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New oscillation criteria for second-order neutral dynamic equations on time scales via Riccati substitution

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Abstract. In this paper, we consider the second-order nonlinear neutral functional dynamic equation

$$(p(t)([y(t)+r(t)y(\tau(t))]^{\Delta})^{\gamma})^{\Delta}+f(t,y(\delta(t)))=0,$$

on a time scale $\mathbf{T}$ and establish some new sufficient conditions for oscillation. Our results improve oscillation results for neutral delay dynamic equations on time scales and are new when $\delta(t)>t$ and/or $0<\gamma<1$. Furthermore our results can be applied on the time scales $\mathbf{T}=h\mathbf{T}$, for $h>0$, $\mathbf{T}=q^{\mathbf{N}}=\{t:t=q^k\}$, $k\in\mathbf{N}$, $q>1$, $\mathbf{T}=\mathbf{N}^2=\{t^2:t\in\mathbf{N}\}$, $\mathbf{T}_2=\{\sqrt{n}:n\in\mathbf{N}_0\}$, $\mathbf{T}_3=\{\sqrt[3]{n}:n\in\mathbf{N}_0\}$, and when $\mathbf{T}=\mathbf{T}_n=\{t_n:n\in\mathbf{N}_0\}$ where $\{t_n\}$ is the set of harmonic numbers, etc.

1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [6], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale $\mathbf{T}$, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [9]), i.e, when $\mathbf{T}=\mathbf{R}$, $\mathbf{T}=\mathbf{N}$ and $\mathbf{T}=q^{\mathbf{N}_0}=\{q^t:t\in\mathbf{N}_0\}$ where $q>1$. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations

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that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [4]). There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics*. A recent cover story article in New Scientist [14] discusses several possible applications. Since then several authors have expounded on various aspects of this new theory [5]. The book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [4] summarizes and organizes much of time scale calculus. For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$. We assume throughout that $\mathbf{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbf{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbf{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbf{T} : s < t\},$$

where $\sup \varnothing = \inf \mathbf{T}$. A point $t \in \mathbf{T}$, is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbf{T}$, is right-dense if $\sigma(t) = t$, is left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbf{T} \to \mathbf{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbf{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbf{T})$. The graininess function μ for a time scale $\mathbf{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbf{T} \to \mathbf{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$.

Definition 1. *Fix $t \in \mathbf{T}$ and let $x : \mathbf{T} \to \mathbf{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t with*

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|, \qquad \textit{for all } s \in U.$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t.

We will frequently use the results in the following theorem which is due to Hilger [6].

Theorem 1. *Assume that $g : \mathbf{T} \to \mathbf{R}$ and let $t \in \mathbf{T}$.*

(i) *If g is differentiable at t, then g is continuous at t.*

(ii) *If g is continuous at t and t is right-scattered, then g is differentiable at t with*

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}.$$

(iii) *If g is differentiable and t is right-dense, then*

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) *If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.*

In this paper, we will refer to the (delta) integral which we can define as follows:

Definition 2. *If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by*

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It can be shown (see [4]) that if $g \in C_{rd}(\mathbf{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbf{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbf{T}$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \qquad \text{and} \qquad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}.$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t,$$

and the integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t)f(t).$$

For oscillation of second-order neutral dynamic equations, we refer the reader to the papers [1], [2], [3], [7], [8], [11], [12], [13], [15] and [16]. We note that all the above results for neutral equations are given in the case when $\gamma \geq 1$ and $\delta(t) \leq t$ and nothing is known regarding the oscillation of neutral dynamic equations when $0 < \gamma < 1$ and $\delta(t) > t$. So the natural question now is: If it is possible to find new oscillation criteria to cover these cases? One of our aims in this paper is to give an affirmative answer to this question.

In this paper, we consider the nonlinear neutral functional dynamic equation

$$(p(t)([y(t) + r(t)y(\tau(t))]^{\Delta})^{\gamma})^{\Delta} + f(t, y(\delta(t))) = 0, \tag{1.1}$$

on a time scale $\mathbf{T}$ and we give some new sufficient conditions for oscillation. Throughout this paper, we will assume the following hypotheses:

(h_1) $\gamma > 0$ is an odd positive integer, $r(t)$ and $p(t)$ are real valued *rd*-continuous positive functions defined on $\mathbf{T}$, $\tau : \mathbf{T} \to \mathbf{T}$, $\delta : \mathbf{T} \to \mathbf{T}$, $\tau(t) \leq t$ for all $t \in \mathbf{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;

(h_2) $\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{1/\gamma} \Delta t = \infty$, $0 \leq r(t) < 1$;

(h_3) $f(t,u) : \mathbf{T} \times \mathbf{R} \to \mathbf{R}$ is continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive *rd*-continuous function $q(t)$ defined on $\mathbf{T}$ such that $|f(t,u)| \geq q(t)|u^{\gamma}|$.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbf{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbf{T}}$ by $[t_0, \infty)_{\mathbf{T}} := [t_0, \infty) \cap \mathbf{T}$. Throughout this paper these assumptions will be supposed to hold. Let $\tau^*(t) = \min\{\tau(t), \delta(t)\}$ and let $T_0 = \min\{\tau^*(t) : t \geq 0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq 0 : \tau^*(s) \leq t\}$ for $t \geq T_0$. Clearly if $\tau^*(t) \leq t$, then $\tau^*_{-1}(t) \geq t$ for $t \geq T_0$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists. Throughout the paper, we will use the following notations:

$$x(t) := y(t) + r(t)y(\tau(t)), \qquad x^{[1]} := p(x^{\Delta})^{\gamma}, \qquad \text{and} \qquad x^{[2]} := (x^{[1]})^{\Delta}. \tag{1.2}$$

By a solution of (1.1) we mean a nontrivial real-valued function y which has the properties $x \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$, and $x^{[1]} \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$ where C_{rd} is the space of *rd*-continuous functions. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution y of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

The results in the subsection 2.1 cover the case when $\delta(t) > t$ and the results in the subsection 2.2 cover the case when $\delta(t) \leq t$. The results in this paper can be applied to the equation (1.1) when $0 < \gamma < 1$ and/or $\delta(t) > t$ and improve the results established in [15], in the sense that the results can be applied on any time scale not only on discrete time scales when $\mu(t) \neq 0$, which is the case considered in [15].

2. Main results

In this section, we state and prove the main oscillation results. We start with the following Lemmas which play important roles in the proofs of the main results.

LEMMA 2.1. *Assume that* (h_1)–(h_3) *hold and* (1.1) *has a nonoscillatory solution* y *on* $[t_0, \infty)_{\mathbf{T}}$ *and* x *is defined as in* (1.2). *Then there exists* $T > t_0$ *such that* $x(t)x^{[1]}(t) > 0$ *for* $t \geq T$.

PROOF. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbf{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbf{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbf{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form). Since y is a positive solution of (1.1) and $q(t) > 0$, we have (see (h_3))

$$(x^{[1]}(t))^{\Delta} \leq -q(t)y^{\gamma}(\delta(t)) < 0, \qquad \text{for } t \in [t_1, \infty)_{\mathbf{T}}. \tag{2.1}$$

Then $x^{[1]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbf{T}}$. We claim that $x^{[1]}(t) > 0$ on $[t_1, \infty)_{\mathbf{T}}$. Assume not. Then there is $t_2 \in [t_1, \infty)_{\mathbf{T}}$ such that (note $x^{[1]}(t)$ is strictly decreasing), $x^{[1]}(t_2) = c < 0$. Then from (2.1), we have $x^{[1]}(t) \leq c$, for $t \geq t_2$ and therefore

$$x^{\Delta}(t) \leq \frac{c}{p^{1/\gamma}(t)}, \qquad \text{for } t \in [t_2, \infty)_{\mathbf{T}}. \tag{2.2}$$

Integrating the last inequality from t_2 to t, we find from (h_2) that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s)\Delta s \leq x(t_2) + c\int_{t_2}^{t} \frac{\Delta s}{p^{1/\gamma}(s)} \to -\infty \qquad \text{as } t \to \infty, \tag{2.3}$$

which implies that x is eventually negative. This contradiction completes the proof.

LEMMA 2.2. *Assume that* (h_1)–(h_3) *hold and* (1.1) *has a nonoscillatory solution* y *on* $[t_0, \infty)_{\mathbf{T}}$ *and* x *is defined as in* (1.2). *Then there exists* $T \geq t_0$ *such that*

$$(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + P(t)x^{\gamma}(\delta(t)) \leq 0, \qquad \textit{for } t \geq T, \tag{2.4}$$

where $P(t) = q(t)(1 - r(\delta(t)))^{\gamma}$.

PROOF. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbf{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbf{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form). Since y is a positive solution of (1.1), then from Lemma 2.1, we see that (note $x^{[1]}(t) > 0$ and $p(t) > 0$)

$$x(t) > 0, \qquad x^{\Delta}(t) > 0, \qquad \text{and} \qquad (x^{[1]}(t))^{\Delta} < 0, \quad \text{for } t \geq t_1. \tag{2.5}$$

Since $\tau(t) \leq t$ and $r(t) \geq 0$, we have from (1.2) and (2.5) that

$$\begin{aligned} x(t) = y(t) + r(t)y(\tau(t)) &\leq y(t) + r(t)x(\tau(t)) \\ &\leq y(t) + r(t)x(t), \qquad \text{for } t \geq t_1. \end{aligned}$$

Thus $y(t) \geq (1 - r(t))x(t)$, for $t \geq t_1$. Then for $t \geq t_2$, where $t_2 > t_1$ is chosen large enough, we have

$$y(\delta(t)) \geq (1 - r(\delta(t)))x(\delta(t)). \tag{2.6}$$

From (2.1) and the last inequality, we have inequality (2.4) and this completes the proof.

2.1. The case when $\delta(t) > t$

In this subsection, we establish some sufficient conditions for oscillation of (1.1) when $\delta(t) > t$. We start with the following theorem.

Theorem 2.1. *Assume that* (h_1)–(h_3) *hold. Let* y *be a nonoscillatory solution of* (1.1) *and make the Riccati substitution*

$$w(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}, \tag{2.7}$$

where x *is defined as in* (1.2). *Then* $w(t) > 0$, *for* $t \geq T$ *(here* T *is as in Lemma* 2.2*) and*

$$w^{\Delta}(t) + Q(t) + \frac{\gamma}{p^{1/\gamma}(t)}(w^{\sigma}(t))^{1+1/\gamma} \leq 0, \qquad \textit{for } t \in [T, \infty)_{\mathbf{T}}, \tag{2.8}$$

where

$$Q(t) := \gamma P(t)\left(\frac{p^{1/\gamma}(t)P(t,T)}{p^{1/\gamma}(t)P(t,T) + \sigma(t) - t}\right)^{\gamma}, \qquad \textit{and} \qquad P(t,T) := \int_T^t \left(\frac{1}{p(s)}\right)^{1/\gamma} \Delta s.$$

Proof. Let y be as above and without loss of generality, we assume that there is $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1$. Then from Lemma 2.1 and (1.2), there exists $T > t_1$ such that

$$x(t) > 0, \qquad x^{[1]}(t) > 0, \qquad \text{and} \qquad x^{[2]}(t) < 0, \quad \text{for } t \geq T.$$

By the quotient rule [4, Theorem 1.20] and the definition of $w(t)$, we have

$$\begin{aligned} w^{\Delta}(t) &= \frac{x^{\gamma}(t)x^{[2]}(t) - (x^{\gamma}(t))^{\Delta}x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \\ &= \frac{x^{[2]}(t)}{(x^{\delta}(t))^{\gamma}}\left(\frac{x^{\delta}(t)}{x^{\sigma}(t)}\right)^{\gamma} - \frac{(x^{\gamma}(t))^{\Delta}x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}. \end{aligned}$$

From Lemma 2.2, we see that

$$w^{\Delta}(t) \le -P(t)\left(\frac{x^{\delta}(t)}{x^{\sigma}(t)}\right)^{\gamma} - \frac{(x^{\gamma}(t))^{\Delta}x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}, \qquad \text{for } t \ge T. \tag{2.9}$$

By the Pötzsche chain rule ([4, Theorem 1.90]), if $f^{\Delta}(t) > 0$ and $\gamma > 1$, (note $f^{\sigma} \ge f$) we obtain

$$\begin{aligned} (f^{\gamma}(t))^{\Delta} &= \gamma \int_0^1 [f(t) + \mu h f^{\Delta}(t)]^{\gamma-1} f^{\Delta}(t) dh \\ &\ge \gamma \int_0^1 (f(t))^{\gamma-1} f^{\Delta}(t) dh = \gamma (f(t))^{\gamma-1} f^{\Delta}(t). \end{aligned} \tag{2.10}$$

Also by the Pötzsche chain rule ([4, Theorem 1.90]), if $f^{\Delta}(t) > 0$ and $0 < \gamma \le 1$, we obtain

$$\begin{aligned} (f^{\gamma}(t))^{\Delta} &= \gamma \int_0^1 [f(t) + h\mu(t) f^{\Delta}(t)]^{\gamma-1} dh f^{\Delta}(t) \\ &= \gamma \int_0^1 [(1-h)f(t) + h f^{\sigma}(t)]^{\gamma-1} dh f^{\Delta}(t) \\ &\ge \gamma \int_0^1 (f^{\sigma}(t))^{\gamma-1} dh f^{\Delta}(t) = \gamma (f^{\sigma}(t))^{\gamma-1} f^{\Delta}(t). \end{aligned} \tag{2.11}$$

Now from (2.10) and (2.11), using $f(t) = x(t)$ and the fact that $x(t)$ is increasing and $x^{[1]}(t)$ is decreasing, we have for $\gamma > 1$, that

$$\begin{aligned} \frac{((x(t))^{\gamma})^{\Delta}x^{[1]}(t)}{(x(t))^{\gamma}(x^{\sigma}(t))^{\gamma}} &\ge \frac{\gamma x^{[1]}(t)(x^{[1]})^{1/\gamma}(t)}{p^{1/\gamma}x(t)(x^{\sigma}(t))^{\gamma}} \\ &\ge \frac{\gamma (x^{[1]}(t))^{\sigma}((x^{[1]}(t))^{\sigma})^{1/\gamma}}{p^{1/\gamma}x^{\sigma}(t)(x^{\sigma}(t))^{\gamma}} = \gamma \frac{1}{p^{1/\gamma}(t)}(w^{\sigma}(t))^{1/\gamma+1}. \end{aligned}$$

Also for $0 < \gamma \le 1$, we have

$$\frac{(x^{\gamma}(t))^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \ge \frac{\gamma x^{[1]}(t)(x^{\sigma}(t))^{\gamma-1}(x^{[1]}(t))^{1/\gamma}}{p^{1/\gamma}(t)x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}$$

$$\ge \frac{\gamma(x^{[1]}(t))^{\sigma}((x^{[1]})^{\sigma}(t))^{1/\gamma}}{p^{1/\gamma}(t)(x^{\sigma}(t))^{\gamma}x^{\sigma}(t)} = \gamma \frac{1}{p^{1/\gamma}(t)}(w^{\sigma}(t))^{1+1/\gamma}.$$

Thus

$$\frac{(x^{\gamma}(t))^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \ge \gamma \frac{1}{p^{1/\gamma}}(w^{\sigma}(t))^{1+1/\gamma}, \qquad \text{for } \gamma > 0.$$

Substituting this inequality into (2.9), we have

$$w^{\Delta}(t) \le -P(t)\left(\frac{x^{\delta}(t)}{x^{\sigma}(t)}\right)^{\gamma} - \gamma \frac{1}{p^{1/\gamma}(t)}(w^{\sigma})^{1+1/\gamma}, \qquad \text{for } t \ge T. \tag{2.12}$$

Next consider the coefficient of P in (2.12). Since $x^{\sigma} = x + \mu x^{\Delta}$, we have

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}}{x(t)} = 1 + \frac{\mu(t)}{p^{1/\gamma}(t)} \frac{(x^{[1]}(t))^{1/\gamma}}{x(t)}.$$

Also since $x^{[1]}(t)$ is decreasing, we have for $t \ge T$, that

$$x(t) = x(T) + \int_T^t (x^{[1]}(s))^{1/\gamma}\left(\frac{1}{p(s)}\right)^{1/\gamma} \Delta s$$

$$\ge x(T) + (x^{[1]}(t))^{1/\gamma}\int_T^t \left(\frac{1}{p(s)}\right)^{1/\gamma} \Delta s > (x^{[1]}(t))^{1/\gamma}\int_T^t \left(\frac{1}{p(s)}\right)^{1/\gamma} \Delta s.$$

It follows that

$$\frac{x(t)}{(x^{[1]}(t))^{1/\gamma}} \ge \int_T^t \left(\frac{1}{p(s)}\right)^{1/\gamma} \Delta s = P(t,T), \qquad \text{for } t \ge T. \tag{2.13}$$

Hence

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)} = 1 + \frac{\mu(t)}{p^{1/\gamma}(t)} \frac{(x^{[1]}(t))^{1/\gamma}}{x(t)}$$

$$\le \frac{p^{1/\gamma}(t)P(t,T) + \mu(t)}{p^{1/\gamma}(t)P(t,T)}, \qquad \text{for } t \ge T.$$

Hence, we have

$$\frac{x(t)}{x^{\sigma}(t)} \ge \frac{p^{1/\gamma}(t)P(t,T)}{p^{1/\gamma}(t)P(t,T) + \sigma(t) - t}, \qquad \text{for } t \ge T.$$

Thus for $t \geq T$, we have

$$(2.14) \qquad \frac{x^{\delta}(t)}{x^{\sigma}(t)} = \frac{x^{\delta}(t)}{x(t)} \frac{x(t)}{x^{\sigma}(t)} \geq \left(\frac{x^{\delta}(t)}{x(t)}\right) \frac{p^{1/\gamma}(t)P(t,T)}{p^{1/\gamma}(t)P(t,T) + \sigma(t) - t}.$$

Now, since $\delta(t) > t$ and $x(t)$ is increasing, we have

$$(2.15) \qquad x^{\delta}(t) > x(t).$$

This and (2.14) guarantee that

$$(2.16) \qquad \frac{x^{\delta}(t)}{x^{\sigma}(t)} \geq \frac{p^{1/\gamma}(t)P(t,T)}{p^{1/\gamma}(t)P(t,T) + \sigma(t) - t}, \qquad \text{for } t \geq T.$$

Put (2.16) into (2.12) and we obtain inequality (2.8) and this completes the proof.

THEOREM 2.2. *Assume that* (h_1)–(h_3) *hold. Furthermore, assume that*

$$(2.17) \qquad \int_{t_0}^{\infty} Q(s)\Delta s = \infty.$$

Then every solution of (1.1) *oscillates.*

PROOF. Suppose to the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq T$ (where T is as in Theorem 2.1). We consider only this case, because the proof when $y(t) < 0$ is similar. Let w be defined as in Theorem 2.1. Then from Theorem 2.1, we see that $w(t) > 0$ for $t \geq T$ and satisfies the inequality

$$(2.18) \qquad -w^{\Delta}(t) \geq Q(t) + \frac{\gamma}{p^{1/\gamma}(t)} (w^{\sigma}(t))^{1+1/\gamma} > Q(t), \qquad \text{for } t \geq T.$$

From the definition of $x^{[1]}(t)$, we see that

$$x^{\Delta}(t) = \left(\frac{x^{[1]}(t)}{p(t)}\right)^{1/\gamma}.$$

Integrating this from T to t, we obtain

$$x(t) = x(T) + \int_T^t \left(\frac{1}{p(s)} x^{[1]}(s)\right)^{1/\gamma} \Delta s, \qquad \text{for } t \geq T.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + (x^{[1]}(t))^{1/\gamma} \int_T^t \left(\frac{1}{p(s)}\right)^{1/\gamma} \Delta s, \qquad \text{for } t \geq T.$$

It follows that

$$w(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int_{t_0}^{t} \left(\frac{1}{p(s)} \right)^{1/\gamma} \Delta s \right)^{-\gamma}, \qquad \text{for } t \in [T, \infty)_{\mathbf{T}},$$

which implies, using (h_2), that $\lim_{t\to\infty} w(t) = 0$. Integrating (2.18) from T to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we obtain

$$w(T) \geq \int_{T}^{\infty} Q(s)\Delta s,$$

which contradicts (2.17). The proof is complete.

In the following, we consider the case when

$$\int_{t_0}^{\infty} Q(s)\Delta s < \infty. \tag{2.19}$$

We introduce the following notations:

$$p_* := \liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s, \qquad q_* := \liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{p(t)} Q(s)\Delta s,$$

$$l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}.$$

Theorem 2.3. *Assume that* (h_1)–(h_3) *hold and* $p^{\Delta} \geq 0$. *Let* y *be a positive solution of* (1.1), *and* x *is defined as in* (1.2). *Define*

$$r_* := \liminf_{t\to\infty} \frac{t^{\gamma} w^{\sigma}(t)}{p(t)}, \qquad R := \limsup_{t\to\infty} \frac{t^{\gamma} w^{\sigma}(t)}{p(t)}.$$

where w *is defined as in* (2.7). *Then*

$$p_* \leq r_* - l^{\gamma} r_*^{1+1/\gamma}, \tag{2.20}$$

and

$$p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.21}$$

Proof. Let y be as above and without loss of generality, we assume that there is $T > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq T$ where T is chosen large enough. From Lemma 2.1, we know that x satisfies (2.5) where x is defined as in (1.2). From Theorem 2.1, we get from (2.8) that

$$-w^{\Delta}(t) \geq Q(t) + \frac{\gamma}{p^{1/\gamma}(t)} (w^{\sigma}(t))^{(\gamma+1)/\gamma}, \qquad \text{for } t \geq T. \tag{2.22}$$

First, we prove (2.20). Integrating (2.22) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$ (see the proof of Theorem 2.2), we obtain

$$(2.23)\qquad w^\sigma(t) \geq \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \gamma \int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{1/\gamma} w^\sigma(s)\Delta s}{p^{1/\gamma}(s)}, \qquad \text{for } t \geq T.$$

It follows from (2.23) that

$$(2.24)\qquad \frac{t^\gamma w^\sigma(t)}{p(t)} \geq \frac{t^\gamma}{p(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \frac{\gamma t^\gamma}{p(t)} \int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{1/\gamma} w^\sigma(s)\Delta s}{p^{1/\gamma}(s)}, \qquad \text{for } t \geq T.$$

Let ε be a sufficiently small positive quantity, then by the definition of p_* and r_* we can pick $T_1 \in [T, \infty)_{\mathbf{T}}$, sufficiently large, so that

$$(2.25)\qquad \frac{t^\gamma}{p(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s \geq p_* - \varepsilon, \qquad \text{and} \qquad \frac{t^\gamma w^\sigma(t)}{p(t)} \geq r_* - \varepsilon, \quad \text{for } t \geq T_1.$$

From (2.24) and (2.25) and using the fact that $p^\Delta \geq 0$, it follows that

$$(2.26)\qquad \begin{aligned} \frac{t^\gamma w^\sigma(t)}{p(t)} &\geq (p_* - \varepsilon) + \gamma \frac{t^\gamma}{p(t)} \int_{\sigma(t)}^{\infty} \frac{s(w^\sigma(s))^{1/\gamma} s^\gamma w^\sigma(s)}{p^{1/\gamma}(s) s^{\gamma+1}} \Delta s \\ &\geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+1/\gamma} \frac{t^\gamma}{p(t)} \int_{\sigma(t)}^{\infty} \frac{\gamma p(s)}{s^{\gamma+1}} \Delta s \\ &\geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+1/\gamma} t^\gamma \int_{\sigma(t)}^{\infty} \frac{\gamma \Delta s}{s^{\gamma+1}}, \qquad \text{for } t \geq T_1. \end{aligned}$$

Using the Pötzsche chain rule ([4, Theorem 1.90]), we see that

$$\left(\frac{-1}{s^\gamma}\right)^\Delta = \gamma \int_0^1 \frac{1}{[s + h\mu(s)]^{\gamma+1}}\, dh \leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh = \frac{\gamma}{s^{\gamma+1}}.$$

This implies that

$$(2.27)\qquad \int_{\sigma(t)}^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s \geq \int_{\sigma(t)}^{\infty} \left(\frac{-1}{s^\gamma}\right)^\Delta \Delta s = \frac{1}{\sigma^\gamma(t)}.$$

Then from (2.26) and (2.27), we have

$$\frac{t^\gamma w^\sigma(t)}{p(t)} \geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+1/\gamma} \left(\frac{t}{\sigma(t)}\right)^\gamma, \qquad \text{for } t \geq T_1.$$

Taking the $\liminf$ of both sides as $t \to \infty$, we have

$$r_* \geq p_* - \varepsilon + (r_* - \varepsilon)^{1+1/\gamma} l^\gamma, \qquad \text{for } t \geq T_1.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$p_* \leq r_* - r_*^{1+1/\gamma} l^\gamma, \tag{2.28}$$

and this completes the proof of (2.20). Next, we prove (2.21). Multiplying both sides (2.22) by $t^{\gamma+1}/p(t)$, and integrating the new inequality from T to t $(t \geq T)$, we get

$$\int_T^t \frac{s^{\gamma+1}}{p(s)} w^\Delta(s)\Delta s \leq -\int_T^t \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{p(s)}\right)^{(\gamma+1)/\gamma} \Delta s.$$

Using integration by parts, we obtain for $t \geq T$ that

$$\begin{aligned} \frac{t^{\gamma+1} w(t)}{p(t)} \leq \frac{T^{\gamma+1} w(T)}{p(T)} &+ \int_T^t \left(\frac{s^{\gamma+1}}{p(s)}\right)^\Delta w^\sigma(s)\Delta s - \int_T^t \frac{s^{\gamma+1} Q(s)\Delta s}{p(s)} \\ &- \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{p(s)}\right)^{(\gamma+1)/\gamma} \Delta s. \end{aligned}$$

By the quotient rule and applying the Pötzsche chain rule, we see that

$$\left(\frac{s^{\gamma+1}}{p(s)}\right)^\Delta = \frac{(s^{\gamma+1})^\Delta}{p^\sigma(s)} - \frac{s^{\gamma+1} p^\Delta(s)}{p(s)p^\sigma(s)} \leq \frac{(\gamma+1)\sigma^\gamma(s)}{p^\sigma(s)} \leq \frac{(\gamma+1)\sigma^\gamma(s)}{p(s)}, \tag{2.29}$$

since $p^\Delta(t) \geq 0$. This leads to

$$\begin{aligned} \frac{t^{\gamma+1} w(t)}{p(t)} \leq \frac{T^{\gamma+1} w(T)}{p(T)} &- \int_T^t \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s \\ &+ \int_T^t (\gamma+1)\left(\frac{\sigma^\gamma(s) w^\sigma(s)}{p(s)}\right)\Delta s \\ &- \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{p(s)}\right)^{(\gamma+1)/\gamma} \Delta s, \qquad \text{for } t \geq T. \end{aligned}$$

Let $\varepsilon > 0$ be given, then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that $\frac{s}{\sigma(s)} > l - \varepsilon$, $s \geq T$. It follows that

$$\sigma(s) \leq Ks, \quad s \geq T \qquad \text{where } K := \frac{1}{l - \varepsilon} > 1.$$

Then we get that

$$\frac{t^{\gamma+1}w(t)}{p(t)} - \frac{T^{\gamma+1}w(T)}{p(T)}$$
$$\leq -\int_T^t \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s$$
$$+ \int_T^t \left\{ (\gamma+1)K^\gamma \frac{s^\gamma w^\sigma(s)}{p(s)} - \gamma\left(\frac{s^\gamma w^\sigma(s)}{p(s)}\right)^{(\gamma+1)/\gamma} \right\}\Delta s, \qquad \text{for } t \geq T.$$

Let $u(s) := s^\gamma w^\sigma(s)/p(s)$, so $u^\lambda(s) = (s^\gamma w^\sigma(s)/p(s))^\lambda$, where $\lambda = \frac{\gamma+1}{\gamma}$. It follows for $t \geq T$ that

$$\frac{t^{\gamma+1}w(t)}{p(t)} \leq \frac{T^{\gamma+1}w(T)}{p(T)} - \int_T^t \frac{s^{\gamma+1}Q(s)\Delta s}{p(s)} + \int_T^t \{(\gamma+1)K^\gamma u(s) - \gamma u^\lambda(s)\}\Delta s.$$

Using the inequality

$$Bu - Au^\lambda \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma},$$

where A, B are constants, we get for $t \geq T$ that

$$\frac{t^{\gamma+1}w(t)}{p(t)} \leq \frac{T^{\gamma+1}w(T)}{p(T)} - \int_T^t \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s$$
$$+ \int_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1)K^\gamma]^{\gamma+1}}{\gamma^\gamma}\Delta s$$
$$\leq \frac{T^{\gamma+1}w(T)}{p(T)} - \int_T^t \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s + K^{\gamma(\gamma+1)}(t-T).$$

It follows from this that

$$\frac{t^\gamma w(t)}{p(t)} \leq \frac{T^{\gamma+1}w(T)}{tp(T)} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}Q(s)\Delta s}{p(s)} + K^{\gamma(\gamma+1)}\left(1 - \frac{T}{t}\right), \qquad \text{for } t \geq T.$$

From (2.8), we see that w is nonincreasing and this implies that $w^\sigma \leq w$, since $\sigma(t) \geq t$. This gives us that

$$\frac{t^\gamma w^\sigma(t)}{p(t)} \leq \frac{T^{\gamma+1}w(T)}{tp(T)} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}Q(s)\Delta s}{p(s)} + K^{\gamma(\gamma+1)}\left(1 - \frac{T}{t}\right), \qquad \text{for } t \geq T.$$

Taking the $\limsup$ of both sides as $t \to \infty$, we obtain

$$R \leq -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\varepsilon)^{\gamma(\gamma+1)}}.$$

Since $\varepsilon > 0$ is arbitrary, we get that $R \le -q_* + (1/l^{\gamma(\gamma+1)})$. Using this and inequality (2.28), we get

$$p_* \le r_* - l^\gamma r_*^{1+1/\gamma} \le r_* \le R \le -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Therefore

$$p_* + q_* \le \frac{1}{l^{\gamma(\gamma+1)}},$$

and this completes the proof of (2.21). The proof is complete.

From Theorem 2.3, we have the following result.

Theorem 2.4. *Assume that* (h_1)–(h_3) *hold and* $p^\Delta \ge 0$. *Furthermore, assume that*

$$p_* > \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}. \tag{2.30}$$

Then every solution of (1.1) *oscillates.*

Proof. Suppose to the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \ge T$ where T is chosen large enough. We consider only this case, because the proof when $y(t) < 0$ is similar. Let w and r_* be as defined in Theorem 2.3. Then from Theorem 2.3, we see that r_* satisfies the inequality

$$p_* \le r_* - l^\gamma r_*^{(\gamma+1)/\gamma}.$$

Using

$$Bu - Au^{(\gamma+1)/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma},$$

we get that

$$p_* \le \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.30). This completes the proof. ■

We also have as a consequence of Theorem 2.3 the following oscillation result.

THEOREM 2.5. *Assume that* (h_1)–(h_3) *hold and* $p^{\Delta} \geq 0$. *Furthermore, assume that*

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.31}$$

Then every solution of (1.1) *oscillates.*

In the following, we give an example to illustrate the results in Theorem 2.4 for a mixed type equation. Note the following facts:

$$\int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} = \infty, \quad \text{if } 0 \leq \nu \leq 1, \qquad \text{and} \qquad \int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} < \infty, \quad \text{if } \nu > 1. \tag{2.32}$$

For more details we refer the reader to [4, Theorem 5.68 and Corollary 5.71].

EXAMPLE 1. *Consider the following second-order neutral dynamic equation*

$$\left[y(t) + \frac{1}{2} y(\tau(t))\right]^{\Delta\Delta} + \frac{\lambda(\sigma(t) - 1)}{t\sigma(t)(t-1)} y(\delta(t)) = 0, \qquad \text{for } t \in [2, \infty)_{\mathbf{T}}, \tag{2.33}$$

where $\mathbf{T}$ *is a time scale,* $\gamma = 1$, $\tau(t) < t$, *and* $\delta(t) > t$, $\tau(t)$ *and* $\delta(t) \in \mathbf{T}$ *and* $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$, *and* $\lambda > 0$ *is a constant. Now* $r(t) = 1/2$, $p(t) = 1$, $f(t,u) = q(t)u$, *where*

$$q(t) = \frac{\lambda(\sigma(t) - 1)}{t\sigma(t)(t-1)}.$$

Take any $T \geq 2$, *and since* $p(t) = 1$, *we have* $P(t,T) = P(t,T) = t - T$. *This gives*

$$\begin{aligned} Q(t) := P(t) \frac{P(t,T)}{P(t,T) + \sigma(t) - t} &= \frac{\lambda(\sigma(t) - 1)}{t\sigma(t)(t-1)} \frac{t - T}{t - T + \sigma(t) - t} \\ &= \frac{\lambda(\sigma(t) - 1)}{2t\sigma(t)(t-1)} \frac{t - T}{\sigma(t) - T}. \end{aligned}$$

It is easy to see that assumptions (h_1)–(h_3) *hold and also* (2.19) *is satisfied, since*

$$\begin{aligned} \int_{t_0}^{\infty} Q(s)\Delta s &= \frac{\lambda}{2} \int_{t_0}^{\infty} \frac{(\sigma(s) - 1)}{s\sigma(s)(s-1)} \frac{s - T}{\sigma(s) - T} \Delta s \\ &\leq \frac{\lambda}{2} \int_2^{\infty} \frac{(\sigma(s) - 1)}{s\sigma(s)(s-1)} \Delta s < \frac{\lambda}{2} \int_2^{\infty} \frac{(\sigma(s) - 1)}{s(\sigma(s) - 1)(s-1)} \Delta s \\ &= \frac{\lambda}{2} \int_2^{\infty} \frac{1}{s(s-1)} \Delta s < \frac{\lambda}{2} \int_2^{\infty} \frac{1}{(s-1)^2} \Delta s < \infty. \end{aligned}$$

To apply Theorem 2.4 *it remains to discuss condition* (2.30). *Note*

$$\begin{aligned}\liminf_{t\to\infty}\frac{t^{\gamma}}{p(t)}\int_{\sigma(t)}^{\infty}Q(s)\Delta s &= \frac{\lambda}{2}\liminf_{t\to\infty}\, t\int_{\sigma(t)}^{\infty}\left(\frac{\lambda(\sigma(s)-1)}{2s\sigma(s)(s-1)}\,\frac{s-T}{\sigma(s)-T}\right)\Delta s\\ &> \frac{\lambda}{2}\liminf_{t\to\infty}\, t\int_{\sigma(t)}^{\infty}\left(\frac{\lambda}{2s\sigma(s)}-\frac{T}{2s\sigma(s)(s-1)}\right)\Delta s\\ &\geq \frac{\lambda}{2}\liminf_{t\to\infty}\, t\int_{\sigma(t)}^{\infty}\left(\frac{-1}{s}\right)^{\Delta}\Delta s = \frac{\lambda}{2}l,\end{aligned}$$

since

$$\begin{aligned}\liminf_{t\to\infty}\, t\int_{\sigma(t)}^{\infty}\frac{1}{s\sigma(s)(s-1)}\Delta s &\geq \liminf_{t\to\infty}\, t\int_{\sigma(t)}^{\infty}\frac{1}{s^2\sigma(s)}\Delta s\\ &\geq \liminf_{t\to\infty}\, t\int_{\sigma(t)}^{\infty}\left(\frac{-1}{2s^2}\right)^{\Delta}\Delta s = \liminf_{t\to\infty}\frac{t}{2\sigma^2(t)} = 0.\end{aligned}$$

Hence, by Theorem 2.4 *every solution of* (2.33) *oscillates if* $\lambda > 1/2l^2$.

2.2. The case when $\delta(t) \leq t$

In this subsection, we establish some sufficient conditions for oscillation of (1.1) when $\delta(t) \leq t$. To prove the main results in this subsection we need the following lemma.

Lemma 2.3. *Assume that* (h_1)–(h_3) *hold. Furthermore assume that*

$$p^{\Delta} \geq 0, \qquad \textit{and} \qquad \int_{t_0}^{\infty}\delta^{\gamma}(s)q(s)[1-r(\delta(s))]^{\gamma}\Delta s = \infty. \tag{2.34}$$

Let y be a nonoscillatory solution of (1.1) *on* $[t_0,\infty)_{\mathbf{T}}$ *and x is defined as in* (1.2). *Then there exists* $T \in [t_0,\infty)_{\mathbf{T}}$, *sufficiently large, so that*

(i) $x(t) > tx^{\Delta}(t)$ *for* $t \in [T,\infty)_{\mathbf{T}}$;

(ii) $x(t)/t$ *is strictly decreasing on* $[T,\infty)_{\mathbf{T}}$.

Proof. Assume that y is a positive solution of (1.1) on $[t_0,\infty)_{\mathbf{T}}$. Pick $t_1 \in [t_0,\infty)_{\mathbf{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ on $[t_1,\infty)_{\mathbf{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form). Since y is a positive solution of (1.1), then from Lemma 2.1, we see that $x(t)$ satisfies (2.5) for $t \geq t_2$ where $t_2 > t_1$ is chosen large enough. Let $U(t) := x(t) - tx^{\Delta}(t)$. This implies that $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t)$, for $t \in [t_2,\infty)_{\mathbf{T}}$.

To determine the sign of $U^{\Delta}(t)$ we need to know the sign of $x^{\Delta\Delta}(t)$. Since (see (2.4) and (2.5)) $(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0$ on $[t_2, \infty)_{\mathbf{T}}$, we have after differentiation that

$$p^{\Delta}(t)(x^{\Delta}(t))^{\gamma} + p^{\sigma}((x^{\Delta}(t))^{\gamma})^{\Delta} < 0, \qquad \text{for } t \geq t_2. \tag{2.35}$$

Using the Pötzsche chain rule ([4, Theorem 1.90]),

$$(f^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [f(t) + \mu(t)hf^{\Delta}(t)]^{\gamma-1} f^{\Delta}(t)dh, \tag{2.36}$$

we have

$$\begin{aligned}((x^{\Delta}(t))^{\gamma})^{\Delta} &= \gamma \int_0^1 [x^{\Delta}(t) + h\mu(t)x^{\Delta\Delta}(t)]^{\gamma-1} dhx^{\Delta\Delta}(t) \\ &= \gamma x^{\Delta\Delta}(t) \int_0^1 [x^{\Delta}(t) + h[x^{\Delta\sigma}(t) - x^{\Delta}(t)]]^{\gamma-1} dh \\ &= \gamma x^{\Delta\Delta}(t) \int_0^1 [hx^{\Delta\sigma}(t) + (1-h)x^{\Delta}(t)]^{\gamma-1} dh.\end{aligned} \tag{2.37}$$

From (2.35), we have that

$$p^{\sigma}((x^{\Delta}(t))^{\gamma})^{\Delta} < -p^{\Delta}(t)(x^{\Delta}(t))^{\gamma} \leq 0, \qquad \text{for } t \geq t_2,$$

since $p^{\Delta}(t) \geq 0$ and $x^{\Delta}(t) > 0$ for $t \geq t_2$. It follows that

$$p^{\sigma}((x^{\Delta}(t))^{\gamma})^{\Delta} < 0, \qquad \text{for } t \geq t_2.$$

This shows, see (2.37), that $x^{\Delta\Delta}(t) < 0$ for $t \geq t_2$, since the integral in (2.37) is positive. This implies that $U(t)$ is strictly increasing on $[t_2, \infty)_{\mathbf{T}}$. To complete the proof, we show that there is $t_4 \in [t_2, \infty)_{\mathbf{T}}$ with $U(t_4) \geq 0$, so since $U(t)$ is strictly increasing, there exists $t_3 \in [t_2, \infty)_{\mathbf{T}}$ with $U(t) > 0$ for $t \geq t_3$. Assume not, then $U(t) < 0$ on $[t_3, \infty)_{\mathbf{T}}$ for any $t_3 \geq t_2$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \qquad t \in [t_3, \infty)_{\mathbf{T}}, \tag{2.38}$$

which implies that $x(t)/t$ is strictly increasing on $[t_3, \infty)_{\mathbf{T}}$. Pick $t_4 \in [t_3, \infty)_{\mathbf{T}}$ so that $\delta(t) \geq \delta(t_3) > 0$, for $t \geq t_4$ (note that $\lim_{t\to\infty} \delta(t) = \infty$). Then $x(\delta(t))/\delta(t) \geq x(\delta(t_3))/\delta(t_3) =: d > 0$, so that $x(\delta(t)) \geq d\delta(t)$ for $t \geq t_4$. Now integrating both sides of the dynamic inequality (2.4) from t_4 to t, we have

$$p(t)(x^{\Delta}(t))^{\gamma} - p(t_4)(x^{\Delta}(t_4))^{\gamma} + \int_{t_4}^{t} P(s)x^{\gamma}(\delta(s))\Delta s \leq 0, \qquad \text{for } t \geq t_4.$$

This implies for $t > t_4$, that

$$p(t_4)(x^{\Delta}(t_4))^{\gamma} \geq \int_{t_4}^{t} P(s)x^{\gamma}(\delta(s))\Delta s \geq d^{\gamma} \int_{t_4}^{t} P(s)\delta^{\gamma}(s)\Delta s. \tag{2.39}$$

Letting $t \to \infty$, we obtain a contradiction to assumption (2.34). Hence there exists $t_3 \in [t_2, \infty)_{\mathbf{T}}$ such that $U(t) > 0$ on $[t_3, \infty)_{\mathbf{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \qquad t \in [t_3, \infty)_{\mathbf{T}}, \tag{2.40}$$

and we have that $x(t)/t$ is strictly decreasing on $[t_3, \infty)_{\mathbf{T}}$. This completes the proof of the Lemma.

Theorem 2.6. *Assume that* (h_1)*–*(h_3) *and* (2.34) *hold. Let* y *be a nonoscillatory solution of* (1.1) *and* x *and* w *are defined as in* (1.2) *and* (2.7). *Then* $w(t) > 0$, *for* $t \geq T$ *(here* T *is as in Lemma* 2.3*) and*

$$w^{\Delta}(t) + A(t) + \gamma \frac{1}{p^{1/\gamma}(t)}(w^{\sigma})^{1+1/\gamma}(t) \leq 0, \qquad \textit{for } t \in [T, \infty)_{\mathbf{T}}. \tag{2.41}$$

where

$$A(t) := P(t)\left(\frac{\delta(t)}{\sigma(t)}\right)^{\gamma} \qquad \textit{and} \qquad P(t) = q(t)(1 - r(\delta(t)))^{\gamma}.$$

Proof. Let y be as above and without loss of generality we assume that there is $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1$. From Lemma 2.3, since $p^{\Delta}(t) \geq 0$, we see that (see the proof of Lemma 2.3) there exists $T > t_1$ such that

$$x(t) > 0, \qquad x^{\Delta}(t) > 0, \qquad \text{and} \qquad x^{\Delta\Delta}(t) \leq 0, \quad \text{for } t \geq T.$$

From the definition of w, by quotient rule [4, Theorem 1.20] and as in the proof of Theorem 2.1, we get

$$w^{\Delta}(t) \leq -P(t)\left(\frac{x^{\delta}(t)}{x^{\sigma}(t)}\right)^{\gamma} - \gamma \frac{1}{p^{1/\gamma}(t)}(w^{\sigma}(t))^{1+1/\gamma}, \qquad \text{for } t \geq T. \tag{2.42}$$

Now, we consider the coefficient of $P(t)$ in (2.42). From Lemma 2.3, since $(x(t)/t)$ is decreasing and $\delta(t) \leq t \leq \sigma(t)$, we have

$$\frac{x^{\delta}(t)}{x^{\sigma}(t)} \geq \frac{\delta(t)}{\sigma(t)}. \tag{2.43}$$

Substituting (2.43) into (2.42), we have the inequality (2.41) and this completes the proof.

The proof of the following theorem is similar to the proof of Theorem 2.2 (use inequality (2.41)).

THEOREM 2.7. *Assume that* (h_1)–(h_2) *and* (2.34) *hold. Furthermore, assume that*

$$\int_{t_0}^{\infty} A(s)\Delta s = \infty.$$

Then every solution of (1.1) *oscillates.*

In the following we consider the case when

$$\int_{t_0}^{\infty} A(s)\Delta s < \infty. \tag{2.44}$$

We will use the following notations:

$$A_* := \liminf_{t\to\infty} \frac{t^\gamma}{p(t)} \int_{\sigma(t)}^{\infty} A(s)\Delta s, \qquad B_* := \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{p(s)} A(s)\Delta s.$$

THEOREM 2.8. *Assume that* (h_1)–(h_2) *and* (2.34) *hold. Furthermore, assume that*

$$A_* > \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}. \tag{2.45}$$

Then every solution of (1.1) *oscillates.*

PROOF. The proof is similar to the proof of Theorem 2.4, by replacing $Q(t)$ by $A(t)$ and so is omitted.

Also we can obtain the following result.

THEOREM 2.9. *Assume that* (h_1)–(h_3) *and* (2.34) *hold. Furthermore, assume that*

$$A_* + B_* > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.46}$$

Then every solution of (1.1) *oscillates.*

In the following, we give an example to illustrate the results in Theorem 2.8. To obtain the conditions for oscillation we will use the facts in (2.32).

EXAMPLE 2. *Consider the following second-order nonlinear neutral delay dynamic equation*

$$(2.47)\qquad \left(t^{\gamma-1}\left(\left(y(t)+\frac{\delta^{-1}(t)-1}{\delta^{-1}(t)}y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+\frac{\beta\sigma^{2\gamma-2}}{\delta^{\gamma}(t)}y^{\gamma}(\delta(t))=0,$$

$$t\in[1,\infty)_{\mathbf{T}},$$

where $\beta>0$, *and* $\gamma-2>0$ *and is an odd positive integer,* $\tau(t)$, $\delta(t)\in\mathbf{T}$ *with* $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty$, *and* $\tau(t)\le t$, $\delta(t)\le t$. *We assume that* $\delta^{-1}(t)$ *(the inverse of the function* $\delta(t)$*) exists, and* $\mathbf{T}$ *is a time scale such*

$$(2.48)\qquad \int_{t_0}^{\infty}\frac{\sigma^{\gamma}}{s^{\gamma+2}}\Delta s<\infty.$$

Here

$$p(t)=t^{\gamma-1},\qquad r(t)=\frac{\delta^{-1}(t)-1}{\delta^{-1}(t)}=1-\frac{1}{\delta^{-1}(t)},\qquad \text{and}\qquad q(t)=\frac{\beta\sigma^{2\gamma-2}}{\delta^{\gamma}(t)}.$$

It is easy to see that the assumptions (h_1)–(h_3) *hold. To apply Theorem* 2.8 *we must show that conditions* (2.34), (2.44) *and* (2.45) *are satisfied. Note* (2.34) *is satisfied, since*

$$\begin{aligned}\int_{t_0}^{\infty}\delta^{\gamma}(s)q(s)[1-r(\delta(s))]^{\gamma}\Delta s&=\beta\int_{1}^{\infty}\delta^{\gamma}(s)\left(\frac{1}{s}\right)^{\gamma}\frac{\sigma^{2\gamma-2}}{\delta^{\gamma}(s)}\Delta s\\&=\beta\int_{1}^{\infty}\left(\frac{\sigma(s)}{s}\right)^{\gamma}\sigma^{\gamma-2}(s)\Delta s\\&\ge\beta\int_{1}^{\infty}\sigma^{\gamma-2}(s)\Delta s\ge\beta\int_{1}^{\infty}s^{\gamma-2}\Delta s=\infty.\end{aligned}$$

Now, we show that (2.44) *holds. To see this note by* (2.48) *that*

$$\begin{aligned}\int_{t_0}^{\infty}A(s)\Delta s&=\int_{t_0}^{\infty}\left(\frac{1}{s}\right)^{\gamma}\frac{\beta\sigma^{2\gamma-2}}{\delta^{\gamma}(s)}\left(\frac{\delta(s)}{\sigma(s)}\right)^{\gamma}\Delta s\\&=\beta\int_{t_0}^{\infty}\frac{\sigma^{\gamma-2}}{s^{\gamma}}\Delta s=\beta\int_{t_0}^{\infty}\frac{\sigma^{\gamma}}{s^{\gamma}\sigma^{2}}\Delta s\\&\le\beta\int_{t_0}^{\infty}\frac{\sigma^{\gamma}}{s^{\gamma+2}}\Delta s<\infty.\end{aligned}$$

Finally we discuss (2.45). *Note*

$$\liminf_{t\to\infty}\frac{1}{t}\int_T^t\frac{s^{\gamma+1}}{p(s)}A(s)\Delta s=\liminf_{t\to\infty}\frac{1}{t}\int_1^t\frac{s^{\gamma+1}}{s^{\gamma-1}}\left(\frac{1}{s}\right)^{\gamma}\frac{\beta\sigma^{2\gamma-2}}{\delta^{\gamma}(s)}\left(\frac{\delta(s)}{\sigma(s)}\right)^{\gamma}\Delta s$$
$$=\beta\liminf_{t\to\infty}\frac{1}{t}\int_1^t\frac{s\sigma^{\gamma-2}}{s^{\gamma-1}}\Delta s=\beta\liminf_{t\to\infty}\frac{1}{t}\int_1^t\frac{\sigma^{\gamma-2}}{s^{\gamma-2}}\Delta s$$
$$>\beta\liminf_{t\to\infty}\frac{1}{t}\int_1^t\Delta s=\beta.$$

Then by Theorem 2.8, *every solution of* (2.47) *oscillates if* $\beta>\gamma^{\gamma}/l^{\gamma^2}(\gamma+1)^{\gamma+1}$.

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