorithm 4. Algorithm 4 solves the Hurwitz equivalence problem if possible and returns the value Undecidable if it fails to solve. As in Algorithm 2, Undecidable simply means we can not solve the problem using this algorithm, so it does not imply the problem is undecidable.

Remark 1. Recall that we wanted a complete presentation so that word-reversing effectively detects the congruence of $W(\mathbf{g})$ and $W(\mathbf{g}')$. Since the presentation $\mathscr{P}$ is homogeneous, to detect the congruence of Coxeter words $W(\mathbf{g})$ and $W(\mathbf{g}')$ we do not need all congruence relations: It is sufficient to know the relations of length $\leq l$ where l be the length of $\mathbf{g}$.

Thus in our purpose, step (2) in Algorithm 3 (step (4) in Algorithm 4) can be simplified so that the Algorithm 4 terminates in finite time as follows: In step (2) of Algorithm 3, if the length of newly-added relations $su \equiv tv$ or $u \equiv v$ become bigger than the length of $\mathbf{g}$, then we stop the completion procedure and go directly to step (5) of Algorithm 4.

Algorithm 4: Hurwitz equivalence and search—modified algorithm

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Input: A finite positive group presentation $\mathscr{P} = \langle \mathscr{S} | \mathscr{R} \rangle$ of G such that all relations in $\mathscr{R}$ are Hurwitz-compatible, a generator G-system $\mathbf{g} = (g_1, \dots, g_m)$, and a G-system $\mathbf{g}' = (g'_1, \dots, g'_m)$.
Output: The truth value of $\mathbf{g} \sim_H \mathbf{g}'$ or Undecidable. In case of $\mathbf{g} \sim_H \mathbf{g}'$, then also return a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$.

(1) If $C(\mathbf{g}) \neq C(\mathbf{g}')$, then return false.
(2) Check whether there is a permutation τ of indices such that g'_i is conjugate to $g_{\tau(i)}$. If such a permutation does not exist, then return false.
(3) Compute the expanded presentation $\mathscr{P}'$ of G.
(4) Make the expanded presentation $\mathscr{P}'$ complete by using modified completion procedure (Algorithm 3) (See Remark 1 below.)
(5) Check whether $W(\mathbf{g})^{-1} W(\mathbf{g}') \curvearrowright \varepsilon$ or not. If $W(\mathbf{g})^{-1} W(\mathbf{g}') \not\curvearrowright \varepsilon$, then return undecidable.
(6) Compute a Van-Kampen diagram for $(W(\mathbf{g}), W(\mathbf{g}'))$ using word-reversing.
(7) Calculate a braid β such that $\mathbf{g} \cdot \beta = \mathbf{g}'$ by using the Van-Kampen diagram.
(8) Return true and the braid β.

Remark 2. There are many variations of Algorithm 2 and Algorithm 4. We can use any Hurwitz-compatible presentation $\mathscr{P}''$ of G whose generating set contains $\{g_1, \dots, g_m, g'_1, \dots, g'_m\}$, as substitutes of an expanded presentation $\mathscr{P}'$ in Algorithm 2 and 4.

We illustrate how our algorithm solves Hurwitz equivalence/search problems, by giving a simple, but illustrative example.

Example 4. Let $G = B_3 = \langle x, y \,|\, xyx \equiv yxy \rangle$ be the 3-string braid group with the standard presentation. The relation $xyx \equiv yxy$ is a word-conjugacy relation, hence the standard presentation is Hurwitz compatible. Let us try to solve Hurwitz equivalence/search problems for two G-systems $\mathbf{g} = (x, x, y, x)$ and $\mathbf{g}' = (y^{-1}xy, x, y, y^{-1}xy)$ using Algorithm 2 and Algorithm 4.

It is directly checked that $\mathbf{g}$ and $\mathbf{g}'$ pass the primary test. That is, the step (1) and (2) do not return false.

In step (3), we compute the expanded presentation. We introduce a new generator $z = y^{-1}xy$, and a new word conjugacy relation $yz \equiv xy$. The expanded presentation $\mathscr{P}'$ is now given as

$$\mathscr{P}' = \langle x, y, z \,|\, xyx \equiv yxy, yz \equiv xy \rangle.$$

Let us apply Algorithm 2. By direct computation, $(xxyx)^{-1}(zxyz) \not\curvearrowright \varepsilon$ in the presentation $\mathscr{P}'$, hence Algorithm 2 returns Undecidable.

Then let us apply Algorithm 4. In step (4) in Algorithm 4, we apply the modified completion procedure, Algorithm 3.

As we have seen in Example 2, $y^{-1}xx^{-1}y \curvearrowright xyx^{-1}z^{-1}$, but $(yxy)^{-1}yzx \not\curvearrowright \varepsilon$. The relation $yxy \equiv yzx$ is reducible and the reduced relation $xy \equiv zx$ is a word-conjugacy relation. Thus, we add the new relation $xy \equiv zx$ to $\mathscr{P}'$, and get the presentation

$$\mathscr{P}_1 = \langle x, y, z \,|\, xyx \equiv yxy, yz \equiv xy \equiv zx \rangle.$$

As we have seen in Example 2, the presentation $\mathscr{P}_1$ is complete, hence we arrived at the complete Hurwitz compatible presentation $\overline{\mathscr{P}'} = \mathscr{P}_1$.

Now we proceed to step (5): We reverse the word $(xxyx)^{-1}(zxyz)$. As we have seen in Example 1, $(xxyx)^{-1}(zxyz) \curvearrowright \varepsilon$. In step (6), we draw the Van-Kampen diagram of $(xxyx, zxyz)$. See Example 1 and Figure 3 again.

From the Van-Kampen diagram, we obtain the sequence of words

$$xxyx \to xyxy \to zxxy \to zxyz.$$

Thus by considering the corresponding braid actions, we conclude that

$$\mathbf{g} \cdot (\sigma_2\sigma_3)(\sigma_1^{-1})(\sigma_3) = \mathbf{g}'.$$

Thus, Algorithm 4 returns true and the braid $\sigma_2\sigma_3\sigma_1^{-1}\sigma_3$, hence solves the Hurwitz equivalence and search problems.

This example also illustrates a usefulness of modified completion algorithm: As we observed in Example 2, the usual completion algorithm, Algorithm 1 does not terminate.

4.3. Attacking Hurwitz equivalences for general cases. We close the paper by explaining an algorithmic approach to attack Hurwitz equivalences for general cases, that is, G-systems which are not generator G-systems.

Let G be a group and consider two G-systems of length n, $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathbf{g}' = (g'_1, \dots, g'_n)$. We will assume that the word and the conjugacy problem of G is solvable.

To apply Algorithms given in previous sections, we use a Hurwitz-compatible group presentation

$$\mathscr{P} = \langle a_1, \dots, a_n \,|\, \mathscr{R} \rangle$$

that satisfies the following properties:

(1) Let A be the group defined by the group presentation $\mathscr{P}$. Then a map $\pi : \{a_1, \dots, a_n\}^* \to G$ defined by $\pi(a_i) = g_i$ is a homomorphism.

(2) For each relation $v \equiv w$ in $\mathscr{R}$, $l(v) = l(w) \le n$.

Such a group presentation can be obtained by taking all Hurwitz-compatible relations which hold for $\{g_1, \dots, g_n\}$.

First we check the primary tests for $\mathbf{g}$ and $\mathbf{g}'$. If $\mathbf{g}$ and $\mathbf{g}'$ passed the primary test, then there exists a permutation τ and words $W_i(\mathscr{G})$ on $\mathscr{G} = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ such that

$$g_i' = W_i(\mathscr{G}) g_{\tau(i)} W_i(\mathscr{G})^{-1}$$

holds.

Now let $a_i' = W_i(\mathscr{A}) a_{\tau_i} W_i(\mathscr{A})^{-1}$, where $W_i(\mathscr{A})$ is a word on $a_1, \dots, a_n$ that is obtained from the word $W_i(\mathscr{G})$ by replacing each $g_i^{\pm 1}$ with $a_i^{\pm 1}$. We consider generator A-systems $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{a}' = (a_1', \dots, a_n')$. Clearly if $\mathbf{a} \cdot \beta = \mathbf{a}'$ $(\beta \in B_n)$ then $\mathbf{g} \cdot \beta = \mathbf{g}'$. Apply Algorithm 2 or Algorithm 4 to attack Hurwitz equivalence/search problem for $\mathbf{a}$ and $\mathbf{a}'$. This might solve the Hurwitz equivalence/search problem for $\mathbf{g}$ and $\mathbf{g}'$.

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