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Atomic decomposition of harmonic Bergman functions

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Abstract. We consider harmonic Bergman functions, i.e., functions which are harmonic and p-th integrable. In the present paper, we shall show that when $1 < p < \infty$, every harmonic Bergman function on a smooth domain is represented as a series using the harmonic Bergman kernel. This representation is called an atomic decomposition.

1. Introduction

Let Ω be a domain in the n-dimensional Euclidean space $\mathbf{R}^n$. For $1 \le p < \infty$, we denote by $b^p = b^p(\Omega)$ the harmonic Bergman space on Ω, i.e., the set of all real-valued harmonic functions f on Ω such that $\|f\|_p := (\int_\Omega |f|^p dx)^{1/p} < \infty$, where dx denotes the usual n-dimensional Lebesgue measure on Ω. As is well-known, b^p is a closed subspace of $L^p = L^p(\Omega)$ and hence, b^p is a Banach space (for example see [1]). Especially, when $p = 2$, b^2 is a Hilbert space, which has the reproducing kernel, i.e., there exists a unique symmetric function $R(\cdot,\cdot)$ on $\Omega \times \Omega$ such that for any $f \in b^2$ and any $x \in \Omega$,

$$f(x) = \int_\Omega R(x,y) f(y) dy. \tag{1}$$

The function $R(\cdot,\cdot)$ is called the harmonic Bergman kernel of Ω. When Ω is the open unit ball B, an explicit form is known:

$$R(x,y) = R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{n|B|(1 - 2x\cdot y + |x|^2|y|^2)^{1+n/2}},$$

where $x \cdot y$ denotes the Euclidean inner product in $\mathbf{R}^n$ and $|B|$ is the Lebesgue measure of B.

There are many papers concerning the harmonic Bergman space on the unit ball, where the above explicit form plays important roles. For example,

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in the paper [3] an atomic decomposition theorem is obtained. The purpose of this paper is to generalize this result for more general domains, smooth bounded domains. A bounded domain Ω is said to be smooth if for every boundary point $\eta \in \partial\Omega$ there exist a neighborhood V of η in $\mathbf{R}^n$ and a C^∞-diffeomorphism $f : V \to f(V) \subset \mathbf{R}^n$ such that $f(\eta) = 0$ and $f(\Omega \cap V) = \{(y_1, \dots, y_n) \in \mathbf{R}^n; y_n > 0\} \cap f(V)$. Our main result is the following.

THEOREM 1. *Let $1 < p < \infty$ and let Ω be a smooth bounded domain. Then we can choose a sequence $\{\lambda_i\}$ in Ω satisfying the following property: For any $f \in b^p(\Omega)$, there exists a sequence $\{a_i\} \in l^p$ such that*

$$f(x) = \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-1/p)n}, \tag{2}$$

where $r(x)$ denotes the distance between x and $\partial\Omega$.

The equation (2) is called an atomic decomposition of f. The above theorem shows the existence of a sequence $\{\lambda_i\} \subset \Omega$ permitting the atomic decomposition for every $f \in b^p$. In the last section, we also discuss a sufficient condition for a given sequence $\{\lambda_i\}$ in Ω to permit the atomic decomposition.

THEOREM 2. *Let $1 < p < \infty$ and Ω be a smooth bounded domain. Then there exists a constant $\delta_1 > 0$ such that if a sequence $\{\lambda_i\}$ in Ω satisfies*

$$\bigcup_i B(\lambda_i, \delta_1 r(\lambda_i)) = \Omega,$$

then every $f \in b^p$ can be represented as

$$f = \sum_{i=1}^{\infty} a_i R(\cdot, \lambda_i) r(\lambda_i)^{n(1-1/p)}$$

in b^p with some sequence $\{a_i\} \in l^p$, where $B(x, r)$ is the open ball of radius r, centered at x.

In what follows, Ω is always assumed to be a smooth bounded domain in $\mathbf{R}^n$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries

Equation (1) is called the reproducing formula for $p = 2$. Unfortunately, for general $p \in [1, \infty)$, the reproducing formula is not always ensured a general domain. However, when Ω is a bounded smooth domain, the reproducing formula holds for $1 \le p < \infty$ ([4]), i.e., for any $f \in b^p$

$$f(x) = \int_{\Omega} R(x, y) f(y) dy.$$

This equality follows from the estimates for the harmonic Bergman kernel. Also in this paper, the estimates of the harmonic Bergman kernel obtained by H. Kang and H. Koo [4] play an important role. In this section, we recall some results in [4] and show some basic lemmas. For an n-tuple $\alpha := (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, called a multi-index, we denote $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $D_x^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

Lemma 1 (Theorem 1.1 in [4]). *Let α, β be multi-indices.*
(1) *There exists a constant $C > 0$ such that*

$$|D_x^\alpha D_y^\beta R(x, y)| \le \frac{C}{d(x, y)^{n+|\alpha|+|\beta|}}$$

for every $x, y \in \Omega$, where $d(x, y) = r(x) + r(y) + |x - y|$.
(2) *There exists a constant $C > 0$ such that*

$$R(x, x) \ge \frac{C}{r(x)^n}$$

for every $x \in \Omega$.

Based on Lemma 1, B. R. Choe, Y. J. Lee and K. Na derived the following lemma for the estimate of the harmonic Bergman kernel.

Lemma 2 (Lemma 2.3 in [2]). *There exist constants $0 < \delta_2 < 1$ and $C > 0$ such that for every $x \in \Omega$ and $y \in B(x, \delta_2 r(x))$,*

$$C^{-1} \le R(x, y) r(x)^n \le C.$$

We generalize Lemma 2 for our later use.

Lemma 3. *There exist constants $0 < \delta_3 < 1$ and $C > 0$ such that for every $x \in \Omega$ and $y, z \in B(x, \delta_3 r(x))$,*

$$C^{-1} \le R(y, z) r(x)^n \le C.$$

Proof. First, we claim that if $0 < \delta < 1$ and $x \in \Omega$ then

$$(1 - \delta) r(x) < r(y) < (1 + \delta) r(x)$$

for any $y \in B(x, \delta r(x))$. In fact, by taking a boundary point η with $r(y) = |\eta - y|$, we have

$$r(x) \le |\eta - x| \le |\eta - y| + |y - x| < r(y) + \delta r(x)$$

for any $y \in B(x, \delta r(x))$. Thus, we obtain $(1 - \delta) r(x) < r(y)$ for any $y \in B(x, \delta r(x))$. Similarly, taking a boundary point η' with $r(x) = |\eta' - x|$, we have

$$r(y) \le |\eta' - y| \le |\eta' - x| + |x - y| < r(x) + \delta r(x) = (1 + \delta) r(x)$$

for any $y \in B(x, \delta r(x))$.

We take a constant $\delta_2 > 0$ in Lemma 2 and choose a constant δ_3 with $0 < \delta_3 < \frac{\delta_2}{2+\delta_2}$. Then we obtain $2\delta_3 < \delta_2(1 - \delta_3)$. Hence, for any $y, z \in B(x, \delta_3 r(x))$, the above assertion shows

$$|y - z| < 2\delta_3 r(x) < \delta_2(1 - \delta_3) r(x) < \delta_2 r(y).$$

These inequalities imply $z \in B(y, \delta_2 r(y))$. By Lemma 2, there exists a constant $C > 0$ such that

$$C^{-1} \le R(y, z) r(y)^n \le C.$$

Since $y \in B(x, \delta_3 r(x))$, the above assertion implies that $r(x)$ and $r(y)$ are comparable. Therefore Lemma 3 is shown. □

LEMMA 4 (Lemma 2.4 in [2]). *Let* $1 < p < \infty$. *Then there exists a constant* $C > 0$ *such that for any* $x \in \Omega$,

$$C^{-1} r(x)^{n(1/p-1)} \le \|R(x, \cdot)\|_p \le C r(x)^{n(1/p-1)}.$$

We need the following calculation.

LEMMA 5 (Lemma 4.1 in [4]). *Let* s *and* t *be nonnegative real numbers. If* $s + t > 0$ *and* $t < 1$, *then there exists a constant* $C > 0$ *such that*

$$\int_\Omega \frac{dy}{d(x, y)^{n+s} r(y)^t} \le \frac{C}{r(x)^{s+t}}$$

for every $x \in \Omega$.

Here, we define an auxiliary integral operator

$$Kf(x) = \int_\Omega \frac{1}{d(x, y)^n} f(y) dy.$$

LEMMA 6. *For* $1 < p < \infty$, K *is a bounded linear operator from* $L^p(\Omega)$ *to* $L^p(\Omega)$.

Proof. We have only to check the Schur test (see p. 42 in [7]). Let $1 < p < \infty$ and let q be the exponent conjugate to p. Putting $h(x) = r(x)^{-1/pq}$, we have estimates

$$\int_{\Omega} \frac{1}{d(x,y)^n} h(y)^p dy \le Cr(x)^{-1/q} = Ch(x)^p$$

and

$$\int_{\Omega} \frac{1}{d(x,y)^n} h(y)^q dy \le Cr(x)^{-1/p} = Ch(x)^q$$

with some constant $C > 0$, by Lemma 5. Hence, the Schur test ensures that K is a bounded operator from $L^p(\Omega)$ to $L^p(\Omega)$. □

Using Lemma 6, we have norm estimates of the derivatives of harmonic Bergman functions.

Lemma 7. *Let $1 < p < \infty$ and let α be a multi-index. Then there exists a constant $C > 0$ such that*

$$\int_{\Omega} |r(x)^{|\alpha|} D_x^{\alpha} f(x)|^p dx \le C \|f\|_p^p$$

for every $f \in b^p$.

Proof. For any $x \in \Omega$, by (1) of Lemma 1 we have

$$\begin{aligned} |r(x)^{|\alpha|} D_x^{\alpha} f(x)| &= r(x)^{|\alpha|} \left| \int_{\Omega} D_x^{\alpha} R(x,y) f(y) dy \right| \\ &\le \int_{\Omega} \frac{Cr(x)^{|\alpha|}}{d(x,y)^{n+|\alpha|}} |f(y)| dy \\ &\le C \int_{\Omega} \frac{1}{d(x,y)^n} |f(y)| dy \\ &\le CK|f|(x). \end{aligned}$$

Then, the lemma follows from Lemma 6. □

Finally, we remark the following duality.

Lemma 8 (Corollary 4.3 in [4]). *Let $1 < p < \infty$ and let q be the exponent conjugate to p. Then $(b^p)^* \cong b^q$, under the pairing*

$$\langle f, g \rangle_L = \int_{\Omega} f(x) g(x) dx$$

for $f \in b^p$ and $g \in b^q$.

3. Covering lemmas

In this section, we consider some properties of sequences $\{\lambda_i\}$ in Ω.

Definition 1. Let $0 < \varepsilon < \delta < 1$.

(1) A sequence $\{\lambda_i\}$ in Ω is called *ε-separated* if $B(\lambda_i, \varepsilon r(\lambda_i)) \cap B(\lambda_j, \varepsilon r(\lambda_j)) = \varnothing$ for $i \neq j$.

(2) A sequence $\{\lambda_i\}$ in Ω is called *an (ε, δ)-lattice* if the following two conditions are satisfied:

$$\text{(a)} \quad \{\lambda_i\} \text{ is } \varepsilon\text{-separated}; \qquad \text{(b)} \quad \bigcup B(\lambda_i, \delta r(\lambda_i)) = \Omega.$$

The following lemma shows that the number of intersection can be bounded above.

Lemma 9. *Let $0 < \varepsilon < \delta < 1$. If a sequence $\{\lambda_i\}$ is ε-separated, then for every $i \in \mathbf{N}$,*

$$\#\{j \in \mathbf{N}; B(\lambda_i, \delta r(\lambda_i)) \cap B(\lambda_j, \delta r(\lambda_j)) \neq \varnothing\} \leq \left(\frac{(1+\delta)(2\delta + \varepsilon(1+\delta))}{(1-\delta)^2 \varepsilon}\right)^n,$$

where for a set A, $\#A$ denotes the number of elements in A.

Proof. Let a sequence $\{\lambda_i\}$ be ε-separated and let i be fixed. If $j \in \mathbf{N}$ satisfies $B(\lambda_i, \delta r(\lambda_i)) \cap B(\lambda_j, \delta r(\lambda_j)) \neq \varnothing$, then

$$|\lambda_i - \lambda_j| < \delta(r(\lambda_i) + r(\lambda_j)). \tag{3}$$

Taking boundary points η_i and η_j with $r(\lambda_i) = |\eta_i - \lambda_i|$ and $r(\lambda_j) = |\eta_j - \lambda_j|$, we have

$$r(\lambda_j) \leq |\eta_i - \lambda_j| \leq |\eta_i - \lambda_i| + |\lambda_i - \lambda_j| < r(\lambda_i) + \delta(r(\lambda_i) + r(\lambda_j)).$$

Similarly, we obtain

$$r(\lambda_i) < r(\lambda_j) + \delta(r(\lambda_i) + r(\lambda_j)),$$

which shows

$$\frac{1-\delta}{1+\delta} r(\lambda_i) < r(\lambda_j) < \frac{1+\delta}{1-\delta} r(\lambda_i). \tag{4}$$

By (3) and (4), we have

$$|\lambda_i - \lambda_j| < \delta\left(r(\lambda_i) + \frac{1+\delta}{1-\delta} r(\lambda_i)\right) = \frac{2\delta}{1-\delta} r(\lambda_i),$$

i.e.,

$$\lambda_j \in B\left(\lambda_i, \frac{2\delta}{1-\delta} r(\lambda_i)\right). \tag{5}$$

Hence

$$B(\lambda_j, \varepsilon r(\lambda_j)) \subset B\left(\lambda_i, \frac{2\delta}{1-\delta} r(\lambda_i) + \varepsilon r(\lambda_j)\right) \subset B\left(\lambda_i, \frac{2\delta + \varepsilon(1+\delta)}{1-\delta} r(\lambda_i)\right),$$

by (4) and (5). Put

$$J(i) := \left\{ j \in \mathbf{N}; B(\lambda_j, \varepsilon r(\lambda_j)) \subset B\left(\lambda_i, \frac{2\delta + \varepsilon(1+\delta)}{1-\delta} r(\lambda_i)\right) \right\}.$$

Then, since the sequence $\{\lambda_i\}$ is ε-separated, we have

$$\begin{aligned}
&\#\{j \in \mathbf{N}; B(\lambda_i, \delta r(\lambda_i)) \cap B(\lambda_j, \delta r(\lambda_j)) \neq \emptyset\} \leq \#J(i) \\
&\quad \leq \sup\left\{ \left| B\left(\lambda_i, \frac{2\delta + \varepsilon(1+\delta)}{1-\delta} r(\lambda_i)\right) \right| \cdot |B(\lambda_j, \varepsilon r(\lambda_j))|^{-1}; j \in J(i) \right\} \\
&\quad \leq \sup\left\{ \left(\frac{(2\delta + \varepsilon(1+\delta)) r(\lambda_i)}{\varepsilon(1-\delta) r(\lambda_j)} \right)^n; j \in J(i) \right\} \\
&\quad < \left(\frac{(1+\delta)(2\delta + \varepsilon(1+\delta))}{\varepsilon(1-\delta)^2} \right)^n. \qquad \square
\end{aligned}$$

The following lemma shows the existence of an (ε, δ)-lattice for some ε and δ.

Lemma 10. *For each $0 < \delta < 1$, there exists a $\left(\frac{\delta}{2}, \delta\right)$-lattice.*

Proof. First, we take a point λ_1 in Ω such that $r(\lambda_1) = \max_{x \in \Omega} r(x)$. Second, we take λ_2 such that $r(\lambda_2) = \max\{r(x); x \in \Omega \backslash B(\lambda_1, \delta r(\lambda_1))\}$. Third, we take λ_3 such that $r(\lambda_3) = \max\{r(x); x \in \Omega \backslash (B(\lambda_1, \delta r(\lambda_1)) \cup B(\lambda_2, \delta r(\lambda_2)))\}$. Proceeding this process, we can obtain a $\left(\frac{\delta}{2}, \delta\right)$-lattice. In fact, the condition (a) follows easily from the way of the construction. We check the condition (b). If we assume $\bigcup B(\lambda_i, \delta r(\lambda_i)) \neq \Omega$, there exists a point $x_0 \in \Omega$ such that $x_0 \notin \bigcup B(\lambda_i, \delta r(\lambda_i))$. Put $r_0 = r(x_0)$. Then $r(\lambda_i) \geq r_0$ for all i. Since the family $\left\{B\left(\lambda_i, \frac{\delta}{2} r_0\right)\right\}_i$ are pairwise disjoint, the volume $|\Omega|$ must be infinite, which contradicts the boundness of Ω. Thus we have the condition (b) in (2) of Definition 1. $\square$

Definition 2. A family $\{U_i\}$ of subsets of Ω is said to have the *uniformly finite intersection* (with bound $N > 0$), if $\#\{i \in \mathbf{N}; x \in U_i\} \leq N$ for any $x \in \Omega$.

By Lemma 9, we easily obtain the following.

Lemma 11. *Let $0 < \delta < \frac{1}{4}$. If a sequence $\{\lambda_i\}$ is $\frac{\delta}{2}$-separated, then $\{B(\lambda_i, 3\delta r(\lambda_i))\}$ has the uniformly finite intersection with bound 100^n.*

PROOF. Let $0 < \delta < \frac{1}{4}$ and let a sequence $\{\lambda_i\}$ be $\frac{\delta}{2}$-separated. Using Lemma 9 as $\varepsilon = \frac{\delta}{2}$ and 3δ, we can calculate $\#\{j \in \mathbf{N}; B(\lambda_i, 3\delta r(\lambda_i)) \cap B(\lambda_j, 3\delta r(\lambda_j)) \neq \emptyset\}$. □

PROPOSITION 1. *There exists a constant $N > 0$ such that for $0 < \delta < \frac{1}{4}$, we can choose a sequence $\{\lambda_i^\delta\}$ and a disjoint covering $\{E_i^\delta\}$ of Ω satisfying the following conditions:*

(a) *E_i^δ is measurable for each $i \in \mathbf{N}$;*

(b) *$E_i^\delta \subset B(\lambda_i^\delta, \delta r(\lambda_i^\delta))$ for each $i \in \mathbf{N}$;*

(c) *$\{B(\lambda_i^\delta, 3\delta r(\lambda_i^\delta))\}$ has the uniformly finite intersection with bound N.*

PROOF. For $0 < \delta < \frac{1}{4}$, we take $\{\lambda_i^\delta\}$ in Lemma 10 and we put $E_1^\delta := B(\lambda_1^\delta, \delta r(\lambda_1^\delta))$, $E_2^\delta := B(\lambda_2^\delta, \delta r(\lambda_2^\delta)) \backslash B(\lambda_1^\delta, \delta r(\lambda_1^\delta))$ and $E_3^\delta := B(\lambda_3^\delta, \delta r(\lambda_3^\delta)) \backslash (E_1^\delta \cup E_2^\delta), \ldots,$ inductively. We can easily check that these $\{\lambda_i^\delta\}$ and $\{E_i^\delta\}$ satisfy the above conditions (a) and (b). By Lemma 11, we can easily check the condition (c) with $N = 100^n$. □

We refer $\{\lambda_i^\delta\}$ and $\{E_i^\delta\}$ obtained in Proposition 1 as the standard δ-sequence of Ω and the standard δ-covering of Ω, respectively.

4. Atomic decomposition

In this section, we prove Theorem 1. We discuss the operator

$$V_{p,\{\lambda_i\}}^\delta f := \{(\delta r(\lambda_i))^{n/p} f(\lambda_i)\},$$

where $1 \leq p < \infty$, $0 < \delta \leq 1$ and $\{\lambda_i\}$ is a sequence in Ω.

LEMMA 12. *Let $1 \leq p < \infty$ and $0 < \delta < 1$. If $\{B(\lambda_i, \delta r(\lambda_i))\}$ has the uniformly finite intersection with bound N, then for each $0 < \varepsilon \leq 1$ the operator $V_{p,\{\lambda_i\}}^\varepsilon : b^p \to l^p$ is bounded:*

$$\|V_{p,\{\lambda_i\}}^\varepsilon\| \leq C\left(\frac{\varepsilon}{\delta}\right)^{n/p} N^{1/p}$$

with some constant $C > 0$ depending only on the dimension n.

PROOF. Let $f \in b^p(\Omega)$. Since $|f|^p$ is subharmonic, the sub-mean value property implies

$$|f(\lambda_i)|^p \leq \frac{1}{(\delta r(\lambda_i))^n |B|} \int_{B(\lambda_i, \delta r(\lambda_i))} |f(y)|^p dy$$

for each i, where $|B|$ is the volume of the unit open ball in $\mathbf{R}^n$. Then we have

$$\sum_{i=1}^{\infty}(\varepsilon r(\lambda_i))^n|f(\lambda_i)|^p \le \left(\frac{\varepsilon}{\delta}\right)^n \frac{1}{|B|}\sum_{i=1}^{\infty}\int_{B(\lambda_i,\delta r(\lambda_i))}|f(y)|^p dy$$
$$\le \left(\frac{\varepsilon}{\delta}\right)^n \frac{N}{|B|}\int_{\Omega}|f(y)|^p dy. \qquad \square$$

Lemma 13. *Let* $1 < p < \infty$ *and* $0 < \delta < \frac{1}{4}$. *If a sequence* $\{\lambda_i\}$ *and a disjoint covering* $\{E_i\}$ *of* Ω *satisfy* $E_i \subset B(\lambda_i, \delta r(\lambda_i))$ *for each i, then there exists a constant* $C = C(n,p) > 0$ *such that*

$$\sum_{i=1}^{\infty}\int_{E_i}|f(x) - f(\lambda_i)|^p dx \le C\delta^p\|f\|_p^p \qquad \text{for all } f \in b^p(\Omega).$$

Proof. Let $1 < p < \infty$, $0 < \delta < \frac{1}{4}$ and $f \in b^p(\Omega)$. For any $x \in E_i$, by Lemma 1,

$$|f(x) - f(\lambda_i)| = \left|\int_{\Omega}(R(x,y) - R(\lambda_i,y))f(y)dy\right|$$
$$\le \int_{\Omega}|R(x,y) - R(\lambda_i,y)|\,|f(y)|dy$$
$$\le \int_{\Omega}|x - \lambda_i|\,|\nabla R(\tilde{x},y)|\,|f(y)|dy$$
$$\le \int_{\Omega}\delta r(\lambda_i)\frac{C_1}{d(\tilde{x},y)^{n+1}}|f(y)|dy$$
$$\le C_2\delta\int_{\Omega}\frac{1}{d(x,y)^n}|f(y)|dy$$
$$= C_2\delta K|f|(x),$$

where ∇ denotes the gradient operator on $\mathbf{R}^n$ and $\tilde{x}$ is given by the mean value property in calculus. Here we remark that constants C_1 and C_2 depend only on the dimension n. By Lemma 6, we obtain

$$\sum_{i=1}^{\infty}\int_{E_i}|f(x) - f(\lambda_i)|^p dx \le C_2^p\delta^p\sum_{i=1}^{\infty}\int_{E_i}(K|f|(x))^p dx$$
$$= C_2^p\delta^p\int_{\Omega}(K|f|(x))^p dx$$
$$\le C_3\delta^p\|f\|_p^p,$$

which shows the lemma. $\square$

In the following, we introduce three operators, which are closely related to our atomic decomposition.

First, we define the operator $A^{\delta}_{p,\{\lambda_i\}}$ by

$$A^{\delta}_{p,\{\lambda_i\}}(\{a_i\})(x) := \sum_{i=1}^{\infty} a_i R(x,\lambda_i)(\delta r(\lambda_i))^{n/q} \tag{6}$$

for $\{a_i\} \in l^p$, where $0 < \delta \leq 1$, $1 < p < \infty$, q is the exponent conjugate to p and $\{\lambda_i\}$ is a sequence in Ω. The following lemma shows the well-definedness of $A^{\delta}_{p,\{\lambda_i\}}$.

LEMMA 14. *Let $0 < \delta \leq 1$, $1 < p < \infty$ and let q be the exponent conjugate to p. If the operator $V^{\delta}_{q,\{\lambda_i\}} : b^q \to l^q$ is bounded, then for any $\{a_i\} \in l^p$ the right hand side of* (6) *converges absolutely for each $x \in \Omega$. Moreover the right hand side of* (6) *converges in b^p as a function of x, and $A^{\delta}_{p,\{\lambda_i\}} : l^p \to b^p$ is a bounded linear operator.*

PROOF. Let $\{a_i\} \in l^p$. By the Hölder inequality and Lemma 1, we have

$$\begin{aligned}\sum_{i=1}^{\infty} |a_i R(x,\lambda_i)(\delta r(\lambda_i))^{n/q}| &\leq C\|\{a_i\}\|_{l^p}\left(\sum_{i=1}^{\infty}\frac{1}{d(x,\lambda_i)^{nq}}(\delta r(\lambda_i))^n\right)^{1/q}\\ &\leq C\|\{a_i\}\|_{l^p}\frac{1}{r(x)^n}\left(\sum_{i=1}^{\infty}(\delta r(\lambda_i))^n\right)^{1/q}\\ &= C\|\{a_i\}\|_{l^p}\frac{\|V^{\delta}_{q,\{\lambda_i\}}1\|_{l^q}}{r(x)^n}.\end{aligned}$$

This implies $A^{\delta}_{p,\{\lambda_i\}}(\{a_i\})$ converges absolutely for each $x \in \Omega$ and uniformly on every compact subset of Ω. Hence $A^{\delta}_{p,\{\lambda_i\}}(\{a_i\})$ is a harmonic function in Ω.

Next, we consider the partial sum

$$g_m(x) := \sum_{k=1}^{m} a_k R(x,\lambda_k)(\delta r(\lambda_k))^{n/q}$$

of the series in (6). For any $f \in b^q$, by the Hölder inequality and Lemma 1, we have

$$\begin{aligned}|\langle g_m, f\rangle_L| &= \left|\int_\Omega \sum_{k=1}^{m} a_k R(x,\lambda_k)(\delta r(\lambda_k))^{n/q} f(x)dx\right|\\ &= \left|\sum_{k=1}^{m} a_k (\delta r(\lambda_k))^{n/q}\int_\Omega R(x,\lambda_k)f(x)dx\right|\end{aligned}$$

$$= \left| \sum_{k=1}^{m} a_k (\delta r(\lambda_k))^{n/q} f(\lambda_k) \right|$$

$$\leq \left(\sum_{k=1}^{m} |a_k|^p \right)^{1/p} \| V_{q,\{\lambda_i\}}^{\delta} f \|_{l^q}$$

$$\leq C \left(\sum_{k=1}^{m} |a_k|^p \right)^{1/p} \|f\|_q,$$

which shows by Lemma 8,

$$\|g_m\|_p \leq C \sup_{\|f\|_q = 1} |\langle g_m, f \rangle_L| \leq C \left(\sum_{k=1}^{m} |a_k|^p \right)^{1/p}. \tag{7}$$

Similarly, for positive integers $s > t$, we have

$$\|g_s - g_t\|_p \leq C \left(\sum_{k=t+1}^{s} |a_k|^p \right)^{1/p},$$

which implies $\{g_m\}$ is a Cauchy sequence in b^p. Hence, there exists $g \in b^p$ such that $g_m \to g$ in b^p. We also have the norm estimate $\|g\|_p = \lim_{m\to\infty} \|g_m\|_p \leq C \|\{a_k\}\|_{l^p}$ from (7), i.e., $\|A_{p,\{\lambda_i\}}^{\delta}(\{a_k\})\| \leq C \|\{a_k\}\|_{l^p}$. This completes the proof.

Next, we define the operator $U_{p,\{\lambda_i\},\{E_i\}}^{\delta}$ by

$$U_{p,\{\lambda_i\},\{E_i\}}^{\delta} f := \{|E_i| f(\lambda_i) (\delta r(\lambda_i))^{-n/q}\},$$

where $0 < \delta \leq 1$, $1 < p < \infty$ and q is the exponent conjugate to p.

Lemma 15. *Let $1 < p < \infty$ and $0 < \delta \leq 1$. Suppose $\{\lambda_i\}$ is a sequence in Ω and the family $\{E_i\}$ satisfies there exists $0 < \varepsilon \leq 1$ such that $E_i \subset B(\lambda_i, \varepsilon r(\lambda_i))$ for each i. If the operator $V_{p,\{\lambda_i\}}^{\delta} : b^p \to l^p$ is bounded, then $U_{p,\{\lambda_i\},\{E_i\}}^{\delta} : b^p \to l^p$ is a bounded linear operator.*

Proof. Let $f \in b^p$ and put $b_i := |E_i| f(\lambda_i)(\delta r(\lambda_i))^{-n/q}$. Inequalities $|E_i| \leq |B(\lambda_i, \varepsilon r(\lambda_i))| \leq C\left(\frac{\varepsilon}{\delta}\right)^n (\delta r(\lambda_i))^n$ imply $|b_i| \leq C |f(\lambda_i)| (\delta r(\lambda_i))^{n/p}$. Then we have

$$\sum_{i=1}^{\infty} |b_i|^p \leq C \sum_{i=1}^{\infty} |f(\lambda_i)|^p (\delta r(\lambda_i))^n = C \| V_{p,\{\lambda_i\}}^{\delta} f \|_{l^p}^p \leq C \|f\|_p^p. \qquad \square$$

Finally, we define the operator $S_{p,\{\lambda_i\},\{E_i\}}$ by

$$S_{p,\{\lambda_i\},\{E_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i|. \tag{8}$$

By a simple calculation, we obtain $S_{p,\{\lambda_i\},\{E_i\}} = A^{\delta}_{p,\{\lambda_i\}} \circ U^{\delta}_{p,\{\lambda_i\},\{E_i\}}$. Thus we have the following:

LEMMA 16. *Let $1 < p < \infty$ and $0 < \delta \le 1$. Suppose $\{\lambda_i\}$ is a sequence in Ω and the family $\{E_i\}$ satisfies there exists $0 < \varepsilon \le 1$ such that $E_i \subset B(\lambda_i, \varepsilon r(\lambda_i))$ for each i. If $\{B(\lambda_i, \delta r(\lambda_i))\}$ has the uniformly finite intersection, then for any $f \in b^p$, the right hand side of (8) converges in b^p as functions of x. Moreover $S_{p,\{\lambda_i\},\{E_i\}} : b^p \to b^p$ is a bounded linear operator.*

Moreover, we have the following lemma for $S_{p,\{\lambda_i\},\{E_i\}}$, which plays an essential role for atomic decompositions.

LEMMA 17. *Let $1 < p < \infty$. Let $\{\lambda_i^{\delta}\}$ and $\{E_i^{\delta}\}$ be a standard δ-sequence of Ω and a standard δ-covering of Ω, respectively. Then there exists a constant $0 < \delta_4 < 1$ such that for any $0 < \delta < \delta_4$, the operator $S_{p,\{\lambda_i^{\delta}\},\{E_i^{\delta}\}} : b^p \to b^p$ is bijective.*

PROOF. We have only to prove $\|I - S_{p,\{\lambda_i^{\delta}\},\{E_i^{\delta}\}}\| < 1$ for sufficiently small $\delta > 0$. Let q be the exponent conjugate to p and take $f \in b^p$ and $g \in b^q$, arbitrarily. Then we have

$$\begin{aligned}
\langle (I - S_{p,\{\lambda_i^{\delta}\},\{E_i^{\delta}\}})f, g\rangle_L &= \int_{\Omega} f(x)g(x)dx - \int_{\Omega} \sum_{i=1}^{\infty} |E_i^{\delta}| f(\lambda_i^{\delta}) R(x, \lambda_i^{\delta}) g(x)dx \\
&= \sum_{i=1}^{\infty} \int_{E_i^{\delta}} f(x)g(x)dx - \sum_{i=1}^{\infty} |E_i^{\delta}| f(\lambda_i^{\delta}) g(\lambda_i^{\delta}) \\
&= \sum_{i=1}^{\infty} \int_{E_i^{\delta}} (f(x)g(x) - f(\lambda_i^{\delta})g(\lambda_i^{\delta}))dx \\
&= \sum_{i=1}^{\infty} \int_{E_i^{\delta}} (f(x)g(x) - f(x)g(\lambda_i^{\delta}))dx \\
&\quad + \sum_{i=1}^{\infty} \int_{E_i^{\delta}} (f(x)g(\lambda_i^{\delta}) - f(\lambda_i^{\delta})g(\lambda_i^{\delta}))dx.
\end{aligned}$$

By Lemmas 12 and 13 and the Hölder inequality, we have estimates

$$\begin{aligned}
\left| \sum_{i=1}^{\infty} \int_{E_i^{\delta}} f(x)(g(x) - g(\lambda_i^{\delta}))dx \right| &\le \left(\sum_{i=1}^{\infty} \int_{E_i^{\delta}} |f(x)|^p dx \right)^{1/p} \\
&\quad \times \left(\sum_{i=1}^{\infty} \int_{E_i^{\delta}} |g(x) - g(\lambda_i^{\delta})|^q dx \right)^{1/q} \\
&\le C_1 \delta \|f\|_p \|g\|_q
\end{aligned}$$

and

$$\left|\sum_{i=1}^{\infty}\int_{E_i^\delta}(f(x)-f(\lambda_i^\delta))g(\lambda_i^\delta)dx\right| \le \sum_{i=1}^{\infty}\left(\int_{E_i^\delta}|f(x)-f(\lambda_i^\delta)|^p dx\right)^{1/p}|g(\lambda_i^\delta)|\,|E_i^\delta|^{1/q}$$
$$\le \left(\sum_{i=1}^{\infty}\int_{E_i^\delta}|f(x)-f(\lambda_i^\delta)|^p dx\right)^{1/p}\|V^\delta_{q,\{\lambda_i^\delta\}}g\|_{l^q}$$
$$\le C_2\delta\|f\|_p\|g\|_q.$$

Hence,

$$|\langle(I-S_{p,\{\lambda_i^\delta\},\{E_i^\delta\}})f,g\rangle_L| \le (C_1+C_2)\delta\|f\|_p\|g\|_q,$$

which implies $\|I-S_{p,\{\lambda_i^\delta\},\{E_i^\delta\}}\| \le (C_1+C_2)\delta$. Since the constants C_1 and C_2 are independent of δ, choosing $0<\delta_4\le\frac{1}{C_1+C_2}$, we obtain $\|I-S_{p,\{\lambda_i^\delta\},\{E_i^\delta\}}\|<1$ for any $0<\delta<\delta_4$. This completes the proof. □

As a consequence, we obtain the following theorem.

Theorem 3. *Let $1<p<\infty$ and let $\{\lambda_i^\delta\}$ be a standard δ-sequence of Ω. There exists a constant $0<\delta_5\le\frac14$ such that for any $0<\delta<\delta_5$, $A^\delta_{p,\{\lambda_i^\delta\}}: l^p\to b^p$ is surjective. In fact, there exists a bounded linear operator $T: b^p\to l^p$ such that $A^\delta_{p,\{\lambda_i^\delta\}}\circ T$ is the identity on b^p.*

Proof. We take a constant $0<\delta_4\le\frac14$ in Lemma 17 and put $\delta_5=\delta_4$. Then for $0<\delta<\delta_4$, we can put $T:=U^\delta_{p,\{\lambda_i^\delta\},\{E_i^\delta\}}\circ(S_{p,\{\lambda_i^\delta\},\{E_i^\delta\}})^{-1}$. By Lemmas 15 and 17, T is a bounded linear operator and $A^\delta_{p,\{\lambda_i^\delta\}}\circ T$ is the identity on b^p. Hence, $A^\delta_{p,\{\lambda_i^\delta\}}: l^p\to b^p$ is surjective. This completes the proof. □

Proof (of Theorem 1). Theorem 3 implies Theorem 1. In fact, we take a constant $\delta_5>0$ in Theorem 3. Then $A^\delta_{p,\{\lambda_i^\delta\}}: l^p\to b^p$ is surjective for $0<\delta<\delta_5$. Hence, for any $f\in b^p$, we can choose a sequence $\{a_i'\}\in l^p$ such that

$$f(x)=A^\delta_{p,\{\lambda_i^\delta\}}(\{a_i'\})(x)=\sum_{i=1}^{\infty}a_i'R(x,\lambda_i^\delta)(\delta r(\lambda_i^\delta))^{n/q}.$$

The atomic decomposition of f in Theorem 1 is given by $\{a_i\}:=\{\delta^{n/q}a_i'\}\in l^p$. This completes the proof.

5. Relation of operators

In this section, we discuss the operators A and V in section 4. We put $A_{p,\{\lambda_i\}}:=A^1_{p,\{\lambda_i\}}$, $V_{p,\{\lambda_i\}}:=V^1_{p,\{\lambda_i\}}$, whose domains are $\mathscr{D}(A_{p,\{\lambda_i\}}):=$

$l_c \subset l^p$ and $\mathscr{D}(V_{p,\{\lambda_i\}}) := \{f \in b^p; V_{p,\{\lambda_i\}} f \in l^p\}$, respectively. Here $l_c := \{\{a_i\}; \#\{i \in \mathbf{N}; a_i \neq 0\} < \infty\}$. For any $1 \le p < \infty$, l_c is dense in l^p.

Lemma 18. *Let $1 < p < \infty$, q be the exponent conjugate to p and $\{\lambda_i\}_i \subset \Omega$ be a sequence. If the operator $A_{p,\{\lambda_i\}} : l_c (\subset l^p) \to b^p$ is bounded, then there exists a constant $\delta > 0$ independent of $\{\lambda_i\}$ such that $\{B(\lambda_i, \delta r(\lambda_i))\}_i$ has the uniformly finite intersection.*

Proof. Let x_0 be any point in Ω. Take a constant $0 < \delta_3 < 1$ in Lemma 3 and fix a constant $0 < \delta < \frac{\delta_3}{1+\delta_3}$. Remark that if $x \in B(y, \delta r(y))$, then $y \in B(x, \delta_3 r(x))$. For $M > 0$, we consider a sequence $\{a_i^M\}$ such that

$$a_i^M = \begin{cases} 1 & \text{if } i \in \Lambda_{x_0,\delta,M} \\ 0 & \text{if } i \notin \Lambda_{x_0,\delta,M}, \end{cases}$$

where $\Lambda_{x_0,\delta,M} := \{i \in \mathbf{N}; x_0 \in B(\lambda_i, \delta r(\lambda_i)), i \le M\}$. Then $\{a_i^M\} \in l_c$ and $\|\{a_i^M\}\|_{l^p} = (\#\Lambda_{x_0,\delta,M})^{1/p}$. Since $A_{p,\{\lambda_i\}}$ is bounded, we have

$$\|A_{p,\{\lambda_i\}}(\{a_i^M\})\|_p \le C(\#\Lambda_{x_0,\delta,M})^{1/p}$$

with some constant $C > 0$. On the other hand, by Lemma 3, we have

$$\begin{aligned}
\|A_{p,\{\lambda_i\}}(\{a_i^M\})\|_p &= \left(\int_\Omega |\Sigma_{i \in \Lambda_{x_0,\delta,M}} R(x,\lambda_i) r(\lambda_i)^{n/q}|^p dx\right)^{1/p} \\
&\ge C\left(\int_{B(x_0,\delta_3 r(x_0))} (\#\Lambda_{x_0,\delta,M} r(x_0)^{-n} r(x_0)^{n/q})^p dx\right)^{1/p} \\
&= C\#\Lambda_{x_0,\delta,M}(r(x_0)^{-n}|B(x_0,\delta_3 r(x_0))|)^{1/p} \\
&= C\#\Lambda_{x_0,\delta,M}(\delta_3^n|B|)^{1/p}.
\end{aligned}$$

Hence, we obtain

$$\#\Lambda_{x_0,\delta,M} \le C(|B|\delta_3^n)^{1/(1-p)}.$$

Since $M > 0$ is arbitrary, we have

$$\#\{i \in \mathbf{N}; x_0 \in B(\lambda_i, \delta r(\lambda_i))\} \le C(\delta_3^n|B|)^{1/(1-p)},$$

which shows that $\{B(\lambda_i, \delta r(\lambda_i))\}$ has the uniformly finite intersection with bound $C(\delta_3^n|B|)^{1/(1-p)}$. This completes the proof. □

We show the relation between $A_{p,\{\lambda_i\}}$ and $V_{p,\{\lambda_i\}}$.

Theorem 4. *Let $1 < p < \infty$ and let q be the exponent conjugate to p. Then the relation $A^*_{p,\{\lambda_i\}} = V_{q,\{\lambda_i\}}$ holds. Moreover, the following conditions are equivalent:*

(a) $A_{p,\{\lambda_i\}} : l^p \to b^p$ *is bounded.*
(b) $V_{q,\{\lambda_i\}} : b^q \to l^q$ *is bounded.*
(c) *There exists a constant* $0 < \delta < 1$ *such that* $\{B(\lambda_i, \delta r(\lambda_i))\}$ *has the uniformly finite intersection.*

Remark 1. *The boundedness of* $A_{p,\{\lambda_i\}}$ *and that of* $V_{p,\{\lambda_i\}}$ *are equivalent for every* $1 < p < \infty$.

Proof. Let $\{a_i\} \in l_c$ and $f \in b^q$. Then we have

$$\begin{aligned}\langle A_{p,\{\lambda_i\}}(\{a_i\}), f\rangle_L &= \int_\Omega \sum_{i=1}^{\infty} a_i R(x,\lambda_i) r(\lambda_i)^{n/q} f(x)dx \\ &= \sum_{i=1}^{\infty} a_i r(\lambda_i)^{n/q} \int_\Omega R(x,\lambda_i) f(x)dx \\ &= \sum_{i=1}^{\infty} a_i r(\lambda_i)^{n/q} f(\lambda_i) \\ &= \sum_{i=1}^{\infty} a_i (V_{q,\{\lambda_i\}} f)_i. \end{aligned} \tag{9}$$

First, we assume $f \in \mathscr{D}(V_{q,\{\lambda_i\}})$. Then $V_{q,\{\lambda_i\}} f \in l^q = (l^p)^*$. By (9), we have

$$\langle A_{p,\{\lambda_i\}}(\{a_i\}), f\rangle_L = \langle \{a_i\}, V_{q,\{\lambda_i\}} f\rangle_l,$$

which implies $f \in \mathscr{D}(A^*_{p,\{\lambda_i\}})$ and $A^*_{p,\{\lambda_i\}} f = V_{q,\{\lambda_i\}} f$. Hence, we obtain $\mathscr{D}(V_{q,\{\lambda_i\}}) \subset \mathscr{D}(A^*_{p,\{\lambda_i\}})$. Next, we assume $f \in \mathscr{D}(A^*_{p,\{\lambda_i\}})$. Then $A^*_{p,\{\lambda_i\}} f \in (l^p)^* = l^q$. By (9), we have

$$\begin{aligned}\langle \{a_i\}, A^*_{p,\{\lambda_i\}} f\rangle_l &= \langle A_{p,\{\lambda_i\}}(\{a_i\}), f\rangle_L \\ &= \sum_{i=1}^{\infty} a_i (V_{q,\{\lambda_i\}} f)_i\end{aligned}$$

for any $\{a_i\} \in l_c$. Hence, $A^*_{p,\{\lambda_i\}} f = V_{q,\{\lambda_i\}} f$ and $f \in \mathscr{D}(V_{q,\{\lambda_i\}})$. Therefore, $A^*_{p,\{\lambda_i\}} = V_{q,\{\lambda_i\}}$. Thus the first part is proved. The second part follows easily from Lemmas 12, 14 and 18. This completes the proof. □

6. Generalization of Theorem 1

In section 4, we studied the atomic decomposition for the standard sequence $\{\lambda_i^\delta\} \subset \Omega$. In this section, we consider a sufficient condition for a sequence $\{\lambda_i\}$ in Ω such that any b^p function has the atomic decomposition. We shall prove a reformulated version of Theorem 2.

THEOREM 5. *Let $1 < p < \infty$. Then there exists a constant $\delta_6 > 0$ with the following property: if a sequence $\{\lambda_i\}$ in Ω satisfies $\bigcup B(\lambda_i, \delta_6 r(\lambda_i)) = \Omega$, then there exists a bounded linear operator $W : b^p \to l^p$ such that $A^{\delta_6}_{p,\{\lambda_i\}} \circ W$ is the identity on b^p.*

PROOF. Let $0 < \delta < \frac{1}{4}$. We take a standard δ-sequence $\{\lambda_i^\delta\}$ in Ω and a standard δ-covering $\{E_i^\delta\}$ of Ω. Suppose that a sequence $\{\lambda_i\}$ satisfies $\bigcup B(\lambda_i, \delta r(\lambda_i)) = \Omega$. Then, for each $i \in \mathbf{N}$, we can choose $\lambda_i' \in \{\lambda_j; j \in \mathbf{N}\}$ such that $\lambda_i^\delta \in B(\lambda_i', \delta r(\lambda_i'))$.

First, we claim that the family $\{B(\lambda_i', \delta r(\lambda_i'))\}$ has the uniformly finite intersection. We show $B(\lambda_i', \delta r(\lambda_i')) \subset B(\lambda_i^\delta, 3\delta r(\lambda_i^\delta))$. In fact, as in the proof of Lemma 3, we have

$$(1-\delta) r(\lambda_i') < r(\lambda_i^\delta) < (1+\delta) r(\lambda_i'). \tag{10}$$

Furthermore, if $x \in B(\lambda_i', \delta r(\lambda_i'))$, then

$$|x - \lambda_i^\delta| \le |x - \lambda_i'| + |\lambda_i' - \lambda_i^\delta| < 2\delta r(\lambda_i') < \frac{2\delta}{1-\delta} r(\lambda_i^\delta) < 3\delta r(\lambda_i^\delta).$$

Thus, we obtain $B(\lambda_i', \delta r(\lambda_i')) \subset B(\lambda_i^\delta, 3\delta r(\lambda_i^\delta))$. Since $\{\lambda_i^\delta\}$ is the standard δ-sequence in Ω, the family $\{B(\lambda_i^\delta, 3\delta r(\lambda_i^\delta))\}$ has the uniformly finite intersection with some bound N. Therefore, $\{B(\lambda_i', \delta r(\lambda_i'))\}$ has also the uniformly finite intersection with the same bound N.

Next, we show that there exists $0 < \varepsilon \le 1$ such that $E_i^\delta \subset B(\lambda_i', \varepsilon r(\lambda_i'))$ for each i. Indeed, we put $\varepsilon = \delta(2+\delta)$. Then, by (10), we have

$$\begin{aligned} E_i^\delta &\subset B(\lambda_i^\delta, \delta r(\lambda_i^\delta)) \subset B(\lambda_i^\delta, \delta(1+\delta) r(\lambda_i')) \\ &\subset B(\lambda_i', \delta(1+\delta) r(\lambda_i') + \delta r(\lambda_i')) = B(\lambda_i', \varepsilon r(\lambda_i')). \end{aligned}$$

Let q be the exponent conjugate to p. Then, since $\{B(\lambda_i', \delta r(\lambda_i'))\}$ has the uniformly finite intersection, Lemmas 12 and 14 imply that the operators $V^\delta_{q,\{\lambda_i'\}} : b^q \to l^q$ and $A^\delta_{p,\{\lambda_i'\}} : l^p \to b^p$ are bounded. Moreover, since there exists $0 < \varepsilon \le 1$ such that $E_i^\delta \subset B(\lambda_i', \varepsilon r(\lambda_i'))$ for each i, Lemmas 15 and 16 imply that the operators $U^\delta_{p,\{\lambda_i'\},\{E_i^\delta\}} : b^p \to l^p$ and $S_{p,\{\lambda_i'\},\{E_i^\delta\}} : b^p \to b^p$ are bounded. Here, we remark $S_{p,\{\lambda_i'\},\{E_i^\delta\}} = A^\delta_{p,\{\lambda_i'\}} \circ U^\delta_{p,\{\lambda_i'\},\{E_i^\delta\}}$. By Lemma 13, for any $f \in b^p$ and $g \in b^q$, we have

$$\sum_{i=1}^\infty \int_{E_i^\delta} |f(x) - f(\lambda_i')|^p dx \le C\varepsilon^p \|f\|_p^p$$

and

$$\sum_{i=1}^{\infty}\int_{E_i^\delta}|g(x)-g(\lambda_i')|^q dx \le C\varepsilon^q\|g\|_q^q.$$

Therefore, as in the proof of Lemma 17, we obtain from Lemma 12

$$\begin{aligned}
&|\langle (I-S_{p,\{\lambda_i'\},\{E_i^\delta\}})f,g\rangle_L| \\
&\le \left(\sum_{i=1}^{\infty}\int_{E_i^\delta}|f(x)|^p dx\right)^{1/p}\left(\sum_{i=1}^{\infty}\int_{E_i^\delta}|g(x)-g(\lambda_i')|^q dx\right)^{1/q} \\
&\quad+\sum_{i=1}^{\infty}\left(\int_{E_i^\delta}|f(x)-f(\lambda_i')|^p dx\right)^{1/p}|g(\lambda_i')|\,|E_i^\delta|^{1/q} \\
&\le C_1\varepsilon\|f\|_p\|g\|_q + C_2\left(\sum_{i=1}^{\infty}\int_{E_i^\delta}|f(x)-f(\lambda_i')|^p dx\right)^{1/p}\left(\sum_{i=1}^{\infty}(\varepsilon r(\lambda_i'))^n|g(\lambda_i')|^q\right)^{1/q} \\
&\le C_1\varepsilon\|f\|_p\|g\|_q + C_2\varepsilon\|f\|_p\|V^{\varepsilon}_{q,\{\lambda_i'\}}g\|_{l^q} \\
&\le C_1\varepsilon\|f\|_p\|g\|_q + C_2\varepsilon\left(\frac{\varepsilon}{\delta}\right)^{n/p}\|f\|_p\|g\|_q \\
&= \delta(2+\delta)(C_1+C_2(2+\delta)^{n/p})\|f\|_p\|g\|_q,
\end{aligned}$$

where the constants C_1 and C_2 are independent of δ. Since we can choose $\delta_6>0$ such that $\delta(2+\delta)(C_1+C_2(2+\delta)^{n/p})<1$ whenever $\delta\le\delta_6$, we have the theorem. In fact, putting

$$T' := U^{\delta_6}_{p,\{\lambda_i'\},\{E_i^{\delta_6}\}}\circ(S_{p,\{\lambda_i'\},\{E_i^{\delta_6}\}})^{-1},$$

we find that T' is bounded and $A^{\delta_6}_{p,\{\lambda_i'\}}\circ T'$ is the identity on b^p. Since $\{\lambda_i'; i\in\mathbf{N}\}$ is a subset of $\{\lambda_j; j\in\mathbf{N}\}$, we can construct the desired operator W from T'. This completes the proof. □

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