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On the monoid in the fundamental group of type $\mathrm{B_{ii}}$

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Abstract. We study the monoid generated by certain Zariski-van Kampen generators in the positive homogeneous presented fundamental group of the complement of the *logarithmic free divisor*, called the type $\mathrm{B_{ii}}$ in the list by Sekiguchi. Although the monoid is cancellative, it turns out that the monoid is not *Gaussian* and, hence, is neither *Garside* nor *Artin*. Nevertheless, we show that the monoid carries certain particular elements similar to the fundamental elements in Artin monoid. Hence, we can solve the word problem and the conjugacy problem in the monoid and determine the center of it and the explicit form of the growth function for it. As a result, we can also solve the word problem and the conjugacy problem in the fundamental group, and determine the center of it (Theorem 5.8).

1. Introduction

A hypersurface D in $\mathbf{C}^l$ ($l \in \mathbf{Z}_{\geq 0}$) is called a *logarithmic free divisor* ([S1, S2]), if the associated module $Der_{\mathbf{C}^l}(-\log(D))$ of logarithmic vector fields is a free $\mathcal{O}_{\mathbf{C}^l}$-module. Classical example of logarithmic free divisors is the discriminant loci of a finite reflection group ([S1], [S2]). The fundamental group of the complement of the discriminant loci is presented ([B]) by certain positive homogeneous relations, called Artin braid relations. The group (resp. monoid) defined by that presentation is called an *Artin group* (resp. *Artin monoid*) of finite type [B-S], for which the word problem and other problems are solved using a particular element Δ, the *fundamental element*, in the corresponding monoid ([B-S], [D], [G]).

In [Se1, 2], Sekiguchi made a list of 17 weighted homogeneous polynomials that define logarithmic free divisors in $\mathbf{C}^3$. They are labeled by the type $X \in \{\mathrm{A_i}, \mathrm{A_{ii}}, \mathrm{B_i}, \mathrm{B_{ii}}, \dots, \mathrm{B_{vii}}, \mathrm{H_i}, \mathrm{H_{ii}}, \dots, \mathrm{H_{viii}}\}$. Then, the fundamental groups of the complements of the divisors are presented by using Zariski-van Kampen method in [I]. In [S-I], it turns out that the defining relations can be rewritten by a system of positive homogeneous relations in the sense explained in §2 of

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the present paper, so that we can introduce monoids defined by them. We have shown in [S-I] that, among 17 monoids, five are *Artin monoids* ([B-S]), and eight are free abelian monoids. However, four remaining types $\mathrm{B_{ii}}$, $\mathrm{B_{vi}}$, $\mathrm{H_{ii}}$, $\mathrm{H_{iii}}$, the monoids are not *Gaussian*, and hence are neither *Garside* ([D-P]) nor Artin. Nevertheless, we have shown that all the 17 monoids carry certain particular elements similar to the fundamental elements in Artin monoids. In this paper, we focus our attention to the type $\mathrm{B_{ii}}$ monoid among the remaining four monoids. As a result, some decision problems in the fundamental group can be solved (Theorem 5.8). Moreover, we show that the fundamental group is a solvable group and admits a faithful 5×5-matrix representation (Corollary 5.12).

Let us explain more details of the content. The explicit form of type $\mathrm{B_{ii}}$ Sekiguchi-polynomial is $z(-2y^3 + 4x^3z + 18xyz + 27z^2)$ and is denoted by $\Delta_{\mathrm{B_{ii}}}(x, y, z)$. We put $D_{\mathrm{B_{ii}}} := \{\Delta_{\mathrm{B_{ii}}}(x, y, z) = 0\}$. Then, the fundamental group of the complement of the divisor $D_{\mathrm{B_{ii}}}$ is presented by Zariski-van Kampen method, where we need to choose a generator system of the fundamental group by fixing pathes in a reference fiber. There is an ambiguity of choosing Zariski-van Kampen generator system, where any two Zariski-van Kampen generator systems can be transformed to each other by an action of braid. In §3, we choose a suitable generator system of the fundamental group $\pi_1(\mathbf{C}^3 \backslash D_{\mathrm{B_{ii}}}, *)$ for solving some decision problems on it (Proposition 3.1). We fix the presentation and denote the presented group by $G_{\mathrm{B_{ii}}}$. For the presented group $G_{\mathrm{B_{ii}}}$, we associate a monoid $G^+_{\mathrm{B_{ii}}}$ defined by it. We will show that the associated monoid $G^+_{\mathrm{B_{ii}}}$ satisfies the cancellation condition (Proposition 5.5) and naturally injects into the group $G_{\mathrm{B_{ii}}}$. Hence, we can say that the solvability of the word problem and the conjugacy problem and determinativeness of the center in the monoid imply those in the group $G_{\mathrm{B_{ii}}}$ (Lemma 4.2). In this way, we solve the word problem and the conjugacy problem in the group $G_{\mathrm{B_{ii}}}$, and determine the center of it. Moreover, we will determine the set $\mathscr{F}(G^+_{\mathrm{B_{ii}}})$ of fundamental elements and the set $\mathscr{QZ}(G^+_{\mathrm{B_{ii}}})$ of quasi-central elements. As a corollary, we will show that the subsemigroup $\mathscr{F}(G^+_{\mathrm{B_{ii}}})(\subset \mathscr{QZ}(G^+_{\mathrm{B_{ii}}}))$ is an infinitely generated idealistic subsemigroup (Remark 5.9). Moreover, we show that the group $G_{\mathrm{B_{ii}}}$ is not *word hyperbolic* ([Gr2]) (Remark 5.10). We will show that the *growth function* for the monoid $G^+_{\mathrm{B_{ii}}}$ is a rational function and the explicit form of it can be determined (Theorem 5.8). By observing the distribution of the zeroes of the denominator polynomial of the growth function for the monoid $G^+_{\mathrm{B_{ii}}}$, it is conjectured that the group $G_{\mathrm{B_{ii}}}$ contains a free abelian subgroup of rank 4 of finite index. Indeed, we can show that the group $G_{\mathrm{B_{ii}}}$ contains a subgroup of index three. By using this proposition, we will show that the group $G_{\mathrm{B_{ii}}}$ is a solvable group and admits a faithful 5×5-matrix representation.

2. Positive homogeneous presentation

In this section, we first recall from [B-S] some basic definitions and notations. Next, for a positive homogeneously finitely presented group

$$G = \langle L|R\rangle,$$

we associate a monoid defined by it.

Let L be a finite set. Let $F(L)$ be the free group generated by L, and let L^* be the free monoid generated by L inside $F(L)$. We call the elements of $F(L)$ *words* and the elements of L^* *positive words*. The empty word ε is the identity element of L^*. If two words A, B are identical letter by letter, we write $A \equiv B$. Let $G = \langle L|R\rangle$ be a positive homogeneously presented group (i.e. the set R of relations consists of those of the form $R_i = S_i$ where R_i and S_i are positive words of the same length), where R is the set of relations. We often denote the images of the letters and words under the quotient homomorphism

$$F(L) \to G$$

by the same symbols and the equivalence relation on elements A and B in G is denoted by $A = B$.

Next, we recall some terminologies and concepts on a monoid M. An element $U \in M$ is said to *divide* $V \in M$ from the left (resp. right), and denoted by $U|_l V$ (resp. $U|_r V$), if there exists $W \in M$ such that $V = UW$ (resp. $V = WU$). We also say that V is *left-divisible* (resp. *right-divisible*) by U, or V is a *right-multiple* (resp. *left-divisible*) of U. We say that M *admits the left* (resp. *right*) *divisibility theory*, if for any two elements U, V of M, there always exists their left (resp. right) least common multiple, i.e. a left (resp. right) common multiple which divides any other left (resp. right) common multiple.

Next, we recall from [S-I] some terminologies and concepts on positive homogeneously presented monoid.

Definition 2.1. *Let $G = \langle L|R\rangle$ be a positive homogeneously finitely presented group, where L is the set of generators (called alphabet) and R is the set of relations. Then we associate a monoid $G^+ = \langle L|R\rangle_{mo}$ defined as the quotient of the free monoid L^* generated by L by the equivalence relation defined as follows:*

i) *two words U and V in L^* are called* elementarily equivalent *if either $U \equiv V$ or V is obtained from U by substituting a substring R_i of U by S_i where $R_i = S_i$ is a relation of R ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),*

ii) *two words U and V in L^* are called* equivalent*, denoted by $U \doteq V$, if there exists a sequence $U \equiv W_0, W_1, \dots, W_n \equiv V$ of words in L^* for $n \in \mathbf{Z}_{\geq 0}$ such that W_i is elementarily equivalent to W_{i-1} for $i = 1, \dots, n$.*

1. *Due to the homogeneity of the relations, we define a homomorphism:*

$$\ell : G^+ \to \mathbf{Z}_{\geq 0}$$

by assigning to each equivalent class of words the length of the words.

2. *We say that G^+ is* cancellative*, if an equality $AXB \doteq AYB$ for $A, B, X, Y \in G^+$ implies $X \doteq Y$.*

3. *The natural homomorphism $\pi : G^+ \to G$ will be called the* localization homomorphism.

4. *An element $\Delta \in G^+$ is called* quasi-central *(also see* [B-S] 7.1*), if there exists a permutation σ_Δ of $L/\sim$ ($:=$ the image of the set L in G^+) such that*

$$s \cdot \Delta \doteq \Delta \cdot \sigma_\Delta(s)$$

holds for all generators $s \in L/\sim$. The set of all quasi-central elements is denoted by $\mathscr{QZ}(G^+)$. The order of an element σ_Δ in the permutation group $\mathfrak{S}(L/\sim)$ is denoted by $\mathrm{ord}(\sigma_\Delta)$*. Note that $\Delta^{\mathrm{ord}(\sigma_\Delta)}$ belongs to the center $\mathscr{Z}(G^+)$ of the monoid G^+.*

5. *An element $\Delta \in G^+$ is called* fundamental *if there exists a permutation σ_Δ of $L/\sim$ such that, for any $s \in L/\sim$, there exists $\Delta_s \in G^+$ satisfying the following relation:*

$$\Delta \doteq s \cdot \Delta_s \doteq \Delta_s \cdot \sigma_\Delta(s).$$

We denote by $\mathscr{F}(G^+)$ the set of all fundamental elements of G^+. Note that $\varepsilon \in \mathscr{QZ}(G^+)$ but $\varepsilon \notin \mathscr{F}(G^+)$. It is easy to show that

$$\mathscr{F}(G^+)\mathscr{QZ}(G^+) = \mathscr{QZ}(G^+)\mathscr{F}(G^+) = \mathscr{F}(G^+).$$

6. *A fundamental element Δ is called a* minimal fundamental element *if any fundamental element dividing Δ from right or left coincides with Δ itself.*

7. *A quasi-central element Δ is called* indecomposable*, if it does not decompose into a product of two nontrivial quasi-central elements. We note that the identity element ε is not indecomposable. We call a fundamental element* prime*, if it is an indecomposable quasi-central element.*

In general, a minimal fundamental element may not be prime. Here is an example.

Example 2.2. *Let us consider the following monoid:*

$$M_1 := \left\langle a, b, c \;\middle|\; \begin{array}{c} cb = ba, \\ bc = ab, \\ ac = ca \end{array} \right\rangle_{mo}.$$

We can easily show that acb is a minimal fundamental element, but ac and b are nontrivial quasi-central elements. Hence, acb is not prime.

3. Positive homogeneous presentation of $G_{\mathrm{B_{ii}}}$

In this section, we recall from [S-I] a positive homogeneous presentation of the fundamental group of the complement of the type $\mathrm{B_{ii}}$ logarithmic free divisor that is given by using Zariski-van Kampen method (see [Ch], [T-S] for instance). There is an ambiguity of choosing Zariski-van Kampen generator system. We choose one of them and consider some dicision problems of words.

In [S-I] §4, we presented the fundamental group of the type $\mathrm{B_{ii}}$ positive homogeneously. We then showed the following proposition:

PROPOSITION 3.1. *For any choice of Zariski-van Kampen generator system $\{a, b, c\}$ (up to a permutation), the fundamental group of type* $\mathrm{B_{ii}}$ *admits only one of the following two presentations* I *and* II

$$\mathrm{I:} \quad \left\langle a, b, c \;\middle|\; \begin{array}{c} cbb = bba, \\ bc = ab, \\ ac = ca \end{array} \right\rangle,$$

$$\mathrm{II:} \quad \left\langle a, b, c \;\middle|\; \begin{array}{c} ababab = bababa, \\ b = c, \\ aabab = baaba \end{array} \right\rangle.$$

For example, an explicit form of an isomorphism from the presentation I *to the presentation* II *is given by the correspondence*

$$a \mapsto baba^{-1}b^{-1}, \quad b \mapsto babab^{-1}a^{-1}b^{-1}, \quad c \mapsto c.$$

In this paper, we adopt the presentation I and denote this presented group by $G_{\mathrm{B_{ii}}}$. For the presented group $G_{\mathrm{B_{ii}}}$, we associate the monoid $G^+_{\mathrm{B_{ii}}}$. We have an important remark on the monoid $G^+_{\mathrm{B_{ii}}}$.

REMARK 3.2. *Since both sides of the defining relations of $G^+_{\mathrm{B_{ii}}}$ contain the same number of the letter b, for arbitrary word W in $G^+_{\mathrm{B_{ii}}}$, the number of the letter b in W ought to be preserved in the process of rewriting W.*

4. Word problem and Conjugacy problem

In the present section, we define the word problem and the conjugacy problem in a monoid.

Definition 4.1. *Let* $G^+ = \langle L|R \rangle_{mo}$ *be a positive homogeneously presented monoid.*

1) *For arbitrary two words* U, V *in* L^*, *give an algorithm that decides whether* $U \doteqdot V$ *in* G^+ *or not.*

2) *For arbitrary two words* U, V *in* L^*, *give an algorithm that decides whether there exists an element* A *in* G^+ *such that* $AU \doteqdot VA$ *(then we write* $U \underset{mo}{\sim} V$*) or not.*

The problems 1), 2) *are called the* word problem *and the* conjugacy problem *Editor in a monoid* G^+, *respectively.*

Lemma 4.2. *Let* G *be a positive homogeneously presented group, and let* G^+ *be the associated monoid. Assume that the monoid* G^+ *is a cancellative monoid and* $\mathscr{F}(G^+) \neq \emptyset$. *Then:*

(1) *The localization homomorphism* $\pi : G^+ \to G$ *is injective.*

(2) *The word problem in* G^+ *is solvable if and only if the word problem in* G *is solvable.*

(3) *The conjugacy problem in* G^+ *is solvable if and only if the conjugacy problem in* G *is solvable.*

Proof. (1) Let $\Delta \in \mathscr{F}(G^+)$ be a fundamental element. We can easily show that, for any $U \in G^+$, U devides $\Delta^{\ell(U)}$ from the left and the right. Hence, we show that the monoid G^+ satisfies Öre's condition (see [C-P]). Therefore, the localization homomorphism π is injective.

(2) We put $\Lambda := \Delta^{\mathrm{ord}(\sigma_\Delta)}$, which belongs to the center $\mathscr{Z}(G^+)$ of the monoid G^+. For any two elements U, V in G, there exists a non-negative integer k in $\mathbf{Z}_{\geq 0}$ such that both $(\pi(\Lambda))^k U$ and $(\pi(\Lambda))^k V$ are equivalent to positive words. Since the localization homomorphism π is injective, there exists a unique element $U' \in G^+$ (resp. $V' \in G^+$) such that

$$\pi(U') = (\pi(\Lambda))^k U \ \ (\text{resp. } \pi(V') = (\pi(\Lambda))^k V).$$

Therefore, we have shown that $U = V$ can be shown in G algorithmically if and only if $U' \doteqdot V'$ can be shown in G^+ algorithmically.

(3) If two elements U and V in G are conjugate, then there exists a word B such that $BU = VB$. There exists a non-negative integer l in $\mathbf{Z}_{\geq 0}$ such that $(\pi(\Lambda))^l B$ is equivalent to a positive word. Since $\pi(\Lambda)$ belongs to the center of the group G, we say that two elements U and V in G are conjugate precisely when there is a positive word A such that AU is equivalent to VA. Therefore, due to the injectivity of the localization homomorphism π, we can show that the conjugacy problem in G^+ is solvable if and only if the conjugacy problem in G is solvable. $\square$

5. Main results

In this section, we state the main results on $G^{+}_{\mathrm{B_{ii}}}$ and $G_{\mathrm{B_{ii}}}$. First we prepare some lemmas.

For each $j \in \mathbf{Z}_{\geq 0}$, let

$$W(j) := \{w \in G^{+}_{\mathrm{B_{ii}}} \mid w \text{ contains the letter } b \text{ just } j\text{-times}\}.$$

For each $k \in \mathbf{Z}_{\geq 0}$, we put

$$\Delta_k := (a^k b)^3, \qquad \Lambda_k := ba^k bb.$$

Lemma 5.1. *The following relations hold for* $i = 1, 2, \dots$:

$$a^i b \doteq bc^i, \qquad bba^i \doteq c^i bb.$$

Proof. By using the defining relations $ab = bc$, $bba = cbb$ repeatedly, we show the equations. □

Lemma 5.2. *If* $w \in W(j)$ $(j \geq 4)$, *then* $b^3|_l w$ *and* $b^3|_r w$.

Proof. First of all, $\Delta_0 = b^3$ belongs to the center $\mathscr{Z}(G^{+}_{\mathrm{B_{ii}}})$ of the monoid $G^{+}_{\mathrm{B_{ii}}}$. Secondly, w inevitably contains a substring whose form is generally written as $ba^p c^q ba^r c^s ba^t c^u b$ $(p, q, r, s, t, u \in \mathbf{Z}_{\geq 0})$. Lastly, we have an equality

$$ba^p c^q b \doteq bc^q a^p b \doteq a^q bbc^p.$$

Therefore, by applying the defining relations to the substring $ba^p c^q ba^r c^s ba^t c^u b$, we have:

$$ba^p c^q ba^r c^s ba^t c^u b \doteq a^q bbc^p a^r c^s a^u bbc^t \doteq a^j c^{r+u} bbbba^{p+s} c^t \doteq bbba^j c^{r+u} ba^{p+s} c^t.$$

Lemma 5.3. $a\Lambda_k \doteq \Lambda_k a$, $c\Lambda_k \doteq \Lambda_k c$.

Proof. We have an equality:

$$aba^k bb \doteq bca^k bb \doteq ba^k cbb \doteq ba^k bba.$$

In the same way, we have an equality:

$$cba^k bb \doteq cbbc^k b \doteq bbac^k b \doteq bbc^k ab \doteq bbc^k bc \doteq ba^k bbc. \qquad □$$

We recall three facts from [S-I] §5, §7 and §8.

Proposition 5.4. *The monoid* $G^{+}_{\mathrm{B_{ii}}}$ *admits neither the left divisibility theory nor the right divisibility theory.*

Proposition 5.5. *The monoid* $G^{+}_{\mathrm{B_{ii}}}$ *is a cancellative monoid.*

Proposition 5.6. *For any* $\Delta \in \mathscr{Q}\mathscr{Z}(G^{+}_{\mathrm{B_{ii}}})$, $ord(\sigma_\Delta)$ *is equal to* 1.

As a consequence of Proposition 5.4, the monoid $G^+_{\mathrm{B_{ii}}}$ is not Gaussian and, hence, is neither Garside monoid nor Artin monoid.

We show that for an arbitrary element w in $G^+_{\mathrm{B_{ii}}}$ we choose a unique word in the free monoid $\{a, b, c\}^*$ that represents w, which we call the normal form of w. Thanks to Lemma 5.2 and Proposition 5.5, for any element w in $W(j)$ $(j \geq 4)$, there exist a unique integer m in $\mathbf{Z}_{\geq 0}$ and a unique element w' in $W(j)$ $(j \leq 3)$ such that $w \doteq b^{3m}w'$. Therefore, it is sufficient to show the existence of this notion for $w \in W(j)$ $(j \leq 3)$.

Lemma 5.7. *If $w \in W(j)$ $(j \leq 3)$, then w has the following normal forms:*
$j = 0$: $a^p c^q =: (p, q)_0$ $(p, q \in \mathbf{Z}_{\geq 0})$
$j = 1$: $a^p c^q b a^r =: (p, q, r)_1$ $(p, q, r \in \mathbf{Z}_{\geq 0})$
$j = 2$: $a^p c^q b b c^r =: (p, q, r)_2$ $(p, q, r \in \mathbf{Z}_{\geq 0})$
$j = 3$: $a^p c^q b a^r b b =: (p, q, r)_3$ $(p, q, r \in \mathbf{Z}_{\geq 0})$

Proof. We can easily show that w can be equivalently transformed into the above form. Therefore, we only prove the uniqueness of the normal form.

$j = 0$: We assume that $a^p c^q \doteq a^s c^t$ $(p + q = s + t)$ and $p \geq s$. Due to the cancellativity, we cancell a^s from left so that we obtain a new relation $a^{p-s} c^q \doteq c^t$. Next, we cancell c^q from right so that we obtain a relation $a^{p-s} \doteq c^{t-q}$. If $p - s$ $(= t - q) \geq 1$, this relation is contrary to Proposition 7.7 in [S-I]. Hence, we conclude $p - s = t - q = 0$.

$j = 1$: We assume that $a^p c^q b a^r \doteq a^s c^t b a^u$ $(p + q + r = s + t + u)$, and $q \geq t$. We cancell c^t from left so that we obtain a relation $c^{q-t} b c^p a^r \doteq b c^s a^u$. If $q - t \geq 1$, this relation is contrary to Proposition 7.7 in [S-I]. Hence, we conclude $q = t$. Next, we cancell b from left so that we obtain a relation $c^p a^r \doteq c^s a^u$. From the case of $j = 0$, we obtain $p = s$, $r = u$.

$j = 2$: We assume $a^p c^q b b c^r \doteq a^s c^t b b c^u$. We can easily show an equivalent relation $a^p c^q b a^r b \doteq a^s c^t b a^u b$ and cancell b from right. From the case of $j = 1$, we obtain $p = s$, $q = t$, $r = u$.

$j = 3$: We assume $a^p c^q b a^r b b \doteq a^s c^t b a^u b b$. We cancell bb from right so that we obtain a relation $a^p c^q b a^r \doteq a^s c^t b a^u$. From the case of $j = 1$, we obtain $p = s$, $q = t$, $r = u$. □

Theorem 5.8. *The following* i), ii), iii), iv), *and* v) *hold.*

i) *The element Δ_0 belongs to $\mathscr{QZ}(G^+_{\mathrm{B_{ii}}}) \backslash \mathscr{F}(G^+_{\mathrm{B_{ii}}})$. The elements Δ_k $(k \geq 1)$ belong to $\mathscr{F}(G^+_{\mathrm{B_{ii}}})$ and $ord(\sigma_{\Delta_k})$ $(k \geq 0)$ is equal to* 1.

ii) a) *The element Δ_0 is an indecomposable quasi-central element.*

b) *The fundamental elements Δ_k $(k \geq 1)$ are prime.*

c) *If Δ is an indecomposable quasi-central element, then there exists a non-negative integer k in $\mathbf{Z}_{\geq 0}$ such that Δ is equivalent to Δ_k.*

d) *We have $\Delta_{k_1} \Delta_{k_2} \doteq \Delta_{k_1+k_2} \Delta_0$ $(k_1, k_2 \in \mathbf{Z}_{\geq 0})$.*

iii) *The center* $\mathscr{Z}(G_{\mathrm{B_{ii}}})$ *is isomorphic to* $\mathbf{Z}^2$ *and generated by* Δ_0 *and* Δ_1.
iv) *The word problem and the conjugacy problem in* $G_{\mathrm{B_{ii}}}$ *are solvable.*
v) *The spherical growth function for the monoid* $G_{\mathrm{B_{ii}}}^+$ *is the following:*

$$\sum_{n=0}^{\infty} a_n t^n = \frac{t^2 - t + 1}{(1-t)^4},$$

where we put $a_n := \#\{w \in G_{\mathrm{B_{ii}}}^+ \mid \ell(w) = n\}$.

Proof. i) Since Δ_0 belongs to the center of $G_{\mathrm{B_{ii}}}^+$, Δ_0 belongs to $\mathscr{QZ}(G_{\mathrm{B_{ii}}}^+)$. Due to the cancellativity of $G_{\mathrm{B_{ii}}}^+$, we can easily show that Δ_0 does not belong to $\mathscr{F}(G_{\mathrm{B_{ii}}}^+)$. Next, we prove Δ_k belong to $\mathscr{F}(G_{\mathrm{B_{ii}}}^+)$. For the proof of this, it is sufficient to show that Δ_k are quasi-central elements which are divisible by the generators a, b and c (see [S-I, Proposition 7.4]). Actually, it is easy to show the following:

$$(a^k b)^3 \doteq (ba^k)^3 \doteq (bc^k)^3 \doteq (c^k b)^3.$$

And, we can also show that

$$a \cdot \Delta_k \doteq a \cdot ba^k ba^k ba^k \doteq ba^k cba^k ba^k \doteq ba^k bbc^k aa^k \doteq ba^k ba^k ba^k \cdot a \doteq \Delta_k \cdot a,$$

$$b \cdot \Delta_k \doteq b \cdot c^k bc^k bc^k b \doteq \Delta_k \cdot b$$

and

$$c \cdot \Delta_k \doteq a^k cba^k ba^k b \doteq a^k cbbc^k a^k b \doteq a^k bbc^k aa^k b \doteq a^k ba^k ba^k b \cdot c \doteq \Delta_k \cdot c.$$

Lastly, according to the Corollary of Theorem 5 in [S-I], we can show that $\mathrm{ord}(\sigma_{\Delta_k})$ is equal to 1.

ii) a),b) Since Δ_k contain the letter b just 3-times, it is sufficient to show that, if $w \in W(j)$ $(j \leq 2)$ is a quasi-central element, then w is equivalent to ε. We consider the following three cases:

$j = 0$: As $\mathrm{ord}(\sigma_w)$ is equal to 1, a relation $b \cdot (p,q)_0 \doteq (p,q)_0 \cdot b$ ought to hold. Thus, an equation $(q,0,p)_1 \doteq (p,q,0)_1$ holds. Due to Lemma 5.7, we conclude $p = q = 0$.

$j = 1$: In the same way, we have a relation $a \cdot (p,q,r)_1 \doteq (p,q,r)_1 \cdot a$. Thus, an equation $(p+1,q,r)_1 \doteq (p,q,r+1)_1$ holds. A contradiction.

$j = 2$: We have a relation $c \cdot (p,q,r)_2 \doteq (p,q,r)_2 \cdot c$. Thus, an equation $(p,q+1,r)_2 \doteq (p,q,r+1)_2$ holds. A contradiction.

c) Due to Lemma 5.2, Δ belongs to $W(j)$ $(j \leq 3)$. If Δ belongs to $W(j)$ $(j \leq 2)$, Δ cannot be a quasi-central element. Thus, Δ belongs to $W(3)$. So, we put $\Delta \equiv (p,q,r)_3$. In particular, we have a relation $b \cdot (p,q,r)_3 \doteq (p,q,r)_3 \cdot b$. Cancelling bbb from left, we have an equation $(q,r,p)_1 \doteq (p,q,r)_1$. Hence, we obtain $p = q = r$. Then, we have $\Delta \doteq (p,p,p)_3 \equiv \Delta_p$.

d) We have

$$\Delta_{k_1}\Delta_{k_2} \doteq a^{k_1}c^{k_1}\Lambda_{k_1}a^{k_2}c^{k_2}\Lambda_{k_2} \doteq a^{k_1+k_2}c^{k_1+k_2}\Lambda_{k_1}\Lambda_{k_2}$$
$$\doteq a^{k_1+k_2}c^{k_1+k_2}\Lambda_{k_1+k_2}bbb \doteq \Delta_{k_1+k_2}\Delta_0.$$

iii) By the consideration in ii), we easily show that, if w belongs to $\mathscr{Z}(G^+_{\mathrm{B_{ii}}})\backslash\{\varepsilon\}$, then there exists a unique pair $(k,l) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ such that w is equivalent to $\Delta_k\Delta_0^l$. And, due to Lemma 4.2, we say that the localization homomorphism $\pi : G^+_{\mathrm{B_{ii}}} \to G_{\mathrm{B_{ii}}}$ is injective. For an arbitrary element U in $\mathscr{Z}(G_{\mathrm{B_{ii}}})$, there exists a positive integer $m \in \mathbf{Z}_{\geq 0}$ such that $\Delta_1^m U$ is equivalent to a positive word. Since we can regard $\Delta_1^m U$ as an element in $\mathscr{Z}(G^+_{\mathrm{B_{ii}}})$, there exists a unique pair $(k,l) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ such that $\Delta_1^m U$ is equivalent to $\Delta_k\Delta_0^l (\doteq \Delta_1^k\Delta_0^{1-k}\Delta_0^l)$. Thus, we show that $\mathscr{Z}(G_{\mathrm{B_{ii}}})$ can be generated by Δ_0 and Δ_1. Next, we consider an equation

$$\Delta_0^{k_1}\Delta_1^{l_1} = \Delta_0^{k_2}\Delta_1^{l_2} \quad (k_1, k_2, l_1, l_2 \in \mathbf{Z})$$

to determine the center $\mathscr{Z}(G_{\mathrm{B_{ii}}})$. Applying Lemma 5.7 to this equation, we obtain $k_1 = k_2$ and $l_1 = l_2$. Hence, the center $\mathscr{Z}(G_{\mathrm{B_{ii}}})$ is isomorphic to $\mathbf{Z}^2$.

iv) By Lemma 5.7, it is sufficient to show that we can solve the word problem and the conjugacy problem in $G^+_{\mathrm{B_{ii}}}$. Because of the homogeneity of the defining relations in $G^+_{\mathrm{B_{ii}}}$, we can obtain algorithmically all the possible expressions of word W in $G^+_{\mathrm{B_{ii}}}$ in a finite number of steps. Hence, for arbitrary two words $U, V \in G^+_{\mathrm{B_{ii}}}$, by comparing two types of complete lists of all the possible expressions of words U and V, we can solve the word problem in $G^+_{\mathrm{B_{ii}}}$. Next, we consider the conjugacy problem in $G^+_{\mathrm{B_{ii}}}$. It is sufficient to show that, for arbitrary two words U, V $(\in W(j)$ $(j \leq 3))$ of the same length n, we decide in a finite number of steps whether $U \underset{mo}{\sim} V$ or not. We consider the following four cases:

$j = 0$: We prove the following Claims:

Claim 1. If $(p, n-p)_0 \underset{mo}{\sim} (q, n-q)_0$ $(0 \leq p < q \leq n)$, then we say that $p = 0$ and $q = n$.

Proof. First, we easily show that $(n,0)_0 \underset{mo}{\sim} (0,n)_0$. Assuming that

$$(p, n-p)_0 \underset{mo}{\sim} (q, n-q)_0 \qquad (1 \leq p < q \leq n),$$

then we say there exists an element w in $G^+_{\mathrm{B_{ii}}}$ such that

$$w \cdot (p, n-p)_0 \doteq (q, n-q)_0 \cdot w.$$

By Lemma 5.2, we may assume $w \in W(j)$ $(j \leq 3)$. Applying Lemma 5.7 to the equality $w \cdot (p, n-p)_0 \doteq (q, n-q)_0 \cdot w$, we show that a contradiction occurs for any $w \in W(j)$ $(j \leq 3)$. □

$j = 1$: We prove the following claim:
Claim 2. If $w, w' \in W(1)$, then $w \underset{mo}{\sim} w'$.

Proof. We have

$$c^{p+r}a^r \cdot (p,q,r)_1 \doteqdot c^{p+r}a^r a^p c^q b a^r \doteqdot c^{p+q+r}a^r a^p b a^r$$
$$\doteqdot c^{p+q+r} b c^{p+r} a^r \doteqdot (0, p+q+r, 0)_1 \cdot c^{p+r}a^r.$$

Hence, we say $(p,q,r)_1 \underset{mo}{\sim} (0, p+q+r, 0)_1$.

$j = 2$: We prove the following claim:
Claim 3. If $w, w' \in W(2)$, then $w \underset{mo}{\sim} w'$.

Proof. We have

$$a^{p+r}c^p b \cdot (p,q,r)_2 \doteqdot a^{p+r}c^p b a^p c^q b b c^r \doteqdot a^{p+q+r} c^p b a^p b b c^r \doteqdot a^{p+q+r} \cdot b a^p b b c^{p+r}$$
$$\doteqdot a^{p+q+r} b b a^{p+r} c^p b \doteqdot (p+q+r, 0, 0)_2 \cdot a^{p+r}c^p b.$$

Hence, we say $(p,q,r)_2 \underset{mo}{\sim} (p+q+r, 0, 0)_2$. □

$j = 3$: We prove the following claim:
Claim 4. $(p,q,r)_3 \underset{mo}{\sim} (r,p,q)_3 \underset{mo}{\sim} (q,r,p)_3$.

Proof. First, we have $b \cdot (p,q,r)_3 \doteqdot b a^p c^q b a^r b b \doteqdot (q,r,p)_3 \cdot b$. Next, we have $bb \cdot (p,q,r)_3 \doteqdot b b a^p c^q b a^r b b \doteqdot c^p b b c^q b b c^r b \doteqdot a^r c^p b a^q b b b b \doteqdot (r,p,q)_3 \cdot bb$. □

Hence, we can choose a representative $(p,q,r)_3$ $(p, q \geq r)$.

As $(p,q,r)_3 \doteqdot (p-r, q-r)_0 \Delta_r$ and Δ_r belongs to the center $\mathscr{Z}(G^+_{\mathrm{B_{ii}}})$, the case $j = 3$ can be reduced to the case $j = 0$.

These complete the proof.

v) First, let $\alpha_{k,n} := \#\{w \in W(k) \mid \ell(w) = n\}$ $(k = 0, 1, \ldots, n)$, and let $\beta_m := \#\{w \in W(3) \mid \ell(w) = m\}$ $(m = 3, 4, \ldots, n, n+1, n+2)$. We consider the following three cases: a_{3n}, a_{3n+1} and a_{3n+2}.

Case a_{3n}: By Lemma 5.7, we show $\beta_m = (1/2)(m-2)(m-1)$. And, due to Lemma 5.2, we can easily show that

$$\alpha_{3n,3n} = \beta_3, \alpha_{3n-1,3n} = \beta_4, \ldots, \alpha_{3,3n} = \beta_{3n}.$$

In the same way, we show that

$$\alpha_{2,3n} = (3n/2)(3n-1)(= \beta_{3n+1}), \qquad \alpha_{1,3n} = (3n/2)(3n+1)(= \beta_{3n+2})$$

and $\alpha_{0,3n} = 3n + 1$. Hence,

$$a_{3n} = \sum_{k=0}^{3n} \alpha_{k,3n} = \alpha_{0,3n} + \sum_{k=1}^{3n} \alpha_{k,3n}$$

$$= 3n + 1 + (1/2) \sum_{k=1}^{3n} k(k+1) = (1/2)(9n^3 + 9n^2 + 8n + 2).$$

Case a_{3n+1}: In the same way, we show that

$$a_{3n+1} = \sum_{k=0}^{3n+1} \alpha_{k,3n+1} = \alpha_{0,3n+1} + \sum_{k=1}^{3n+1} \alpha_{k,3n+1}$$

$$= 3n + 2 + (1/2) \sum_{k=1}^{3n+1} k(k+1) = (1/2)(9n^3 + 18n^2 + 17n + 6).$$

Case a_{3n+2}: In the same way, we show that

$$a_{3n+2} = \sum_{k=0}^{3n+2} \alpha_{k,3n+2} = \alpha_{0,3n+2} + \sum_{k=1}^{3n+2} \alpha_{k,3n+2}$$

$$= 3n + 3 + (1/2) \sum_{k=1}^{3n+2} k(k+1) = (1/2)(9n^3 + 27n^2 + 32n + 14).$$

Next, we easily show:

$$(1-t)^4 \left\{ \sum_{k=0}^{\infty} a_{3k} t^{3k} + \sum_{k=0}^{\infty} a_{3k+1} t^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} t^{3k+2} \right\} = t^2 - t + 1.$$

Hence, we obtain the explicit formula. $\square$

Remark 5.9. *In* [S-I] §6, *we raised a question: let G be a positive homogeneously presented group and let G^+ be the associated monoid. Then, are there finitely many elements $\Delta_1, \Delta_2, \ldots, \Delta_k \in \mathscr{F}(G^+)$ such that the following holds?*

$$\mathscr{F}(G^+) = \mathscr{QZ}(G^+)\Delta_1 \cup \mathscr{QZ}(G^+)\Delta_2 \cup \cdots \cup \mathscr{QZ}(G^+)\Delta_k.$$

As a consequence of Theorem 5.8, *we have constructed a counterexample. We claim that $\mathscr{F}(G^+_{\mathrm{B_{ii}}})(\subset \mathscr{QZ}(G^+_{\mathrm{B_{ii}}}))$ is an infinitely generated idealistic subsemigroup. For the proof of this, we assume that $\mathscr{F}(G^+_{\mathrm{B_{ii}}})$ is finitely generated: $\mathscr{QZ}(G^+_{\mathrm{B_{ii}}})\Delta_{k_1} \cup \mathscr{QZ}(G^+_{\mathrm{B_{ii}}})\Delta_{k_2} \cup \cdots \cup \mathscr{QZ}(G^+_{\mathrm{B_{ii}}})\Delta_{k_m}$. However, when we take an integer l large enough, Δ_l cannot belong to*

$$\mathscr{QZ}(G^+_{\mathrm{B_{ii}}})\Delta_{k_1} \cup \mathscr{QZ}(G^+_{\mathrm{B_{ii}}})\Delta_{k_2} \cup \cdots \cup \mathscr{QZ}(G^+_{\mathrm{B_{ii}}})\Delta_{k_m}.$$

A contradiction.

REMARK 5.10. *Since the group $G_{\mathrm{B_{ii}}}$ contains $\mathbf{Z}^2$ as a subgroup, $G_{\mathrm{B_{ii}}}$ is not* word hyperbolic ([Gr2]).

In [S3, S4], the distribution of the zeroes of the denominator polynomials of the growth functions associated with Artin monoids is investigated. Since the zeroes of the denominator polynomials of the growth function for the monoid $G^+_{\mathrm{B_{ii}}}$ only consist of 1 with multiplicity 4, he asked whether the group $G_{\mathrm{B_{ii}}}$ contains a free abelian subgroup of rank 4 of finite index. Actually, we show this in the following Lemma.

LEMMA 5.11. *For an arbitrary element w in $G_{\mathrm{B_{ii}}}$, the element w has the following normal form:*

$$\langle p,q,r,s\rangle := b^p(\Delta_1\Delta_0^{-1})^q a^r c^s \qquad (p,q,r,s \in \mathbf{Z}).$$

PROOF. We assume that

$$\langle p,q,r,s\rangle = \langle p',q',r',s'\rangle \qquad (p,p',q,q',r,r',s,s' \in \mathbf{Z}).$$

Since $\Delta_1\Delta_0^{-1}$ belongs to the center of the group $G_{\mathrm{B_{ii}}}$ and $ac = ca$, we say that $\langle p-p', q-q', r-r', s-s'\rangle = \varepsilon$. Without loss of generality, we assume that $q - q' \geq 0$. Then, an equation

$$b^{p-p'}\Delta_1^{q-q'}a^{r-r'}c^{s-s'} = b^{3q-3q'}$$

holds. If $p - p' \geq 0$, then we multiply a^tc^u $(t,u \gg 0)$ from the right. Thus, both sides of the equation are equivalent to positive words. Since the localization homomorphism π is injective, an equation

$$b^{p-p'}\Delta_1^{q-q'}a^{r-r'+t}c^{s-s'+u} \doteqdot b^{3q-3q'}a^tc^u$$

holds. Due to Remark 3.4, we can easily show that $p = p'$. Thus, an equation

$$\Delta_1^{q-q'}a^{r-r'+t}c^{s-s'+u} \doteqdot b^{3q-3q'}a^tc^u$$

holds. Due to Limma 5.7, we show that $q = q'$, $r = r'$, $s = s'$. Similarly, if $p - p' \leq 0$, we conclude that $p = p'$, $q = q'$, $r = r'$, $s = s'$. Therefore, we have shown the uniqueness of the normal form.

For an arbitrary element w in $G_{\mathrm{B_{ii}}}$, there exists a non-negative integer k in $\mathbf{Z}_{\geq 0}$ such that $(\pi(\Delta_1))^k w$ is equivalent to a positive word. Since the localization homomorphism π is injective, there exists a unique element w' in $G^+_{\mathrm{B_{ii}}}$ such that $\pi(w') = (\pi(\Delta_1))^k w$. Applying Lemma 5.7 to the element w', we can easily show that w can be equivalently transformed into the above form. $\square$

As a corollary of the theorem, we show the following.

Corollary 5.12. *The following* i), ii), iii), iv) *and* v) *hold.*

i) *The group* $G_{\mathrm{B_{ii}}}$ *contains a subgroup of index three isomorphic to* $\mathbf{Z}^4$.

ii) *The group* $G_{\mathrm{B_{ii}}}$ *has a polynomial growth rate.*

iii) *The group* $G_{\mathrm{B_{ii}}}$ *is solvable.*

iv) *A faithful* 5×5*-matrix representation* $\rho : G_{\mathrm{B_{ii}}} \to GL(5, \mathbf{Z})$ *of the group* $G_{\mathrm{B_{ii}}}$ *is constructed:*

$$\langle p, q, r, s \rangle \mapsto \left(\begin{array}{cc|ccc} 1 & p & q & r & s \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & & & \\ 0 & 0 & & N^p & \\ 0 & 0 & & & \end{array} \right), \qquad \text{where } N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

v) *The group* $G_{\mathrm{B_{ii}}}$ *is torsion free.*

Proof. i) Let H be the subgroup of $G_{\mathrm{B_{ii}}}$ generated by Δ_0, $\Delta_1\Delta_0^{-1}$, a and c. Due to the commutativity of each pair of the generators and Lemma 5.11, we show that H is isomorphic to $\mathbf{Z}^4$. It is easy to show that H is a subgroup of index three.

ii) Due to the Gromov's theorem on groups of polynomial growth ([Gr1]), the group $G_{\mathrm{B_{ii}}}$ has a polynomial growth rate.

iii) Since there is a sequence of subgroups

$$\{1\} \lhd H \lhd G_{\mathrm{B_{ii}}}$$

such that $G_{\mathrm{B_{ii}}}/H$ is an abelian group, the group $G_{\mathrm{B_{ii}}}$ is solvable.

iv) For any integers p, q, r, s, t in $\mathbf{Z}$, we have three equalities:

$$\langle p, q, r, s \rangle \cdot b^{3t} \doteqdot b^{3t} \cdot \langle p, q, r, s \rangle,$$

$$\langle p, q, r, s \rangle \cdot b^{3t+1} \doteqdot b^{3t+1} \cdot \langle p, q + s, -s, r - s \rangle,$$

$$\langle p, q, r, s \rangle \cdot b^{3t+2} \doteqdot b^{3t+2} \cdot \langle p, q + r, s - r, -r \rangle.$$

Therefore, we have three equalities:

$$\langle p, q, r, s \rangle \cdot \langle 3t, q', r', s' \rangle \doteqdot \langle p + 3t, q + q', r + r', s + s' \rangle,$$

$$\langle p, q, r, s \rangle \cdot \langle 3t + 1, q', r', s' \rangle \doteqdot \langle p + 3t + 1, q' + q + s, r' - s, s' + r - s \rangle,$$

$$\langle p, q, r, s \rangle \cdot \langle 3t + 2, q', r', s' \rangle \doteqdot \langle p + 3t + 2, q' + q + r, r' + s - r, s' - r \rangle.$$

Hence, we show that the map ρ is a group homomorphism. Due to Lemma 5.11, we show that ρ is a faithful representation.

v) We assume that $\langle p, q, r, s \rangle^k = \varepsilon$ for integers p, q, r, s in $\mathbf{Z}$ and k in $\mathbf{Z}_{>0}$. Since the first 2×2-matrix of the normal form $\langle p, q, r, s \rangle$ is a unipotent matrix with the $(1,2)$ entry equal to p, we have $p=0$. Since $\Delta_1\Delta_0^{-1}$ belongs to the center of the group $G_{\mathrm{B_{ii}}}$ and $ac=ca$, we show that

$$\langle 0,q,r,s\rangle^k = \langle 0,kq,kr,ks\rangle.$$

Hence, we say that $q=0$, $r=0$, $s=0$. □

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