Transversal symmetries on real hypersurfaces in a complex space form

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ABSTRACT. We classify real hypersurfaces in a complex space form whose structural reflections are isometries. We also determine real hypersurfaces in a complex space form whose transversal Jacobi operators have constant eigenvalues and at the same time their eigenspaces are parallel (along transversal geodesics).

1. Introduction

Let $(\tilde{M}_n(c), J, \tilde{g})$ be an n-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature c and let M be an orientable real hypersurface in $\tilde{M}_n(c)$. Then M has an almost contact metric structure (η, ϕ, ξ, g) induced from $(J, \tilde{g})$ (see section 1). U-H. Ki [13] proved that there are no real hypersurfaces with parallel Ricci tensors in a non-flat complex space form $\tilde{M}_n(c)$ ($c \neq 0$) when dim $n \geq 3$. U. K. Kim [15] proved that this is also true when $n = 2$. These results say that there do not exist locally symmetric ($\nabla R = 0$) real hypersurfaces in a non-flat complex space form.

On the one hand, it is well-known ([7]) that a Riemannian manifold is locally symmetric if and only if every geodesic symmetry is an isometry. In contact geometry there is the so-called ϕ-symmetry (or the transversal symmetry). T. Takahashi ([29]) introduced *Sasakian locally ϕ-symmetric spaces,* which may be considered as the analogues of locally Hermitian symmetric spaces. In fact, a Sasakian manifold is called locally ϕ-symmetric if the Riemannian curvature tensor R satisfies

$$(C_1) \quad g((\nabla_U R)(X, V)W, Y) = 0$$

for all vector fields U, V, W, X, Y orthogonal to ξ. He proves that this condition is equivalent to having ϕ-geodesic (or transversal geodesic) symmetries which are local automorphisms, i.e., local diffeomorphisms leaving all structure tensor fields invariant. Later, it was proved in [5] that the isometric property of ϕ-geodesic symmetry is already sufficient. From this geometric reality, E. Boeckx and L. Vanhecke ([6]) gave another definition for a locally ϕ-symmetric contact manifold: they require the structural reflections (i.e., the reflections with respect to the integral curves of the structure vector ξ) to be local isometries. This notion may be extended to almost contact metric spaces. Then it leads to an infinite number of curvature conditions, including (C_1). Indeed, they are given by

$$(C_2) \quad \begin{cases} \xi \text{ is a geodesic vector field}, \\ g((\nabla^{2k}_{U...U}R)(V, U)U, \xi) = 0, \\ g((\nabla^{2k+1}_{U...U}R)(V, U)U, W) = 0, \\ g((\nabla^{2k+1}_{U...U}R)(\xi, U)U, \xi) = 0, \end{cases}$$

for all vector fields U, V and W orthogonal to ξ and $k = 0, 1, 2, \ldots$.

On the other hand, J. Berndt and L. Vanhecke ([3]) found a different remarkable property of a locally symmetric space. That is, a Riemannian manifold M is locally symmetric if and only if the Jacobi operators $R_\gamma = R(\cdot, \gamma)\gamma$ are diagonalizable by parallel orthonormal frame fields and their eigenvalues are constant along any geodesics γ, which can be expressed by the condition: $(\nabla_{\dot\gamma}R)(\cdot, \dot\gamma)\dot\gamma = 0$ for any geodesic γ. This nature of local symmetry leads us to define a class of almost contact metric manifolds whose *transversal Jacobi operator R_γ* is diagonalizable by a parallel orthonormal frame field and their eigenvalues are constant along each transversal geodesic γ. It can be interpreted by the following condition:

$$(C_3) \quad \begin{cases} - & \text{a geodesic } \gamma \text{ initially belonging to } D_p \text{ remains } D_\gamma \text{ along } \gamma, \\ & \text{(we call it a } transversal\ geodesic) \\ - & (\nabla_{\dot\gamma}R)(\cdot, \dot\gamma)\dot\gamma = 0 \text{ along any transversal geodesic } \gamma, \end{cases}$$

where $D_p = \{v \in T_pM \mid \eta(v) = 0\}$ and $D : p \to D_p$ defines a distribution orthogonal to ξ. We can see that the condition (C_1) is common in the 2nd condition of (C_3) and the 3rd condition for $k = 0$ in (C_2). It is intriguing to determine real hypersurfaces in a complex space form which satisfy (C_2) or (C_3), respectively. Then we prove

THEOREM 1. *Let M be a real hypersurface in a non-flat complex space form $\tilde{M}_n(c)$ $(c \neq 0)$. Then the structural reflections on M are isometries if and only if M is locally congruent to a homogeneous hypersurface of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.*

THEOREM 2. *Let M be a real hypersurface in a non-flat complex space form $\tilde{M}_n(c)$ $(c \neq 0)$. Then the eigenvalues of the transversal Jacobi operators are constant and their eigenspaces are parallel along each transversal geodesic if and only if M is locally congruent to a ruled real hypersurface in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.*

2. Almost contact geometry

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

First, we give a brief review of several fundamental notions and formulas which we will need later on. An odd-dimensional differentiable manifold M has an *almost contact structure* if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1. \tag{1}$$

We call ξ the *Reeb vector field* (*or structural vector field*) and its integral curve the *Reeb flow* (*or structural flow*). Then we can always find a compatible Riemannian metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold*. From (1) and (2) we easily get

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(X) = g(X, \xi). \tag{3}$$

The tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \mathrm{Span}\{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. For an almost contact metric manifold M, we define its fundamental 2-form Φ by $\Phi(X, Y) = g(\phi X, Y)$. If M satisfies in addition

$$\Phi = d\eta, \tag{4}$$

M is called a *contact metric manifold*. For more details about the general theory of almost contact metric manifolds, we refer to [4].

Let $\gamma : [a, b] \to M$ be an embedded curve in a Riemannian manifold (M, g) and let v be a unit vector field orthogonal to $\gamma'(t)$ at $\gamma(t) = p$ for any fixed $t \in [a, b]$. The reflection $\exp_p rv \mapsto \exp_p(-rv)$ with respect to γ is defined for sufficiently small $r > 0$. In [9], B. Y. Chen and L. Vanhecke obtained a criterion for a reflection with respect to a curve to be locally isometric.

Applying this notion to the structural curve in an almost contact metric manifold and we may consider the criterion for the reflection with respect to the structural curve, which is called *the structural reflection*, to be isometric. Actually, E. Boeckx and L. Vanhecke ([6]) have shown the following.

PROPOSITION 1.   *Let $M = (M; \eta, \phi, \xi, g)$ be an almost contact metric manifold.   If the structural reflections on M are isometries, then*

$$
\begin{array}{ll}
(\,i\,) & \xi \text{ is a geodesic vector field}, \\[6pt]
(ii) & g((\nabla^{2k}_{U\ldots U}R)(V, U)U, \xi) = 0, \\[6pt]
(iii) & g((\nabla^{2k+1}_{U\ldots U}R)(V, U)U, W) = 0, \\[6pt]
(iv) & g((\nabla^{2k+1}_{U\ldots U}R)(\xi, U)U, \xi) = 0,
\end{array}
\tag{5}
$$

for all vector fields U, V and W orthogonal to ξ and $k = 0, 1, 2, \ldots$.   Moreover, if M is real analytic, these conditions are also sufficient for the structural reflections on M to be isometries.

In [24], S. Nagai proved the classification theorem for real hypersurfaces in $P_n\mathbf{C}$ whose structural reflections are isometries.   In Section 5, we extend his result to a non-flat complex space form $\tilde{M}_n(c)$ $(c \neq 0)$ in a different way.

On the other hand, J. Berndt and L. Vanhecke ([3]) proved that a Riemannian manifold M is locally symmetric if and only if the Jacobi operator is diagonalizable by a parallel orthonormal frame field and their eigenvalues are constant along each geodesic γ.   In that process the following lemma has an important role.

LEMMA 1 ([12], [30]).   *For a Riemannian manifold M, the following two conditions are equivalent.*
 (i) $\nabla R = 0$,
 (ii) $(\nabla_X R)(\cdot, X)X = 0$ *for any vector field X on M.*

To prove the above lemma, we mainly use the fundamental symmetries of R and their polarization technique.   For almost contact metric manifolds, we consider the analogous condition: the transversal Jacobi operator R_γ has constant eigenvalues and at the same time their eigenspaces are parallel along each transversal geodesic γ.   Equivalently, this property is interpreted as the following two conditions:

$$
\left\{
\begin{array}{ll}
(\,i\,) & \text{a geodesic } \gamma \text{ initially belonging to } D_p \text{ remains } D_\gamma \text{ along } \gamma, \\
 & \text{(we call it a \emph{transversal geodesic})} \\
(ii) & (\nabla_{\dot\gamma}R)(\cdot, \dot\gamma)\dot\gamma = 0 \text{ along any transversal geodesic (or } \phi\text{-geodesic) } \gamma.
\end{array}
\right.
\tag{6}
$$

3. Real hypersurfaces in a complex space form

Let $\tilde{M} = \tilde{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature c, M be a real hypersurface of $\tilde{M}$ and N be a unit normal vector field of M in $\tilde{M}$. We denote by $\tilde{g}$ and J a Kähler metric tensor and its complex structure tensor on $\tilde{M}$, respectively. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi, \tag{7}$$

where ϕX is the tangential part of JX, ϕ a $(1,1)$-type tensor field, η is a 1-form, and ξ is a unit vector field on M. The induced Riemannian metric on M is denoted by g. Then by properties of $(\tilde{g}, J)$ we see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, from (7) we can deduce (1) and (2).

The Gauss and Weingarten formula for M are given as

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

$$\tilde{\nabla}_X N = -AX$$

for any tangent vector fields X, Y, where $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of $(M_n(c), \tilde{g})$ and (M, g), respectively, and A is the shape operator field. An eigenvalue and an eigenvector of the shape operator A is called a principal curvature and a principal curvature vector, respectively. From (7) and $\tilde{\nabla}J = 0$, we then obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{8}$$

$$\nabla_X \xi = \phi AX. \tag{9}$$

From (9) it follows easily that

LEMMA 2. *The structural vector field ξ is principal if and only if its integral curves are geodesics.*

We have the following Gauss and Codazzi equations:

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y$$

$$+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}$$

$$+ g(AY, Z)AX - g(AX, Z)AY, \tag{10}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{11}$$

for any tangent vector fields X, Y, Z on M. From (10) we get for the Ricci tensor S of type $(1,1)$:

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \qquad (12)$$

where $h = \mathrm{tr}\, A$ denotes the mean curvature.

The following facts are needed later to prove our results.

LEMMA 3 ([14], [19], [21]). *If ξ is a principal curvature vector, then the associated principal curvature $\alpha_1 = g(A\xi, \xi)$ is constant.*

R. Takagi [27], [28] classified the homogeneous real hypersurfaces of P_nC into six types. T. E. Cecil and P. J. Ryan [8] extensively studied a Hopf hypersurface (whose Reeb vector ξ is a principal curvature vector), which is realized as tubes over certain submanifolds in P_nC, by using its focal map. By making use of those results, M. Kimura [16] proved the local classification theorem for Hopf hypersurfaces of P_nC whose all principal curvatures are constant.

THEOREM 3 ([16]). *Let M be a Hopf hypersurface of P_nC. Then M has constant principal curvatures if and only if M is locally congruent to one of the following:*
   (A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*
   (A_2) *a tube of radius r over a totally geodesic P_lC $(1 \le l \le n-2)$, where $0 < r < \frac{\pi}{2}$,*
   (B) *a tube of radius r over a complex quadric Q^{n-1} and P_nR, where $0 < r < \frac{\pi}{4}$,*
   (C) *a tube of radius r over $P_1C \times P_{(n-1)/2}C$, where $0 < r < \frac{\pi}{4}$ and $n(\ge 5)$ is odd,*
   (D) *a tube of radius r over a complex Grassmann $G_{2,5}C$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
   (E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

For the case H_nC, J. Berndt [2] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

THEOREM 4 ([2]). *Let M be a Hopf hypersurface of H_nC. Then M has constant principal curvatures if and only if M is locally congruent to one of the following:*
   (A_0) *a horosphere,*
   (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}C$,*

(A$_2$)   *a tube over a totally geodesic $H_l\mathbf{C}$ $(1 \le l \le n-2)$,*

(B)   *a tube over a totally real hyperbolic space $H_n\mathbf{R}$.*

We call simply type (A) for real hypersurfaces of type (A$_1$), (A$_2$) in $P_n\mathbf{C}$ and ones of type (A$_0$), (A$_1$) or (A$_2$) in $H_n\mathbf{C}$.

M. Kimura [17] constructed ruled real hypersurfaces, which are foliated real hypersurfaces with totally geodesic submanifolds of $P_n\mathbf{C}$ as leaves of codimension 1. Let $\overline{M}$ be a hypersurface in S^{2n+1} defined by

$$\{(re^{it}\cos\theta, re^{it}\sin\theta, (1-r^2)^{1/2}z_2, \ldots, (1-r^2)^{1/2}z_n) \in \mathbf{C}^{n+1} \,|$$

$$\sum_{j=2}^{n}|z_j|^2 = 1,\ 0 < r < 1,\ 0 \le t,\ \theta < 2\pi, \}.$$

Then the Hopf image M of $\overline{M}$ is a minimal ruled hypersurface in $P_n(\mathbf{C})$. Actually, the shape operator is given as follows: $A\xi = r^{-1}(1-r^2)^{1/2}U$, $AU = r^{-1}(1-r^2)^{1/2}\xi$ and $AZ = 0$ for $Z \perp \xi, U$. We note that the above example of a ruled real hypersurface is not complete. In a similar way, S.-S. Ahn, S.-B. Lee and Y. J. Suh ([1]) gave a minimal complete ruled real hypersurface in $H_n(\mathbf{C})$. Furthermore, they are characterized by the following.

THEOREM 5 ([1]). *Let M be a real hypersurface in a non-flat complex space form $\tilde{M}$. Then M is a ruled real hypersurface if and only if $g(AX,Y) = 0$ for any tangent vectors X, Y of M with $X, Y \perp \xi$.*

The shape operator of ruled real hypersurfaces in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ is written as follows:

$$\begin{aligned}
A\xi &= \alpha_1\xi + \mu U \qquad (\mu \ne 0),\\
AU &= \mu\xi,\\
AZ &= 0
\end{aligned} \tag{13}$$

for any $Z \perp \{\xi, U\}$, where $U \perp \xi$ is a unit vector field, α_1 and μ are functions on M.

4.   Real hypersurfaces of type (A) or (B)

Real hypersurfaces of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ have a following characterization.

THEOREM 6 ([18], [26]). *Let M be a Hopf hypersurface in a non-flat complex space form $\tilde{M} = \tilde{M}_n(c)$. Then the shape operator A is η-parallel, that is $g((\nabla_X A)Y, Z) = 0$ for $X, Y, Z \perp \xi$ if and only if M is locally congruent to a real hypersurface of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.*

By using the above result, we can find the full expression of ∇A for real hypersurfaces of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$, which is very useful to prove Theorem 1. Namely, we have

THEOREM 7. *Let M be a real hypersurface in a non-flat complex space form $\tilde{M} = \tilde{M}_n(c)$. Then M is locally congruent to a real hypersurface of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ if and only if M satisfies*

$$(\nabla_X A)Y = \eta(X)\left(\frac{c}{4}\phi Y + FY\right) + \eta(Y)FX + g(FX, Y)\xi, \qquad (14)$$

for any vector fields X, Y tangent to M, where $F = \eta(A\xi)\phi A - A\phi A$.

PROOF. If we denote by X^T the component of X orthogonal to ξ, then we have for arbitrary vector fields X, Y, Z on M:

$$g((\nabla_{X^T}A)Y^T, Z^T)$$

$$= g((\nabla_{X-\eta(X)\xi}A)(Y - \eta(Y)\xi, Z - \eta(Z)\xi)$$

$$= g((\nabla_X A)Y, Z) - \eta(X)g((\nabla_\xi A)Y, Z) - \eta(Y)g((\nabla_X A)\xi, Z)$$

$$\quad - \eta(Z)g((\nabla_X A)Y, \xi) + \eta(X)\eta(Y)g((\nabla_\xi A)\xi, Z) + \eta(Y)\eta(Z)g((\nabla_X A)\xi, \xi)$$

$$\quad + \eta(Z)\eta(X)g((\nabla_\xi A)Y, \xi) - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi A)\xi, \xi). \qquad (15)$$

Let M be a real hypersurface of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$. Then since ξ is a principal curvature vector, that is $A\xi = \alpha_1\xi$ on M, then differentiating this covariantly, and then using Lemma 6 and (9) we have

$$(\nabla_X A)\xi = \alpha_1\phi AX - A\phi AX, \qquad (16)$$

and further using (11) we obtain

$$(\nabla_\xi A)X = \frac{c}{4}\phi X + \alpha_1\phi AX - A\phi AX \qquad (17)$$

for any vector field X on M. Hence, we see that $(\nabla_\xi A)\xi = 0$. Furthermore, use the η-parallelity of A, the symmetry of $\nabla_X A$, (16) and (17) in (15) to obtain (14).

Conversely, we suppose that real hypersurface M satisfies (14). Then we obtain

$$(\nabla_X A)Y - (\nabla_Y A)X$$

$$= \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X\} + \eta(A\xi)g((\phi A + A\phi)X, Y)\xi - 2g(A\phi AX, Y)\xi$$

for any vector fields X and Y on M. Together with (11) we get

$$\alpha_1 g((\phi A + A\phi)X, Y) - 2g(A\phi AX, Y) = -\frac{c}{2}g(\phi X, Y) \qquad (18)$$

for any vector fields X and Y on M, where $\alpha_1 = \eta(A\xi)$. If we put $X = \xi$ in (18), then we have

$$2A\phi A\xi = \alpha_1 \phi A\xi. \qquad (19)$$

Put $Y = \phi A\xi$ in (18) and use (2) and (19) to obtain

$$(\alpha_1^2 + c)g(\phi X, \phi A\xi) = (\alpha_1^2 + c)(\eta(AX) - \alpha_1 \eta(X)) = 0,$$

which says that ξ is a principal curvature vector field on M in $P_n\mathbf{C}$. So, we consider the case $\alpha_1^2 + c = 0$, which occur only in $H_n\mathbf{C}$. If we put $X = \xi$ in (14) and use (19), then we get

$$(\nabla_\xi A)Y = \frac{c}{4}\phi Y + \alpha_1 \phi AY - A\phi AY + \frac{\alpha_1}{2}\eta(Y)\phi A\xi + \frac{\alpha_1}{2}g(\phi A\xi, Y)\xi \qquad (20)$$

for any vector field Y on M. From (19) and (20), it follows that

$$(\nabla_\xi A)\xi = \alpha_1 \phi A\xi, \qquad (21)$$

and

$$(\nabla_\xi A)\phi A\xi = \frac{\alpha_1}{2}A^2\xi - \frac{3}{4}\alpha_1^2 A\xi + \left(-\frac{\alpha_1^3}{4} + \frac{\alpha_1\alpha_2}{2}\right)\xi, \qquad (22)$$

where we have used the 1st equation of (1) and (2). Here, we denote $\alpha_2 = g(A^2\xi, \xi)$.

If we differentiate (19) covariantly for ξ, then we get

$$2((\nabla_\xi A)\phi A\xi + A(\nabla_\xi \phi)A\xi + A\phi(\nabla_\xi A)\xi + A\phi A\nabla_\xi\xi)$$
$$= \alpha_1((\nabla_\xi \phi)A\xi + \phi(\nabla_\xi A)\xi + \phi A\nabla_\xi\xi). \qquad (23)$$

By using (19), (21) and (22), we compute (23) again. Then we finally have

$$(\alpha_1^2 - \alpha_2)A\xi = \alpha_1(\alpha_1^2 - \alpha_2)\xi,$$

where we have used $(\nabla_\xi \phi)A\xi = \alpha_1 A\xi - \alpha_2\xi$. Since $\|\phi A\xi\|^2 = \alpha_2 - \alpha_1^2$, we easily show that $\alpha_1^2 - \alpha_2 = 0$ implies $A\xi = \alpha_1\xi$. Hence, we conclude that ξ is a principal curvature vector field on M in $P_n\mathbf{C}$ or $H_n\mathbf{C}$. From (14) we get further that A is η-parallel. Due to Theorem 6, we show that M is locally congruent to a real hypersurface of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$. $\square$

The above Theorem 7 gives a new characterization of real hypersurfaces of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.

5. Real hypersurfaces whose structural reflections are isometric

In this section, we prove Theorem 1. The following lemma can be verified by the induction argument using (3), (8), (9) and (14) (Theorem 7). (For $P_n\mathbf{C}$, this was proved [24] in a different way.)

LEMMA 4. *In a real hypersurface of type* (A) *or* (B) *in* $P_n\mathbf{C}$ *or* $H_n\mathbf{C}$, *the following relations are satisfied:*

$$\begin{cases} (\nabla_{u...u}^{2n+1}\phi)v \in \Gamma(D^\perp), & (\nabla_{u...u}^{2n}\phi)v \in \Gamma(D) \\ (\nabla_{u...u}^{2n+1}\phi)\xi \in \Gamma(D), & (\nabla_{u...u}^{2n}\phi)\xi \in \Gamma(D^\perp) \\ (\nabla_{u...u}^{2n+1}A)v \in \Gamma(D^\perp), & (\nabla_{u...u}^{2n}A)v \in \Gamma(D) \\ (\nabla_{u...u}^{2n+1}A)\xi \in \Gamma(D), & (\nabla_{u...u}^{2n}A)\xi \in \Gamma(D^\perp) \\ \nabla_{u...u}^{2n+1}\xi \in \Gamma(D), & \nabla_{u...u}^{2n}\xi \in \Gamma(D^\perp), \end{cases} \quad (24)$$

for $u, v \in \Gamma(D)$ *and* $n \in \mathbf{N}$, *where* $TM = D \oplus D^\perp$, $D^\perp$ *is the orthogonal complement of* D *in* TM, $\Gamma(D)$ *and* $\Gamma(D^\perp)$ *denotes the space of all sections of* D *and* $D^\perp$, *respectively.*

By using Lemma 4, we can show that

$$(\nabla_{U...U}^{2k}R)(V,U)U \in \Gamma(D), \qquad (\nabla_{U...U}^{2k+1}R)(V,U)U \in \Gamma(D^\perp),$$

$$(\nabla_{U...U}^{2k+1}R)(\xi,U)U \in \Gamma(D)$$

for all vector fields U, V orthogonal to ξ and $k = 0, 1, 2, \ldots$. It is notable that a homogeneous manifold has an real analytic structure. Thus, we have

PROPOSITION 2. *The structural reflections on real hypersurfaces of type* (A) *or* (B) *in* $P_n\mathbf{C}$ *or* $H_n\mathbf{C}$ *are isometric.*

In order to complete the proof of Theorem 1, by using (8) and (10) we have for $U \in \Gamma(D)$:

$$(\nabla_U R)(\xi, U)U = -\frac{3}{4}cg(AU, U)\phi U + \alpha_1 g((\nabla_U A)U, U)\xi$$

$$+ g(AU, U)(\alpha_1 \phi AU - A\phi AU)$$

$$- \alpha_1 g(\phi AU, U)AV + g(A\phi AU, U)AU.$$

From (iv) of (5), the above equation yields that

$$\alpha_1 g((\nabla_U A)U, U) = 0 \quad (25)$$

for any $U \in \Gamma(D)$. Here, we divide our arguments into the two cases: (I) $\alpha_1 \neq 0$, (II) $\alpha_1 = 0$.

(I) Using polarization and symmetry of $(\nabla_U A)$ in (25), then we obtain

$$g((\nabla_U A)V, W) = 0 \qquad (26)$$

for any $U, V, W \in \Gamma(D)$. So, due to Theorem 6, it follows that M is locally congruent to a real hypersurface of type (A) or (B) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.

(II) When $\alpha_1 = 0$, we can prove that all principal curvatures are constant. We refer [24] for the details of proof. But, for $H_n\mathbf{C}$ we already know that $\alpha_1 \neq 0$ in the list of Theorem 4 (cf. [2], [22]). For the case $P_n\mathbf{C}$, among the list of Theorem 3 only a real hypersurface of type (A) with $r = \frac{\pi}{4}$ holds $\alpha_1 = 0$.

We finally have proved Theorem 1.

6. Transversal Jacobi operators

In the present section, we prove Theorem 2. Before proving it, we prepare a useful result.

THEOREM 8 ([20]). *Let M be a real hypersurface in a non-flat complex space form $\tilde{M}_n(c)$ ($c \neq 0$), $n \geq 3$. Then M satisfies*

$$g((\phi A - A\phi)X, Y) = 0 \qquad (27)$$

and

$$g((\nabla_Z R)(X, V)W, Y) = 0 \qquad (28)$$

for $Z, V, W, X, Y \perp \xi$ if and only if M is locally congruent to a hypersurface of type (A) or a ruled real hypersurface in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.

In proving Theorem 8, J. G. Lee, J. D. Pérez and Y. J. Suh made use of a following fact which holds in general (without the dimension restriction): if a real hypersurface M in a complex space form $\tilde{M}_n(c)$ satisfies (27), then M satisfies

$$g((\nabla_X A)Y, Z) = \mathfrak{S}_{X, Y, Z} g(AX, Y)g(Z, T), \qquad (29)$$

where $\mathfrak{S}_{X, Y, Z}$ denotes the cyclic sum with respect to $X, Y, Z \in \Gamma(D)$ and $T = \nabla_\xi \xi$.

Now we extend the above Theorem 8 to the case that $n = 2$.

Extension of Theorem 8 to the case $n = 2$

Let M be a 3-dimensional real hypersurface in $\mathbf{C}P^2$ or $\mathbf{C}H^2$. Then for a local orthonormal frame field $\{\xi, U, \phi U\}$ of M, we may put

$$\begin{cases} A\xi = \alpha\xi + \mu U + \nu\phi U, \\ AU = \mu\xi + \beta U + \delta\phi U, \\ A\phi U = \nu\xi + \delta U + \gamma\phi U, \end{cases} \tag{30}$$

where α, β, γ, μ, ν, δ are smooth functions.

From (30) we get

$$(\phi A - A\phi)U = -\nu\xi - 2\delta U + (\beta - \gamma)\phi U.$$

Together with this and the condition (27), we obtain $\delta = 0$ and $\beta = \gamma$. Hence, (30) is reduced to

$$\begin{cases} A\xi = \alpha\xi + \mu U + \nu\phi U, \\ AU = \mu\xi + \beta U, \\ A\phi U = \nu\xi + \beta\phi U. \end{cases} \tag{31}$$

From (28) using (8) and (10), then it follows that

$$0 = g((\nabla_Z A)V, W)g(AX, Y) + g(AV, W)g((\nabla_Z A)X, Y)$$
$$- g((\nabla_Z A)X, W)g(AV, Y) - g(AX, W)g((\nabla_Z A)V, Y)$$

for any $U, V, W, X, Y \in \Gamma(D)$. Here, we use the relation (29). Then we get again

$$(g(AZ, V)g(W, T) + g(AV, W)g(Z, T) + g(AW, Z)g(V, T))g(AX, Y)$$
$$+ g(AV, W)(g(AZ, X)g(Y, T) + g(AX, Y)g(Z, T) + g(AY, Z)g(X, T))$$
$$- (g(AZ, X)g(W, T) + g(AX, W)g(Z, T) + g(AW, Z)g(X, T))g(AV, Y)$$
$$- g(AX, W)(g(AZ, V)g(Y, T) + g(AV, Y)g(Z, T)$$
$$+ g(AY, Z)g(V, T)) = 0. \tag{32}$$

We put $Z = X = W = U$, $V = Y = \phi U$, and in turn we put $Z = V = Y = \phi U$, $X = W = U$, respectively in (32). Then together with (31) we have

$$\beta^2\nu = 0 \qquad \text{and} \qquad \beta^2\mu = 0,$$

respectively, where we have used $T(= \nabla_\xi\xi = \phi A\xi) = \mu\phi U - \nu U$. So, we consider the two cases: (i) $\beta = 0$, (ii) $\mu = \nu = 0$.

(i)

$$\begin{cases} A\xi = \alpha\xi + \mu U + \nu\phi U, \\ AU = \mu\xi, \\ A\phi U = \nu\xi. \end{cases}$$

Taking account of Theorem 5, we see that M is locally congruent to a ruled real hypersurface.

(ii) Then, ξ is a principal curvature vector field, furthermore, we see that M satisfies $\phi A = A\phi$. This yields that M is locally congruent to a real hypersurface of type (A) in $\mathbf{C}P^2$ or $\mathbf{C}H^2$ (cf. [25], [23]).

Proof of Theorem 2

Now, we prove Theorem 2. We first look at the 1st condition of (6). In order to define a transversal geodesic $\gamma : I \to M$, parametrized with the arclength s, we need the property that $\eta(\dot{\gamma}(0)) = 0$ and $\eta(\dot{\gamma}(s))$ is constant along γ, or using (9) we have $g(\phi A\dot{\gamma}, \dot{\gamma}) = 0$ along γ with $\dot{\gamma}(0) \perp \xi_p$, $\gamma(0) = p$. Thus, we see that the 1st condition of (6) is equivalent to (27) for any vector field $X, Y \perp \xi$. Next, we deal with the 2nd condition of (6). Then we find that

$$(\nabla_U R)(\cdot, U)U = 0 \tag{33}$$

for any $U \perp \xi$. Then using (8) and (10), the condition (33) gives

$$-\frac{3}{4}c\{\eta(X)g(AU, U)g(\phi U, Y) - \eta(Y)g(\phi X, U)g(AU, U)\}$$

$$+ g((\nabla_U A)U, U)g(AX, Y) + g(AU, U)g((\nabla_U A)X, Y)$$

$$- g((\nabla_U A)X, U)g(AU, Y) - g(AX, U)g((\nabla_U A)U, Y) = 0 \tag{34}$$

for any vector field X, Y on M and any vector field $U \perp \xi$.

By the same way as to prove Lemma 1, we see that the condition (33) is equivalent to

$$g((\nabla_U R)(X, V)W, Y) = 0$$

for any vector field X, Y and any vector fields $U, V, W \perp \xi$.

So, due to Theorem 8 and the extension of it to $n = 2$, it needs for us to consider only the two cases: (I) M is of type (A) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$. (II) M is a ruled real hypersurface.

(I) We put $X = \xi$ in (34). Then since ξ is a principal curvature vector and its corresponding principal curvature α_1 is constant (Lemma 6), we get

$$-\frac{3}{4}cg(AU, U)g(\phi U, Y) + \alpha_1 g((\nabla_U A)U, U)\eta(Y)$$

$$+ g(AU, U)g(\alpha_1 \phi AU - A\phi AU, Y)$$

$$- g(\alpha_1 \phi AU - A\phi AU, U)g(AU, Y) = 0 \tag{35}$$

for any vector field Y and $U \perp \xi$. Then we find that $\alpha_1 g((\nabla_U A)U, U) = 0$, and successively (35) reduces to

$$-\frac{3}{4}cg(AU, U)\phi U + g(AU, U)(\alpha_1\phi AU - A\phi AU)$$

$$- g(\alpha_1\phi AU - A\phi AU, U)AU = 0. \tag{36}$$

Assume $AU = \lambda U$ and $g(U, U) = 1$. Then from (36) it follows that

$$-\frac{3}{4}c\lambda + \alpha_1\lambda^2 - \lambda^3 = 0. \tag{37}$$

But, we know that M is of type (A) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ is determined by the equation:

$$\lambda^2 - \alpha_1\lambda - \frac{c}{4} = 0 \qquad (A\phi = \phi A)$$

(cf. [25], [23]). Comparing this with (37) we can see that M of type (A) does not satisfy (33).

(II) With the result of Theorem 5, we see that the shape operator of a ruled real hypersurface in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ is also η-parallel. Then we can find that in (34) it remains only the following term to be considered:

$$g((\nabla_U A)\xi, U)g(AU, \xi) = 0 \tag{38}$$

for a vector field $U(\perp \xi)$, which appeared in (13). Together with Theorem 8, we see that the condition $U\mu = 0$ is necessary and sufficient for a ruled real hypersurface in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ to satisfy the two conditions (27) and (33). But, we know that $\mathrm{grad}(\mu) = (\mu^2 + c/4)\phi U$ on a ruled real hypersurface in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ (cf. [10]). After all, we have proved Theorem 2.

REMARK 1. J. T. Cho and L. Vanhecke ([11]) classified all Hopf hypersurfaces in a non-flat complex space form which are D'Atri spaces (that is, Riemannian manifolds all of whose geodesic symmetries are volume-preserving up to sign) or C-spaces (that is, their Jacobi operators have constant eigenvalues along the corresponding geodesics). In fact, such properties are occurred only in real hypersurfaces of type (A) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$. This also yields a classification of Hopf hypersurfaces which are naturally reductive, g.o., weakly symmetric or commutative spaces. We refer to [11] for the detail.

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