Hiroshima Math. J.
42 (2012), 127–142

N-degeneracy in rack homology and link invariants

Mohamed Elhamdadi and Sam Nelson

(Received July 23, 2010)
(Revised May 23, 2011)

Abstract. The aim of this paper is to define a homology theory for racks with finite rank N and use it to define invariants of knots generalizing the CJKLS 2-cocycle invariants related to the invariants defined in [15]. For this purpose, we prove that N-degenerate chains form a sub-complex of the classical complex defining rack homology. If a rack has rack rank $N = 1$ then it is a quandle and our homology theory coincides with the CKJLS homology theory [6]. Nontrivial cocycles are used to define invariants of knots and examples of calculations for classical knots with up to 8 crossings and classical links with up to 7 crossings are provided.

1. Introduction

Racks are algebraic structures with axioms derived from Reidemeister moves type II and type III. They have been considered by knot theorists in order to construct knot and link invariants and their higher analogues (see for example [7] and references therein). Racks allow a refined and a complete algebraic framework in which ones investigates links and 3-manifolds. They have been studied by many authors and appeared in the literature with different names such as automorphic sets and in a special case quandles, distributive groupoids, crystals etc. Rack cohomology was introduced by Fenn, Rourke and Sanderson [11]. For each rack X and abelian group A, they defined cohomology groups $H^n(X,A)$. Since then there has been a number of results about this cohomology (see [9], [13] and [10]) with studies from different perspectives. A modification of rack cohomology theory led to quandle cohomology theory which was developed in [6] in order to define invariants of classical knots and knotted surfaces in state-sum form, called quandle cocycle (knot) invariants. These invariants can be understood as enhancements of the quandle counting invariant, where quandle colorings of a link diagram are counted with a weight determined by a quandle cocycle.

2010 *Mathematics Subject Classification*. 57M27, 57M25.
Key words and phrases. Finite racks, rack homology, enhancements of counting invariants, cocycle invariants.

In [5], the quandle homology theory was generalized to the case when the coefficient groups admit the structure of Alexander quandles, by including an action of the infinite cyclic group in the boundary operator. Using generalizations of quandle homology theory provided by Andruskiewitsch and Graña [1], the quandle cocycle invariants were generalized in three different directions in [3]. The second author has studied several enhancements of quandle counting invariants of a link L with respect to a finite target quandle T (see for example [15] and [14]).

The paper is organized as follows. After reviewing the basics of racks and recalling the notion of rank for racks in section 2, rack homology is considered in section 3. As our main result, we prove that the degenerate chains form a sub-complex of the classical complex defining rack homology and define a homology theory analogous to quandle homology for non-quandle racks which reduces to the usual quandle homology when the rack is a quandle. In Section 4 we use cocycles in $H^2_{R/ND}$ to enhance the rack counting invariant, obtaining a family of link invariants which generalize the CJKLS invariants to allow non-quandle racks. Note that while a related invariant was defined in [15], the fact that N-degenerate chains form a subcomplex was not proved in [15], only invariance under blackboard framed isotopy and the N-phone cord move. In section 5 explicit reduced cocycles are given and used to perform computations of the invariants for prime knots and links. We end the paper with section 6 in which we make some remarks and suggest some open questions for future investigations.

2. Review of racks

We begin with a definition from [12], give examples and then recall from [15] the notion of rank for racks.

Definition 1. A *rack* is a set X with two binary operations $\triangleright$ and $\triangleright^{-1}$ satisfying for all $x, y, z \in X$

(i) $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$ and

(ii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

A rack which further satisfies $x \triangleright x = x$ for all $x \in X$ is a *quandle.*

Example 1. Let V be any $\mathbf{Z}_4$-module and define

$$\mathbf{u} \triangleright \mathbf{v} = \mathbf{u} + 2\mathbf{v}.$$

Then V is a rack. The inverse operation $\mathbf{u} \triangleright^{-1} \mathbf{v} = \mathbf{u} + 2\mathbf{v}$ coincides with the triangle operation, and we have

$$(\mathbf{u} \triangleright \mathbf{v}) \triangleright \mathbf{w} = (\mathbf{u} + 2\mathbf{v}) + 2\mathbf{w} = \mathbf{u} + 2\mathbf{v} + 2\mathbf{w},$$

while

$$(\mathbf{u} \triangleright \mathbf{w}) \triangleright (\mathbf{v} \triangleright \mathbf{w}) = (\mathbf{u} + 2\mathbf{w}) + 2(\mathbf{v} + 2\mathbf{w}) = \mathbf{u} + 2\mathbf{v} + 6\mathbf{w} = \mathbf{u} + 2\mathbf{v} + 2\mathbf{w}.$$

Example 2. More generally, let V be any module over the ring $\ddot{\Lambda} = \mathbf{Z}[t^{\pm 1}, s]/(s^2 - (1-t)s)$. Then V is a rack with rack operations

$$\mathbf{u} \triangleright \mathbf{v} = t\mathbf{u} + s\mathbf{v} \qquad \text{and} \qquad \mathbf{u} \triangleright^{-1} \mathbf{v} = t^{-1}(\mathbf{u} - s\mathbf{v})$$

since

$$(\mathbf{u} \triangleright \mathbf{v}) \triangleright \mathbf{w} = t(t\mathbf{u} + s\mathbf{v}) + s\mathbf{w} = t^2\mathbf{u} + ts\mathbf{v} + s\mathbf{w}$$

while

$$(\mathbf{u} \triangleright \mathbf{w}) \triangleright (\mathbf{v} \triangleright \mathbf{w}) = t(t\mathbf{u} + s\mathbf{w}) + s(t\mathbf{v} + s\mathbf{w}) = t^2\mathbf{u} + st\mathbf{v} + (st + s^2)\mathbf{w}$$

and since $s^2 = (1-t)s$ we have $s = st + s^2$. Racks of this type are known as (t,s)-*racks* in [12]. Setting $s = 1 - t$ yields a quandle known as an *Alexander quandle*. In particular, the rack in example 1 is a (t,s)-rack with $t = 1$ and $s = 2$.

The rack operations are non-associative in general, with self-distributivity on the right taking the place of associativity. It is a standard exercise to show that in any rack X we also have

- $(x \triangleright^{-1} y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright^{-1} (y \triangleright^{-1} z)$,
- $(x \triangleright y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright (y \triangleright^{-1} z)$, and
- $(x \triangleright^{-1} y) \triangleright z = (x \triangleright z) \triangleright^{-1} (y \triangleright z)$.

To minimize parentheses, any rack word not containing parentheses will be associated left-to-right, so that

$$x_1 \triangleright x_2 \triangleright x_3 \cdots \triangleright x_n = (\ldots((x_1 \triangleright x_2) \triangleright x_3)\ldots) \triangleright x_n.$$

In particular, the expression $x \triangleright^n y$ is an abbreviation for

$$(\ldots((x \triangleright y) \triangleright y)\ldots) \triangleright y$$

where we have n total $\triangleright$s.

A similar notion is a *rack power* $x^{\triangleright k}$, defined recursively by the rules

(i) $x^{\triangleright 1} = x \triangleright x$ and

(ii) $x^{\triangleright(k+1)} = x^{\triangleright k} \triangleright x^{\triangleright k}$.

The map $\pi : X \to X$ given by $\pi(x) = x \triangleright x$ is a bijection known as a *kink map*; we have $x^{\triangleright k} = \pi^k(x)$.

Given any $x \in X$, we can ask what is the minimal $N \in \mathbf{N}$ such that $x^{\triangleright N} = x$.

Definition 2. Let X be a rack. For each $x \in X$, let $N(x)$ be the smallest integer $N(x) \in \mathbf{N}$ such that $x^{\triangleright N} = x$, or ∞ if there is no such $N(x)$. Then the *rack rank* of X is

$$N(X) = \mathrm{lcm}\{N(x) \mid x \in X\}.$$

If any $N(x) = \infty$ then we have $N(X) = \infty$.

Note that a quandle is a rack with rack rank $N(X) = 1$. We will often write N in place of $N(X)$ when the rack in question is understood. The rack rank is analogous to the characteristic of a field or ring; indeed, we might consider the alternative term "rack characteristic."

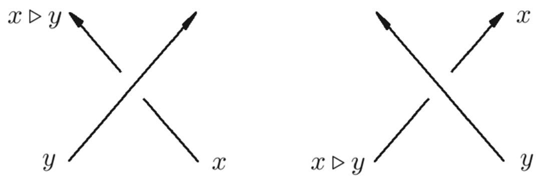
For ease of computation, a finite rack $X = \{x_1, \dots, x_n\}$ can be represented with a *rack matrix* M_X whose (i, j) entry is k where $x_k = x_i \triangleright x_j$. That is, the rack matrix encodes the operation table of $(X, \triangleright)$. Note that the operation table of $(X, \triangleright^{-1})$ can be recovered from M_X, so we do not need to specify both operation tables to determine a rack structure on X. As observed in [15], every finite rack has finite rack rank equal to the order in S_n of the permutation π given by the diagonal of the rack matrix.

Example 3. Let $X = \mathbf{Z}_4$ with $\mathbf{u} \triangleright \mathbf{v} = \mathbf{u} + 2\mathbf{v}$. If we write $\{x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4\}$ then the matrix of X is

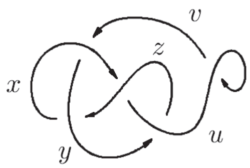
$$M_X = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

Since the diagonal permutation is the transposition $\pi = (13) \in S_4$, this rack has rack rank $N = 2$.

Every framed oriented link L has a *fundamental rack* $FR(L)$ with generators corresponding to arcs in a blackboard-framed diagram of L and relations determined at crossings as pictured:

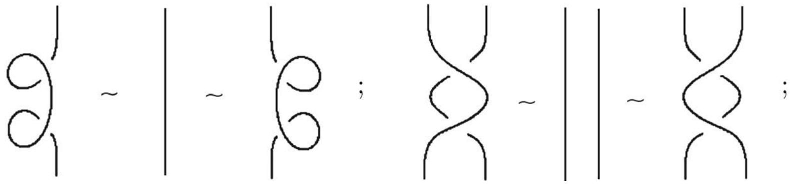


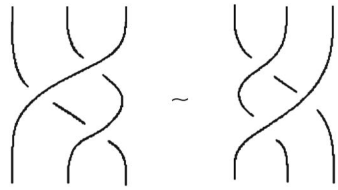
Example 4. The blackboard-framed oriented knot below has listed fundamental rack presentation.



$$FR(L) = \langle x, y, z, u, v \mid u \triangleright u = v,\ y \triangleright x = v,\ x \triangleright y = z,\ y \triangleright u = z,\ x \triangleright z = u \rangle.$$

The rack axioms encode blackboard-framed isotopy, so that two link diagrams which are framed isotopic have isomorphic fundamental racks. Indeed, in [12] it is shown that the fundamental rack is a complete invariant of unsplit oriented links in homology 3-spheres.





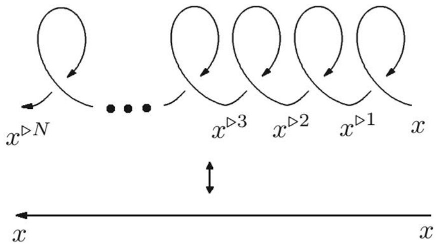
For any rack X, a rack homomorphism $f: FR(L) \to X$ can be visualized as a *coloring* or labeling of a diagram L by elements of X where each arc, say x, in L gets a label $f(x) \in X$. The set of homomorphisms $\mathrm{Hom}(FR(L), X)$ is invariant under blackboard-framed isotopy. By summing these numbers of colorings over a complete set of framings modulo N, we obtain an invariant of ambient isotopy ([15]).

Theorem 1. *Let X be a finite rack with rack rank N, $L = \bigcup_{k=1}^{c} L_k$ a link with c components, $W = (\mathbf{Z}_N)^c$, and $FR(L, \mathbf{w})$ the fundamental rack of L with framing vector $\mathbf{w} = (w_1, \ldots, w_c) \in W$. Then the sum*

$$\Phi_X^{\mathbf{Z}}(L) = \sum_{\mathbf{w} \in W} |\mathrm{Hom}(FR(L, \mathbf{w}), X)|$$

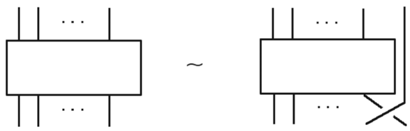
is an invariant of links, known as the integral rack counting invariant.

The theorem follows from the observation that while colorings of link diagrams by racks are not preserved under Reidemeister I moves, they are preserved by the *N-phone cord move* where N is the rack rank of X:

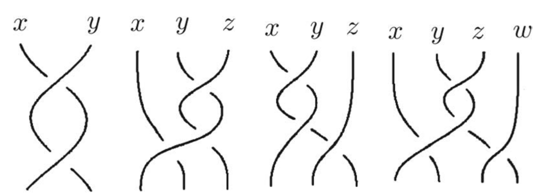


Thus, if two links are equivalent by blackboard-framed moves and N-phone cord moves, their sets of colorings by a rack X with rack rank N are in one-to-one correspondence.

We can compute the set of rack colorings of a link L by putting the link in braid form, assigning a generator to each strand at the top and pushing the colors down the braid; closing the braid, we obtain a system of equations with one equation for each strand. We must repeat this computation with N stabilization moves on each component to get the total set of colorings.



EXAMPLE 5. The rack in example 3 has rack rank $N = 2$. The Hopf link has braid presentations with the four possible writhe vectors in $W = (\mathbf{Z}_2)^2$ as pictured below; closing each braid gives us the listed system of equations in X. The total number of solutions is the integral rack counting invariant $\Phi_X^{\mathbf{Z}}$, so we have $\Phi_X^{\mathbf{Z}}(L) = 4 + 4 + 4 + 8 = 20$.

w	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Braid Diagram				
Equations	$2x = 0$ $2y = 0$	$2x = 0$ $y = x$ $2z = 0$	$2x = 0$ $y = z$ $2y = 0$	$x = 3y$ $w = z$ $2y = 2z$
# solutions	4	4	4	8

3. Rack homology

Let X be a finite rack with rack rank N and let $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1, \dots, x_n)$ of elements of X; for $n < 1$ set $C_n^R(X) = \{0\}$. Recall that for an n-chain, the differential $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ (see [6] for example) is defined on the generator $(x_1, \dots, x_n)$ by

$$\partial_n(x_1, x_2, \dots, x_n) = \sum_{i=2}^{n} (-1)^i [(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - (x_1 \triangleright x_i, x_2 \triangleright x_i, \dots, x_{i-1} \triangleright x_i, x_{i+1}, \dots, x_n)]$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$ and extended to $C_n^R(X)$ by linearity.

Definition 3. Let X be a finite rack of rack rank N and let $n \geq 2$ be an integer. A rack n-chain $\mathbf{x} \in C_n^R(X)$ is *N-degenerate* if $\mathbf{x}$ is a linear combination of chains of the form

$$\sum_{k=1}^{N} (x_1, \dots, x_{i-1}, x_i^{\triangleright k}, x_i^{\triangleright(k+1)}, x_{i+2}, \dots, x_n).$$

We now come to our main result:

Theorem 2. *Let X be a finite rack with rack rank N. Then the N-degenerate n-chains form a subcomplex of the complex $C_*^R(X)$.*

Proof. Let

$$\mathbf{x} = \sum_{k=1}^{N} (x_1, \dots, x_{i-1}, x_i^{\triangleright k}, x_i^{\triangleright(k+1)}, x_{i+2}, \dots, x_n).$$

One checks that the $(n-1)$-chain $\partial_n \mathbf{x}$ is N-degenerate:

$$\begin{aligned}
\partial_n \mathbf{x} &= \partial_n \left[\sum_{k=1}^{N} (x_1, \dots, x_{i-1}, x_i^{\triangleright k}, x_i^{\triangleright(k+1)}, x_{i+2}, \dots, x_n) \right] \\
&= \sum_{k=1}^{N} \partial_n (x_1, \dots, x_{i-1}, x_i^{\triangleright k}, x_i^{\triangleright(k+1)}, x_{i+2}, \dots, x_n) \\
&= \sum_{k=1}^{N} \left\{ \sum_{j=2}^{i-1} (-1)^j [(x_1, \dots, \widehat{x_j}, \dots, x_i^{\triangleright k}, x_i^{\triangleright(k+1)}, x_{i+2}, \dots, x_n) \right. \\
&\qquad \left. - (x_1 \triangleright x_j, \dots, x_{j-1} \triangleright x_j, x_{j+1}, \dots, x_i^{\triangleright k}, x_i^{\triangleright(k+1)}, x_{i+2}, \dots, x_n)] \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg\{\sum_{k=1}^{N}(-1)^{i}[(x_1,\dots,x_{i-1},x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)\\
&\quad -(x_1\triangleright x_i^{\triangleright k},\dots,x_{i-1}\triangleright x_i^{\triangleright k},x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)]\\
&\quad +(-1)^{i+1}[(x_1,\dots,x_{i-1},x_i^{\triangleright k},x_{i+2},\dots,x_n)\\
&\quad -(x_1\triangleright x_i^{\triangleright(k+1)},\dots,x_{i-1}\triangleright x_i^{\triangleright(k+1)},x_i^{\triangleright k}\triangleright x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)]\Bigg\}\\
&+\sum_{k=1}^{N}\Bigg\{\sum_{j=i+2}^{n}(-1)^{j}[(x_1,\dots,x_{i-1},x_i^{\triangleright k},x_i^{\triangleright(k+1)},\dots,\widehat{x_j},\dots,x_n)\\
&\qquad -(x_1\triangleright x_j,\dots,x_{i-1}\triangleright x_j,x_i^{\triangleright k}\triangleright x_j,x_i^{\triangleright(k+1)}\triangleright x_j,\dots,x_{j-1}\triangleright x_j,\\
&\qquad x_{j+1},\dots,x_n)]\Bigg\} \qquad (1)
\end{aligned}
$$

where as usual $(x_1,\dots,\widehat{x_j},\dots,x_n)$ means $(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$. One observes that the following sum vanishes:

$$
\begin{aligned}
&\sum_{k=1}^{N}\{[(x_1,\dots,x_{i-1},x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)\\
&\quad -(x_1\triangleright x_i^{\triangleright k},\dots,x_{i-1}\triangleright x_i^{\triangleright k},x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)]\\
&\quad -[(x_1,\dots,x_{i-1},x_i^{\triangleright k},x_{i+2},\dots,x_n)\\
&\quad -(x_1\triangleright x_i^{\triangleright(k+1)},\dots,x_{i-1}\triangleright x_i^{\triangleright(k+1)},x_i^{\triangleright k}\triangleright x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)]\}
\end{aligned}
$$

because $x^{\triangleright N}=x$ and thus

$$
\sum_{k=1}^{N}[(x_1,\dots,x_{i-1},x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)-(x_1,\dots,x_{i-1},x_i^{\triangleright k},x_{i+2},\dots,x_n)]=0
$$

and

$$
\begin{aligned}
&(x_1\triangleright x_i^{\triangleright k},\dots,x_{i-1}\triangleright x_i^{\triangleright k},x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n)\\
&\quad=(x_1\triangleright x_i^{\triangleright(k+1)},\dots,x_{i-1}\triangleright x_i^{\triangleright(k+1)},x_i^{\triangleright k}\triangleright x_i^{\triangleright(k+1)},x_{i+2},\dots,x_n).
\end{aligned}
$$

This last difference is zero since by [15, Lemma 1] we have $\forall u\in X$, $u\triangleright x_i^{\triangleright(k+1)}=u\triangleright(x_i^{\triangleright k}\triangleright x_i^{\triangleright k})=u\triangleright x_i^{\triangleright k}$ and similarly $x_i^{\triangleright k}\triangleright x_i^{\triangleright(k+1)}=x_i^{\triangleright k}\triangleright x_i^{\triangleright k}$.

Using self-distributivity, the last term in equation (1) can be re-written as (so it makes the last sum fit the definition of degenerate chains)

$$(x_1 \triangleright x_j, \dots, x_{i-1} \triangleright x_j, x_i^{\triangleright k} \triangleright x_j, (x_i^{\triangleright k} \triangleright x_j) \triangleright (x_i^{\triangleright k} \triangleright x_j), \dots, x_{j-1} \triangleright x_j, x_{j+1}, \dots, x_n)$$

Recall from [6] that $C_*^{\mathrm{R}}(X) = \{C_n^{\mathrm{R}}(X), \partial_n\}$ is a chain complex. Let $C_n^{\mathrm{ND}}(X)$ be the subset of $C_n^{\mathrm{R}}(X)$ generated by the N-degenerate n-chains as in Definition 3 if $n \geq 2$; otherwise let $C_n^{\mathrm{ND}}(X) = 0$. If X is a rack, then $\partial_n(C_n^{\mathrm{ND}}(X)) \subset C_{n-1}^{\mathrm{ND}}(X)$ and $C_*^{\mathrm{ND}}(X) = \{C_n^{\mathrm{ND}}(X), \partial_n\}$ is a sub-complex of $C_*^{\mathrm{R}}(X)$. Put $C_n^{\mathrm{R/ND}}(X) = C_n^{\mathrm{R}}(X)/C_n^{\mathrm{ND}}(X)$ and $C_*^{\mathrm{R/ND}}(X) = \{C_n^{\mathrm{R/ND}}(X), \partial'_n\}$, where ∂'_n is the induced homomorphism. Henceforth, all boundary maps will be denoted by ∂_n.

For an abelian group G, define the chain and cochain complexes

$$C_*^{\mathrm{W}}(X;G) = C_*^{\mathrm{W}}(X) \otimes G, \qquad \partial = \partial \otimes \mathrm{id}; \tag{2}$$

$$C^*_{\mathrm{W}}(X;G) = \mathrm{Hom}(C_*^{\mathrm{W}}(X), G), \qquad \delta = \mathrm{Hom}(\partial, \mathrm{id}) \tag{3}$$

in the usual way, where $\mathrm{W} = \mathrm{ND}$, R, R/ND.

Definition 4. The nth *reduced rack homology group* and the nth *reduced rack cohomology group* of a rack X with coefficient group G are

$$H_n^{R/ND}(X;G) = H_n(C_*^{R/ND}(X;G)), \qquad H^n_{R/ND}(X;G) = H^n(C^*_{R/ND}(X;G)). \tag{4}$$

As an example, let us now compute the first and second homology groups of the (t,s)-rack $X = \mathbf{Z}_4$ with $t = 1$ and $s = 2$ as in example 3. We have the following

Lemma 1. *Let $X = \mathbf{Z}_4$ with $\mathbf{u} \triangleright \mathbf{v} = \mathbf{u} + 2\mathbf{v}$, the rack in example 3. The group of 2-degenerate 2-chains $C_2^{ND}(X)$ is generated by*

$$(3,1) + (1,3), \qquad 2(2,2), \qquad \text{and} \qquad 2(4,4).$$

Now since $\mathbf{u} \triangleright \mathbf{v} = \mathbf{u} + 2\mathbf{v}$ and $\partial_2(i,j) = (i) - (i \triangleright j)$, the image group, $\mathrm{Im}(\partial_2)$, is two-dimensional, generated by two generators (1)–(3) and (2)–(4). This implies the following

Lemma 2. *The groups $H_1^R(X)$ and $H_1^{R/ND}(X)$ are two-dimensional.*

Lemma 3. *The group $\mathrm{Ker}(\partial_2)$ is 14-dimensional.*

Proof. A straightforward computation gives the following fourteen generators $(i,2)$, $(i,4)$, $(i,1) + (i+2,3)$ for $i = 1,2,3,4$ and $(i,3) + (i+2,3)$ for $i = 1,2$, of the group $\mathrm{Ker}(\partial_2)$.

Now we calculate the dimension of the second homology group.

Lemma 4. *The group $\mathrm{Im}(\partial_3)$ is 11-dimensional, giving dim $H_2(X) = 3$.*

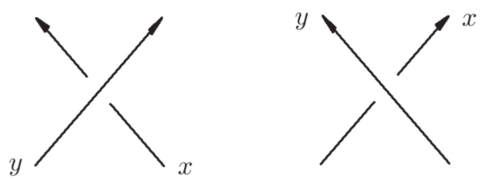
Proof. Again a straightforward computation gives the following eleven generators

$$(i,3)-(i,1),\ (i,2)-(i+2,4), \quad \text{for } i=1,2,3,4 \qquad \text{and}$$
$$(i,2)-(i+2,2) \quad \text{for } i=1,2,3.$$

4. Enhancing the rack counting invariant

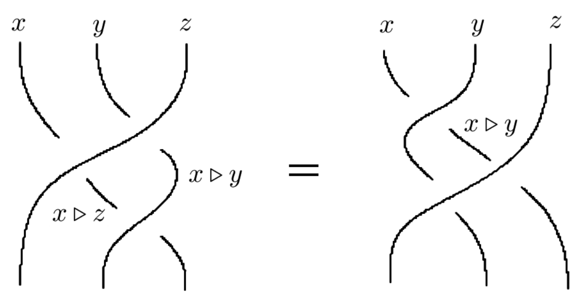
In this section we use cocycles in $H^2_{R/ND}(X;G)$ to define an enhancement of the rack counting invariant.

Let L be a link diagram with c components, X a finite rack with rack rank N, $W=(\mathbf{Z}_N)^c$ and $\phi \in H^2_{R/ND}(X;G)$ a nondegenerate 2-cocycle with coefficients in an abelian group G. For a given coloring $f \in \mathrm{Hom}(FR(L),X)$, we define the *Boltzmann weight* of f in the following way. At each positive crossing in L, we have a contribution of $\phi(x,y)$ where x is the color on the inbound underarc and y is the color on the overarc; at each negative crossing, we have a contribution of $-\phi(x,y)$ where x is the color on the outbound underarc and y is the color on the overarc. The Boltzmann weight $BW(f)$ is then the sum over all crossings in L of these contributions.

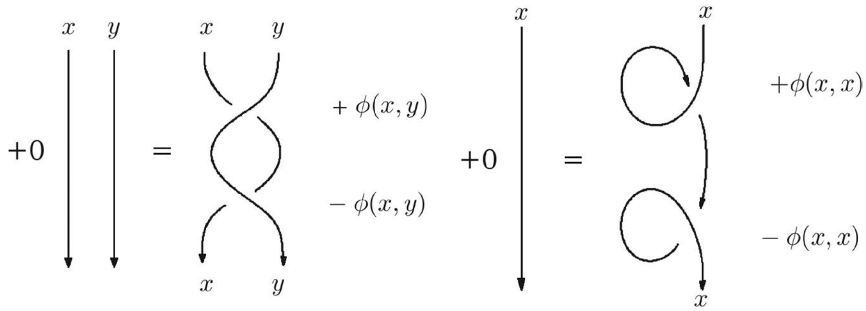


$+\phi(x,y)$ $\qquad\qquad$ $-\phi(x,y)$

The rack cocycle condition is precisely the condition required so that the Boltzmann weight of a rack-colored diagram is unchanged by Reidemeister III moves; the coloring condition is chosen to guarantee invariance under Reidemeister II and blackboard-framed type I moves.



$\phi(y,z)+\phi(x,z)+\phi(x \triangleright z, y \triangleright z)$ $\qquad$ $\phi(x,y)+\phi(x \triangleright y, z)+\phi(y,z)$



The N-degenerate subcomplex is generated by the characteristic chains of N-phone cord tangles. By setting $H^*_{ND}(X;G)$ equal to zero, i.e. selecting $\phi \in H^2_{R/ND}(X;G)$ as opposed to $\phi \in H^2_R(X;G)$, we obtain invariance under the N-phone cord move. Thus, we have

Theorem 3. *For any cocycle* $\phi \in H^2_{R/ND}(X;G)$, *the multiset*

$$\Phi^M_\phi(L) = \{BW(f) \mid f \in \mathrm{Hom}(FR(L, \mathbf{w}), X), \mathbf{w} \in W\}$$

is an invariant of ambient isotopy.

We can also use generating function style notation to rewrite the multiset as a one-variable polynomial with $\mathbf{Z}$ coefficients and G exponents:

$$\Phi_\phi(L) = \sum_{\mathbf{w}\in W}\left(\sum_{f \in \mathrm{Hom}(FR(L,\mathbf{w}),X)} u^{BW(f)}\right).$$

Remark 1. The invariant $\Phi^M_\phi(L)$ is the specialiation of the invariant $\Phi_\phi(L, T)$ defined in [15] (Definition 8) obtained by setting $q_k = 1$ for $k = 1, \ldots, c$.

Remark 2. If $N = 1$ so that X is a quandle, then Φ_ϕ is the CJKLS invariant defined in [6].

Note that there are four oriented N-phone cord moves; we have used only one to define our degenerate subcomplex. One of the other moves yields the same degenerate subcomplex, but the other two yield a slightly different subcomplex, generated by chains of the form

$$\sum_{k=1}^{N}(x_1, \ldots, x_{i-1}, x_i^{\triangleright k}, x_i^{\triangleright k}, x_{i+2}, \ldots, x_n).$$

Note that these chains are called N-*reduced* in [15]. The phone cord moves which yield this degeneracy condition are related to the one we chose through a combination of Reidemeister II and III moves together with blackboard-framed I moves, so it follows that such alternative degenerate cocycles are homologous to cocycles in C^R_*. In particular, we have:

Lemma 5. *If* $\phi \in H^2_R(X)$ *then* $\phi(x,x) = \phi(x, x \triangleright x)$.

Proof. First note that since ϕ is a 2-cocycle we have

$$\phi(a,c) - \phi(a \triangleright b, c) - \phi(a,b) + \phi(a \triangleright c, b \triangleright c) = 0$$

for all $a,b,c \in X$. Then in particular, setting $a = x \triangleright^{-1} x$, $b = x$ and $c = x$ we have

$$\begin{aligned} 0 &= \phi(x \triangleright^{-1} x, x) - \phi((x \triangleright^{-1} x) \triangleright x, x) - \phi(x \triangleright^{-1} x, x) \\ &\quad + \phi((x \triangleright^{-1} x) \triangleright x, x \triangleright x) \\ &= \phi(x \triangleright^{-1} x, x) - \phi(x,x) - \phi(x \triangleright^{-1} x, x) + \phi(x, x \triangleright x) \\ &= -\phi(x,x) + \phi(x, x \triangleright x). \end{aligned}$$

Corollary 1. *Let* X *be a rack of rack rank* $1 \le N < \infty$ *and let* $C^{ND'}_n(X)$ *be the set of linear combinations of chains of the form*

$$\sum_{k=1}^{N} (x_1, \dots, x_{i-1}, x_i^{\triangleright k}, x_i^{\triangleright k}, x_{i+2}, \dots, x_n).$$

Then $C^{ND'}_*(X)$ *is also a subcomplex of* $C^R_*(X)$ *and the quotient complexes* $C^{R/ND}_*(X)$ *and* $C^{R/ND'}_*(X)$ *are isomorphic.*

Remark 3. It is noted in [6] that 2-coboundaries have a Boltzmann weight contribution of zero, and hence cohomologous cocycles define the same invariant. We note that the exactly the same proof applies in this more general setting, and we have

$$\phi_1 \text{ cohomologous to } \phi_2 \quad \Rightarrow \quad \Phi_{\phi_1} = \Phi_{\phi_2}.$$

5. Computations and examples

In this section we collect a few examples of the rack cocycle invariants. These examples were computed using python code available at www.esotericka.org with signed Gauss codes transcribed by hand from diagrams at the Knot Atlas [2] and checked with Maple.

Example 6. Let X be the rack with rack matrix given by

$$M_X = \begin{bmatrix} 1 & 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 2 & 2 \\ 2 & 1 & 3 & 3 & 3 \\ 4 & 4 & 4 & 5 & 5 \\ 5 & 5 & 5 & 4 & 4 \end{bmatrix}.$$

Via computations in python, we selected at random a reduced 2-cocycle $\phi \in H^2_{R/ND}(X; \mathbf{Z}_4)$ given by

$$\phi = (\chi_{1,3} + \chi_{3,2} + \chi_{5,4} + \chi_{5,5}) + 2(\chi_{1,1} + \chi_{2,2} + \chi_{3,3})$$
$$+ 3(\chi_{1,2} + \chi_{1,4} + \chi_{1,5} + \chi_{2,3} + \chi_{2,4} + \chi_{2,5} + \chi_{3,4} + \chi_{3,5} + \chi_{4,4} + \chi_{4,5})$$

and computed $\Phi_\phi(L)$ for all prime classical knots with up to eight crossings and prime classical links with up to seven crossings. The results are collected in the table below; note that multiple entries with equal rack counting invariant are distinguished by the cocycle enhancement.

$\Phi_\phi(L)$	L
$5 + 3u^2$	$4_1, 5_1, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_8, 8_9, 8_{12}, 8_{13}, 8_{14}, 8_{16}, 8_{17}$
$11 + 9u^2$	$3_1, 6_1, 7_4, 7_7, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}, 8_{21}$
$29 + 27u^2$	8_{18}
$10 + 12u + 6u^2 + 12u^3$	$L2a1, L6a2, L7a6$
$22 + 12u + 18u^2 + 12u^3$	$L6a3, L7a5$
$22 + 18u^2$	$L4a1, L5a1, L7a4$
$34 + 30u^2$	$L6a1, L7a1, L7a2, L7a3, L7n1, L7n2$
$92 + 84u^2$	$L6n1, L7a7$
$116 + 108u^2$	$L6a5$
$164 + 156u^2$	$L6a4$

Example 7. Let X be the rack with rack matrix given by

$$M_X = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 5 & 6 & 4 \\ 4 & 4 & 6 & 4 & 3 & 5 \\ 5 & 5 & 4 & 6 & 5 & 3 \\ 6 & 6 & 5 & 3 & 4 & 6 \end{bmatrix}$$

and let ϕ be the reduced 2-cocycle $\phi \in H^2_{R/ND}(X; \mathbf{Z}_4)$ given by

$$\phi = \chi_{1,3} + \chi_{1,4} + \chi_{1,5} + \chi_{1,6} + \chi_{2,3} + \chi_{2,4} + \chi_{2,5} + \chi_{2,6} + \chi_{3,1}$$
$$+ \chi_{3,2} + \chi_{3,4} + \chi_{3,5} + \chi_{4,1} + \chi_{4,2} + \chi_{4,3} + \chi_{5,1} + \chi_{5,2} + \chi_{5,6}$$
$$+ \chi_{6,1} + \chi_{6,2} + \chi_{6,4} + \chi_{6,5} + 2(\chi_{4,5} + \chi_{5,4}) + 3(\chi_{4,6} + \chi_{5,3});$$

The invariant values for knots and links are listed in the tables below. Unlike the previous example, this example includes single-component knots which are distinguished by the cocycle enhancement but not the counting invariant.

$\Phi_\phi(L)$	L
10	$5_1, 5_2, 6_1, 6_2, 6_3, 7_1, 7_4, 7_5, 7_6, 7_7, 8_2,$
	$8_3, 8_6, 8_7, 8_8, 8_9, 8_{10}, 8_{12}, 8_{14}, 8_{16}, 8_{17}$
$10 + 24u^2$	$3_1, 4_1, 7_2, 7_3, 8_1, 8_4, 8_{11}, 8_{13}, 8_{18}$
34	$8_5, 8_{15}, 8_{19}, 8_{20}, 8_{21}$
$34 + 96u^2$	8_{18}

$\Phi_\phi(L)$	L
52	$L5a1, L6a1$
100	$L4a1, L7a4$
232	$L6a4$
$20 + 32u^2$	$L2a1, L6a2, L7a5, L7a6$
$52 + 48u^2$	$L7a1, L7a2, L7a3, L7n1, L7n2$
$68 + 32u^2$	$L6a3$
$104 + 128u^2$	$L6a5$
$232 + 96u^2$	$L6n1, L7a7$

The previous examples both use racks which are disjoint unions of one rack and one quandle, with each orbit acting trivially on the other. For our final example, we use a more "pure rack" with no elements of rack rank 1.

Example 8. Consider the rack with rack matrix

$$M_X = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

and let $\phi \in H^2_{R/ND}(X; \mathbf{Z}_6)$ be given by

$$\phi = \chi_{2,1} + \chi_{2,2} + 3(\chi_{3,3} + \chi_{3,4} + \chi_{4,3} + \chi_{4,4})$$
$$+ 5(\chi_{1,1} + \chi_{1,2} + \chi_{3,1} + \chi_{3,2} + \chi_{4,1} + \chi_{4,2})$$

The invariant has value $\Phi_\phi(K) = 4$ for all of the knots K in our list; however, the invariant is nontrivial on links:

$\Phi_\phi(L)$	L
16	$L5a1, L6a3, L7a1, L7a3, L7a4, L7n2$
$8 + 8u^2$	$L2a1, L7a5, L7a6, L7n1$
$8 + 8u^4$	$L4a1, L6a1, L7a2$
$12 + 4u^4$	$L6a2$
$16 + 48u^4$	$L6a4, L6n1$
$48 + 16u^4$	$L6a5, L7a7$

6. A compendium of questions

We conclude the paper by the following remarks and open questions:

- In [4] a homology theory was developed for set-theoretic Yang-Baxter equations and used to define invariants of classical and virtual knots. This approach was used in [8] and extended to detect non-classicality of some virtual links. Within these lines generalize the cohomology theory in this paper to biracks and use it study invariants of classical and virtual knots.
- How can we extend these rack cocycle invariants to surface knots and links, or more generally, knotted compact oriented n-manifolds in S^{n+2}?
- Recent work (e.g., [16]) has found that quandle homology groups sometimes have additional algebraic structure. How can we use these structures to further enhance the rack 2-cocycle invariants?
- More generally, what kinds of patterns (long exact sequences, etc.) exist in the R/ND homology groups of racks of various types?
- Our computer experiments suggest that small cardinality racks define 2-cocycle invariants which are stronger on links than on knots, due to the orbit subracks being fairly trivial constant action racks. Fast algorithms for computing the homology of large-cardinality racks should improve the practical utility of the rack cocycle invariants.
- Given any rack X, there are a number of quandles associated to X such as the *maximal subquandle* of X consisting of all elements of X of rack rank 0 (this may be the empty quandle) and the *operator quandle* obtained by taking the quotient of X by the congruence $\{x \sim y \Leftrightarrow z \triangleright x = z \triangleright y \ \forall z \in X.\}$ What relationship, if any, can be found between the quandle homology of these quandles and the R/ND homology of X?

References

[1] N. Andruskiewitsch and M. Graña, From racks to pointed Hopf algebras, Adv. Math. **178** (2003), no. 2, 177–243.

[2] D. Bar-Natan, Ed. The Knot Atlas, http://katlas.math.toronto.edu/wiki/Main_Page.
[3] S. Carter, M. Elhamdadi, M. Graña and M. Saito, Cocycle knot invariants from quandle modules and generalized quandle homology, Osaka J. Math. **42** (2005), no. 3, 499–541.
[4] S. Carter, M. Elhamdadi and M. Saito, Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles, Fund. Math. **184** (2004), 31–51.
[5] S. Carter, M. Elhamdadi and M. Saito, Twisted quandle homology theory and cocycle knot invariants, Algebr. Geom. Topol. **2** (2002), 95–135.
[6] S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. **355** (2003), no. 10, 3947–3989.
[7] S. Carter, S. Kamada and M. Saito, Surfaces in 4-space, Encyclopaedia of Mathematical Sciences, **142**. Low-Dimensional Topology, III. Springer-Verlag, Berlin, 2004.
[8] J. Ceniceros and S. Nelson, Virtual Yang-Baxter Cocycle Invariants, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5263–5283.
[9] P. Etingof and M. Graña, On rack cohomology, J. Pure Appl. Algebra **177** (2003), no. 1, 49–59.
[10] R. Fenn, C. Rourke and B. Sanderson, The rack space, Trans. Amer. Math. Soc. **359**, (2007), no. 2, 701–740.
[11] R. Fenn, C. Rourke and B. Sanderson, James bundles and applications, Proc. London Math. Soc. **3**, 89 (2004), no. 1, 217–240.
[12] R. Fenn and C. Rourke, Racks and links in codimension two, J. Knot Theory Ramifications **1** (1992), 343–406.
[13] R. Litherland and S. Nelson, The Betti numbers of some finite racks, J. Pure Appl. Algebra **178** (2003), no. 2, 187–202.
[14] A. Navas and S. Nelson, On symplectic quandles, Osaka J. Math. **45** (2008), no. 4, 973–985.
[15] S. Nelson, Link invariants from finite racks, arXiv:0808.0029.
[16] M. Niebrzydowski and J. Przytycki, The second quandle homology of the Takasaki quandle of an odd abelian group is an exterior square of the group, arXiv:1006.0258.

Mohamed Elhamdadi
Department of Mathematics
University of South Florida
4202 *E Fowler Ave.*
Tampa, FL 33620
E-mail: emohamed@math.usf.edu

Sam Nelson
Department of Mathematical Sciences
Claremont McKenna College
850 *Columbia Ave.*
Claremont, CA 91711
E-mail: knots@esotericka.org