Hiroshima Math. J.
42 (2012), 37–75

Paperfolding sequences, paperfolding curves and local isomorphism

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(Received June 28, 2010)

Abstract. For each integer n, an n-folding curve is obtained by folding n times a strip of paper in two, possibly up or down, and unfolding it with right angles. Generalizing the usual notion of infinite folding curve, we define complete folding curves as the curves without endpoint which are unions of increasing sequences of n-folding curves for n integer.

We prove that there exists a standard way to extend any complete folding curve into a covering of $\mathbf{R}^2$ by disjoint such curves, which satisfies the local isomorphism property introduced to investigate aperiodic tiling systems. This covering contains at most six curves.

The infinite folding sequences (resp. curves) usually considered are sequences $(a_k)_{k \in \mathbf{N}^*} \subset \{+1, -1\}$ (resp. infinite curves with one endpoint) obtained as direct limits of n-folding sequences (resp. curves) for $n \in \mathbf{N}$. It is well known (see [3] and [4]) that paperfolding curves are self-avoiding and that, in some cases, including the Heighway Dragon curve, a small number of copies of the same infinite folding curve can be used to cover $\mathbf{R}^2$ without overlapping. On the other hand, the last property is not true in some other cases.

In the present paper, we define complete folding sequences (resp. curves) as the sequences $(a_k)_{k \in \mathbf{Z}} \subset \{+1, -1\}$ (resp. the infinite curves without endpoint) which are direct limits of n-folding sequences (resp. curves) for $n \in \mathbf{N}$. Any infinite folding sequence (resp. curve) in the classical sense can be extended into a complete folding sequence (resp. curve). On the other hand, most of the complete folding sequences (resp. curves) cannot be obtained in that way.

We prove that any complete folding curve, and therefore any infinite folding curve, can be extended in an essentially unique way into a covering of $\mathbf{R}^2$ by disjoint complete folding curves which satisfies the local isomorphism property. We show that a covering obtained from an infinite folding curve can contain complete folding curves which are not extensions of infinite folding curves.

2010 *Mathematics Subject Classification*. Primary 05B45; Secondary 52C20, 52C23.

Key words and phrases. Paperfolding sequence, paperfolding curve, tiling, local isomorphism, aperiodic.

One important argument in the proofs is the derivation of paperfolding curves, which is investigated in Section 2. Another one is the local isomorphism property for complete folding sequences (cf. Section 1) and for coverings of $\mathbf{R}^2$ by sets of disjoint complete folding curves (cf. Section 3). The local isomorphism property was originally used to investigate aperiodic tiling systems. Actually, we have an interpretation of complete folding sequences as tilings of $\mathbf{R}$, and an interpretation of coverings of $\mathbf{R}^2$ by disjoint complete folding curves as tilings of $\mathbf{R}^2$.

1. Paperfolding sequences

The notions usually considered (see for instance [5]), and which we define first, are those of n-folding sequence (sequence obtained by folding n times a strip of paper in two), and ∞-folding sequence (sequence indexed by $\mathbf{N}^* = \mathbf{N} - \{0\}$ which is obtained as a direct limit of n-folding sequences for $n \in \mathbf{N}$).

Then we introduce complete folding sequences, which are sequences indexed by $\mathbf{Z}$, also obtained as direct limits of n-folding sequences for $n \in \mathbf{N}$. We describe the finite subwords of each such sequence. Using this description, we show that complete folding sequences satisfy properties similar to those of aperiodic tiling systems: they form a class defined by a set of local rules, neither of them is periodic, but all of them satisfy the local isomorphism property introduced for tilings. It follows that, for each such sequence, there exist 2^ω isomorphism classes of sequences which are locally isomorphic to it.

Definitions. For each $n \in \mathbf{N}$ and each sequence $S = (a_1, \dots, a_n) \subset \{+1, -1\}$, we write $|S| = n$ and $\overline{S} = (-a_n, \dots, -a_1)$. We say that a sequence $(a_1, \dots, a_n)$ is a *subword* of a sequence $(b_1, \dots, b_p)$ or $(b_k)_{k \in \mathbf{N}^*}$ or $(b_k)_{k \in \mathbf{Z}}$ if there exists h such that $a_k = b_{k+h}$ for $1 \leq k \leq n$.

Definition. For each $n \in \mathbf{N}$, an *n-folding sequence* is a sequence $(a_1, \dots, a_{2^n - 1}) \subset \{+1, -1\}$ obtained by folding n times a strip of paper in two, with each folding being done independently up or down, unfolding it, and writing $a_k = +1$ (resp. $a_k = -1$) for each $k \in \{1, \dots, 2^n - 1\}$ such that the k-th fold from the left has the shape of a $\vee$ (resp. $\wedge$) (we obtain the empty sequence for $n = 0$).

Properties. The following properties are true for each $n \in \mathbf{N}$:
1) If S is an n-folding sequence, then $\overline{S}$ is also an n-folding sequence.
2) The $(n+1)$-folding sequences are the sequences $(\overline{S}, +1, S)$ and the sequences $(\overline{S}, -1, S)$, where S is an n-folding sequence.
3) There exist 2^n n-folding sequences (proof by induction on n using 2)).

4) Any sequence $(a_1, \dots, a_{2^{n+1}-1})$ is an $(n+1)$-folding sequence if and only if $(a_{2k})_{1 \le k \le 2^n - 1}$ is an n-folding sequence and $a_{1+2k} = (-1)^k a_1$ for $0 \le k \le 2^n - 1$.
5) If $n \ge 2$ and if $(a_1, \dots, a_{2^n - 1})$ is an n-folding sequence, then $a_{2^r(1+2k)} = (-1)^k a_{2^r}$ for $0 \le r \le n-2$ and $0 \le k \le 2^{n-r-1} - 1$ (proof by induction on n using 4)).

Definition. An ∞-*folding sequence* is a sequence $(a_n)_{n \in \mathbf{N}^*}$ such that $(a_1, \dots, a_{2^n-1})$ is an n-folding sequence for each $n \in \mathbf{N}^*$.

Definition. A *finite folding sequence* is a subword of an n-folding sequence for an integer n.

Examples. The sequence $(+1, +1, +1)$ is a finite folding sequence since it is a subword of the 3-folding sequence $(-1, +1, +1, +1, -1, -1, +1)$. On the other hand, $(+1, +1, +1)$ is not a 2-folding sequence and $(+1, +1, +1, +1)$ is not a finite folding sequence.

Definition. A *complete folding sequence* is a sequence $(a_k)_{k \in \mathbf{Z}} \subset \{+1, -1\}$ such that its finite subwords are finite folding sequences.

Examples. For each ∞-folding sequence $S = (a_n)_{n \in \mathbf{N}^*}$, write $\overline{S} = (-a_{-n})_{n \in -\mathbf{N}^*}$. Then $(\overline{S}, +1, S)$ and $(\overline{S}, -1, S)$ are complete folding sequences since $(-a_{2^n-1}, \dots, -a_1, +1, a_1, \dots, a_{2^n-1})$ and $(-a_{2^n-1}, \dots, -a_1, -1, a_1, \dots, a_{2^n-1})$ are $(n+1)$-folding sequences for each $n \in \mathbf{N}$. In Section 3, we give examples of complete folding sequences which are not obtained in that way.

It follows from the property 5) above that, for each complete folding sequence $(a_h)_{h \in \mathbf{Z}}$ and each $n \in \mathbf{N}$, there exists $k \in \mathbf{Z}$ such that $a_{k + l \cdot 2^{n+1}} = (-1)^l a_k$ for each $l \in \mathbf{Z}$. Moreover we have:

Proposition 1.1. *Consider a sequence* $S = (a_h)_{h \in \mathbf{Z}} \subset \{+1, -1\}$. *For each* $n \in \mathbf{N}$, *suppose that there exists* $h_n \in \mathbf{Z}$ *such that* $a_{h_n + k \cdot 2^{n+1}} = (-1)^k a_{h_n}$ *for each* $k \in \mathbf{Z}$, *and consider* $E_n = h_n + 2^n \mathbf{Z}$ *and* $F_n = h_n + 2^{n+1}\mathbf{Z}$. *Then, for each* $n \in \mathbf{N}$:
1) $\mathbf{Z} - E_n = \{h \in \mathbf{Z} \mid a_{h + k \cdot 2^{n+1}} = a_h$ *for each* $k \in \mathbf{Z}\}$ *and* $\mathbf{Z} - E_n$ *is the disjoint union of* $F_0, \dots, F_{n-1}$;
2) *for each* $h \in E_n$, $(a_{h - 2^n + 1}, \dots, a_{h + 2^n - 1})$ *is an* $(n+1)$-*folding sequence;*
3) *for each* $h \in \mathbf{Z}$, *if* $(a_{h - 2^{n+1} + 1}, \dots, a_{h + 2^{n+1} - 1})$ *is an* $(n+2)$-*folding sequence, then* $h \in E_n$.

Proof. It follows from the definition of the integers h_n that the sets F_n are disjoint. For each $n \in \mathbf{N}$, we have $E_n = \mathbf{Z} - (F_0 \cup \dots \cup F_{n-1})$ since $\mathbf{Z} - (F_0 \cup \dots \cup F_{n-1})$ is of the form $h + 2^n \mathbf{Z}$ and h_n does not belong to $F_0 \cup \dots \cup F_{n-1}$.

For each $n \in \mathbf{N}$, there exists no $h \in E_n$ such that $a_{h+k\cdot 2^{n+1}} = a_h$ for each $k \in \mathbf{Z}$, since we have $E_n = (h_n + 2^{n+1}\mathbf{Z}) \cup (h_{n+1} + 2^{n+1}\mathbf{Z})$, $a_{h_n+2^{n+1}} = -a_{h_n}$ and $a_{h_{n+1}+2^{n+2}} = -a_{h_{n+1}}$. On the other hand, we have $a_{h_m+k\cdot 2^{m+2}} = a_{h_m}$ for $0 \leq m \leq n-1$ and $k \in \mathbf{Z}$, and therefore $a_{h+k\cdot 2^{n+1}} = a_h$ for $h \in F_0 \cup \cdots \cup F_{n-1}$ and $k \in \mathbf{Z}$, which completes the proof of 1).

We show 2) by induction on n. The case $n = 0$ is clear. If 2) is true for n, then, for each $h \in E_{n+1}$, the induction hypothesis applied to $(a_{h+2k})_{k\in\mathbf{Z}}$ implies that $(a_{h-2^{n+1}+2k})_{1\leq k\leq 2^{n+1}-1}$ is an $(n+1)$-folding sequence; it follows that $(a_{h-2^{n+1}+1}, \dots, a_{h+2^{n+1}-1})$ is an $(n+2)$-folding sequence, since $a_{h-2^{n+1}+1+2k} = (-1)^k a_{h-2^{n+1}+1}$ for $0 \leq k \leq 2^{n+1}-1$.

Concerning 3), we observe that, for each $h \in \mathbf{Z}$, if $(a_{h-2^{n+1}+1}, \dots, a_{h+2^{n+1}-1})$ is an $(n+2)$-folding sequence, then $a_{h+2^n} = -a_{h-2^n}$. According to 1), it follows $h - 2^n \in E_n$, and therefore $h \in E_n$. ■

Corollary 1.2. *Any sequence $(a_h)_{h\in\mathbf{Z}} \subset \{+1, -1\}$ is a complete folding sequence if and only if, for each $n \in \mathbf{N}$, there exists $h_n \in \mathbf{Z}$ such that $a_{h_n+k\cdot 2^{n+1}} = (-1)^k a_{h_n}$ for each $k \in \mathbf{Z}$.*

For each complete folding sequence S and each $n \in \mathbf{N}$, the sets E_n and F_n of Proposition 1.1 do not depend on the choice of h_n. We denote them by $E_n(S)$ and $F_n(S)$. We write E_n and F_n instead of $E_n(S)$ and $F_n(S)$ if it creates no ambiguity.

Corollary 1.3. *Any complete folding sequence is nonperiodic.*

Proof. Let $S = (a_h)_{h\in\mathbf{Z}}$ be such a sequence, and let r be an integer such that $a_{h+r} = a_h$ for each $h \in \mathbf{Z}$. For each $n \in \mathbf{N}$, it follows from 1) of Proposition 1.1 that $r + (\mathbf{Z} - E_n) = \mathbf{Z} - E_n$, whence $r + E_n = E_n$ and $r \in 2^n\mathbf{Z}$. Consequently, we have $r = 0$. ■

Now, for each complete folding sequence $S = (a_h)_{h\in\mathbf{Z}}$, we describe the finite subwords of S and we count those which have a given length.

Lemma 1.4. *For each $n \in \mathbf{N}$ and for any $r, s \in \mathbf{Z}$, we have $r - s \in 2^{n+1}\mathbf{Z}$ if $(a_{r+1}, \dots, a_{r+t}) = (a_{s+1}, \dots, a_{s+t})$ for $t = \sup(2^n, 7)$.*

Proof. If $r - s \notin 2\mathbf{Z}$, then we have for instance $r \in E_1$ and $s \in F_0$. It follows $a_{r+1} = -a_{r+3} = a_{r+5} = -a_{r+7}$ since $r + 1 \in F_0$. Moreover, we have $a_{s+5} = -a_{s+1}$ if $s + 1 \in F_1$, and $a_{s+7} = -a_{s+3}$ if $s + 3 \in F_1$. One of these two possibilities is necessarily realized since $s + 1 \in E_1$, which contradicts $(a_{r+1}, \dots, a_{r+7}) = (a_{s+1}, \dots, a_{s+7})$.

If $r - s \in 2^k\mathbf{Z} - 2^{k+1}\mathbf{Z}$ with $1 \leq k \leq n$, then we consider $h \in \{1, \dots, 2^k\}$ such that $r + h \in F_{k-1}$. We have $a_{r+h+m\cdot 2^k} = (-1)^m a_{r+h}$ for each $m \in \mathbf{Z}$, and in particular $a_{s+h} = -a_{r+h}$, which contradicts $(a_{r+1}, \dots, a_{r+t}) = (a_{s+1}, \dots, a_{s+t})$ since $1 \leq h \leq 2^k \leq t$. ■

Proposition 1.5. *Consider $n \in \mathbf{N}$ and write $T = (a_{h+1}, \dots, a_{h+2^n-1})$ with $h \in E_n$. Then any sequence of length $\leq 2^{n+1} - 1$ is a subword of S if and only if there exist $\zeta, \eta \in \{-1, +1\}$ such that it can be written in one of the forms:*
(1) (T_1, ζ, T_2) *with T_1 final segment of T and T_2 initial segment of $\overline{T}$;*
(2) (T_1, ζ, T_2) *with T_1 final segment of $\overline{T}$ and T_2 initial segment of T;*
(3) $(T_1, \zeta, \overline{T}, \eta, T_2)$ *with T_1 final segment and T_2 initial segment of T;*
(4) $(T_1, \zeta, T, \eta, T_2)$ *with T_1 final segment and T_2 initial segment of $\overline{T}$.*
If $\sup(2^n, 7) \leq t \leq 2^{n+1} - 1$, then any subword of length t of S can be written in exactly one way in one of the forms (1), (2), (3), (4).

Proof. We can suppose $h \in F_n$ since T and $\overline{T}$ play symmetric roles in the Proposition. Then we have $(a_{k+1}, \dots, a_{k+2^n-1}) = T$ for $k \in F_n$ and $(a_{k+1}, \dots, a_{k+2^n-1}) = \overline{T}$ for $k \in E_{n+1}$. It follows that each subword of S of length $\leq 2^{n+1} - 1$ can be written in one of the forms (1), (2), (3), (4).

Now, we are going to prove that each sequence of one of these forms can be expressed as a subword of S in such a way that the part T_1 is associated to the final segment of a sequence $(a_{k+1}, \dots, a_{k+2^n-1})$ with $k \in E_n$.

First we show this property for the sequences of the form (1) or (3). It suffices to prove that, for any $\zeta, \eta \in \{-1, +1\}$, there exists $k \in F_n$ such that $a_{k-2^n} = \zeta$ and $a_k = \eta$, since these two equalities imply $(a_{k-2^{n+1}+1}, \dots, a_{k+2^n-1}) = (T, \zeta, \overline{T}, \eta, T)$. We consider $l \in F_n$ such that $a_l = \eta$. We have $a_{l+r \cdot 2^{n+2}} = \eta$ for each $r \in \mathbf{Z}$. Moreover, $\{l + r \cdot 2^{n+2} - 2^n \mid r \in \mathbf{Z}\}$ is equal to F_{n+1} or E_{n+2}. In both cases, there exists $r \in \mathbf{Z}$ such that $a_{l+r \cdot 2^{n+2} - 2^n} = \zeta$, and it suffices to take $k = l + r \cdot 2^{n+2}$ for such an r.

Now we show the same property for the sequences of the form (2) or (4). It suffices to prove that, for any $\zeta, \eta \in \{-1, +1\}$, there exists $k \in F_n$ such that $a_k = \zeta$ and $a_{k+2^n} = \eta$, since these two equalities imply $(a_{k-2^n+1}, \dots, a_{k+2^{n+1}-1}) = (\overline{T}, \zeta, T, \eta, \overline{T})$. We consider $l \in F_n$ such that $a_l = \zeta$. We have $a_{l+r \cdot 2^{n+2}} = \zeta$ for each $r \in \mathbf{Z}$. Moreover, $\{l + r \cdot 2^{n+2} + 2^n \mid r \in \mathbf{Z}\}$ is equal to F_{n+1} or E_{n+2}. In both cases, there exists $r \in \mathbf{Z}$ such that $a_{l+r \cdot 2^{n+2} + 2^n} = \eta$, and it suffices to take $k = l + r \cdot 2^{n+2}$ for such an r.

Now, suppose that two expressions of the forms (1), (2), (3), (4) give the same sequence of length t with $\sup(2^n, 7) \leq t \leq 2^{n+1} - 1$. Consider two sequences $(a_{r+1}, \dots, a_{r+t})$ and $(a_{s+1}, \dots, a_{s+t})$ which realize these expressions in such a way that, in each of them, the part T_1 of the expression is associated to a final segment of a sequence $(a_{k+1}, \dots, a_{k+2^n-1})$ with $k \in E_n$, while the part T_2 is associated to an initial segment of a sequence $(a_{l+1}, \dots, a_{l+2^n-1})$ with $l = k + 2^n$ or $l = k + 2^{n+1}$. Then, by Lemma 1.4, the equality $(a_{r+1}, \dots, a_{r+t})$

$= (a_{s+1}, \ldots, a_{s+t})$ implies $r - s \in 2^{n+1}\mathbf{Z}$. It follows that the two expressions are equal. ■

Corollary 1.6. *Any finite folding sequence U is a subword of S if and only if $\overline{U}$ is a subword of S.*

Proof. For each sequence $T = (a_{h+1}, \ldots, a_{h+2^n-1})$ with $n \in \mathbf{N}$ and $h \in E_n$, the sequence U is of the form (1) (resp. (2), (3), (4)) relative to T if and only if $\overline{U}$ is of the form (2) (resp. (1), (4), (3)) relative to T. ■

It follows from the Corollary below that, for each integer $n \geq 3$, each complete folding sequence has exactly 8 subwords which are n-folding sequences:

Corollary 1.7. *Consider $n \in \mathbf{N}$ and write $T = (a_{h+1}, \ldots, a_{h+2^n-1})$ with $h \in E_n$. Then any $(n+2)$-folding sequence is a subword of S if and only if it can be written in the form $(\overline{T}, -\zeta, T, \eta, \overline{T}, \zeta, T)$ or $(T, -\zeta, \overline{T}, \eta, T, \zeta, \overline{T})$ with $\zeta, \eta \in \{+1, -1\}$.*

Proof. For each $k \in \mathbf{Z}$, if $(a_{k-2^{n+1}+1}, \ldots, a_{k+2^{n+1}-1})$ is an $(n+2)$-folding sequence, then $k \in E_n$ by 3) of Proposition 1.1. Consequently, we have $(a_{k+1}, \ldots, a_{k+2^n-1}) = T$ or $(a_{k+1}, \ldots, a_{k+2^n-1}) = \overline{T}$, and $(a_{k-2^{n+1}+1}, \ldots, a_{k+2^{n+1}-1})$ is of the required form.

In order to prove that each sequence of that form is a subword of S, we consider $k \in E_{n+1}$ and we write $U = (a_{k+1}, \ldots, a_{k+2^{n+1}-1})$. We have $U = (\overline{T}, \varepsilon, T)$ or $U = (T, \varepsilon, \overline{T})$ with $\varepsilon = \mp 1$. Here we only consider the first case; the second one can be treated in the same way since T and $\overline{T}$ play symmetric roles in the Corollary.

We apply Proposition 1.5 for $n+1$ instead of n, and we consider the forms (1), (2), (3), (4) relative to U. For any $\zeta, \eta \in \{+1, -1\}$, the sequence $(\overline{T}, -\zeta, T, \eta, \overline{T}, \zeta, T)$ is a subword of S because it is equal to $(U, \eta, \overline{U})$ or to $(\overline{U}, \eta, U)$, and therefore of the form (1) or (2) relative to U. The sequence $(T, -\zeta, \overline{T}, \eta, T, \zeta, \overline{T})$ is also a subword of S because it is of the form $(T, \alpha, \overline{U}, \beta, \overline{T})$ or $(T, \alpha, U, \beta, \overline{T})$ with $\alpha, \beta \in \{+1, -1\}$, and therefore of the form (3) or (4) relative to U. ■

The following result generalizes [1, Th., p. 27] to complete folding sequences:

Theorem 1.8. *The sequence S has $4t$ subwords of length t for each integer $t \geq 7$ and 2, 4, 8, 12, 18, 23 subwords of length $t = 1, 2, 3, 4, 5, 6$.*

Proof. The proof of the Theorem for $t = 1, 2, 3, 4, 5, 6$ is based on Proposition 1.5. We leave it to the reader.

For $t \geq 7$, we consider the integer $n \geq 2$ such that $2^n \leq t \leq 2^{n+1} - 1$, and we write $T = (a_{h+1}, \ldots, a_{h+2^n-1})$ with $h \in E_n$. By Proposition 1.5, it suffices to count the subwords of length t of S which are in each of the forms (1), (2), (3), (4) relative to T.

Each of the forms (1), (2) gives $|T_1| + |T_2| = t - 1$, and therefore $|T_1| \geq (t-1) - (2^n - 1) = t - 2^n$. As $|T_1| \leq 2^n - 1$, we have $(2^n - 1) - (t - 2^n) + 1 = 2^{n+1} - t$ possible values for $|T_1|$. Consequently, there exist $4(2^{n+1} - t)$ sequences associated to these two forms, since there are 2 possible values for ζ.

Each of the forms (3), (4) gives $|T_1| + |T_2| = t - 2^n - 1$, and therefore $|T_1| \leq t - 2^n - 1$. We have $t - 2^n$ possible values for $|T_1|$. Consequently, there exist $8(t - 2^n)$ sequences associated to these two forms, since there are 4 possible values for (ζ, η). Now, the total number of subwords of length t in S is $4(2^{n+1} - t) + 8(t - 2^n) = 4t$. ∎

For each sequence $(a_h)_{h \in \mathbf{Z}} \subset \{+1, -1\}$, we define a tiling of $\mathbf{R}$ as follows: the tiles are the intervals $[k, k+1]$ for $k \in \mathbf{Z}$, where the "colour" of the endpoint k (resp. $k+1$) is the sign of a_k (resp. a_{k+1}). Each tile is of one of the forms $[+,+]$, $[+,-]$, $[-,+]$, $[-,-]$, and each pair of consecutive tiles is of one of the forms $([+,+],[+,+])$, $([+,+],[+,-])$, $([+,-],[-,+])$, $([+,-],[-,-])$, $([-,+],[+,+])$, $([-,+],[+,-])$, $([-,-],[-,+])$, $([-,-],[-,-])$.

Concerning the theory of tilings, the reader is referred to [7], which presents classical results and gives generalizations based on mathematical logic. Two tilings of $\mathbf{R}^n$ are said to be *isomorphic* if they are equivalent up to translation, and *locally isomorphic* if they contain the same bounded sets of tiles modulo translations.

We say that two sequences $(a_h)_{h \in \mathbf{Z}}, (b_h)_{h \in \mathbf{Z}} \subset \{+1, -1\}$ are *isomorphic* (resp. *locally isomorphic*) if they are equivalent up to translation (resp. they have the same finite subwords). This property is true if and only if the associated tilings are isomorphic (resp. locally isomorphic). It follows from the definitions that any sequence $(a_h)_{h \in \mathbf{Z}} \subset \{+1, -1\}$ is a complete folding sequence if it is locally isomorphic to such a sequence.

Corollary 1.9. *Any complete folding sequence $S = (a_h)_{h \in \mathbf{Z}}$ is locally isomorphic to $\overline{S} = (-a_{-h})_{h \in \mathbf{Z}}$, but not locally isomorphic to $-S = (-a_h)_{h \in \mathbf{Z}}$.*

Proof. The first statement is a consequence of Corollary 1.6 since each finite folding sequence T is a subword of S if and only if $\overline{T}$ is a subword of $\overline{S}$.

In order to prove the second statement, we consider $T = (a_{h+1}, a_{h+2}, a_{h+3})$ with $h \in E_2$. We have $-T \neq \overline{T}$ since $a_{h+3} = -a_{h+1}$. By Corollary 1.7, the 4-folding sequences which are subwords of S are the sequences $(\overline{T}, -\zeta, T, \eta, \overline{T}, \zeta, T)$ and $(T, -\zeta, \overline{T}, \eta, T, \zeta, \overline{T})$ for $\zeta, \eta \in \{+1, -1\}$, while the 4-folding sequences which are subwords of $-S$ are the sequences $(-\overline{T}, -\zeta, -T, \eta, -\overline{T}, \zeta, -T)$ and $(-T, -\zeta, -\overline{T}, \eta, -T, \zeta, -\overline{T})$ for $\zeta, \eta \in \{+1, -1\}$. Consequently, S and $-S$ have no 4-folding sequence in common. ■

REMARK. For each ∞-folding sequence S, it follows from Corollary 1.9 that $T = (\overline{S}, +1, S)$ and $U = (\overline{S}, -1, S)$ are locally isomorphic, since $U = \overline{T}$.

We say that a tiling $\mathscr{T}$ of $\mathbf{R}^n$ *satisfies the local isomorphism property* if, for each bounded set of tiles $\mathscr{F} \subset \mathscr{T}$, there exists $r \in \mathbf{R}_*^+$ such that each ball of radius r in $\mathbf{R}^n$ contains the image of $\mathscr{F}$ under a translation. Then any tiling $\mathscr{U}$ is locally isomorphic to $\mathscr{T}$ provided that each bounded set of tiles contained in $\mathscr{U}$ is the image under a translation of a set of tiles contained in $\mathscr{T}$.

We say that a sequence $(a_h)_{h \in \mathbf{Z}} \subset \{+1, -1\}$ *satisfies the local isomorphism property* if the associated tiling satisfies the local isomorphism property.

Like Robinson tilings and Penrose tilings, complete folding sequences are *aperiodic* in the following sense:
1) they form a class defined by a set of rules which can be expressed by first-order sentences (for each $n \in \mathbf{N}$, we write a sentence which says that each subword of length 2^n of the sequence considered is a subword of an $(n+1)$-folding sequence);
2) neither of them is periodic, but all of them satisfy the local isomorphism property.

The second statement of 2) follows from the Theorem below:

THEOREM 1.10. *Let $S = (a_h)_{h \in \mathbf{Z}}$ be a complete folding sequence, let T be a finite subword of S, and let r be an integer such that $|T| \leq 2^r$. Then T is a subword of $(a_{h+1}, \dots, a_{h+10 \cdot 2^r - 2})$ for each $h \in \mathbf{Z}$.*

PROOF. There exists $k \in E_r$ such that T is a subword of $(a_{k-2^r+1}, \dots, a_{k+2^r-1})$. We have $(a_{k-2^r+1}, \dots, a_{k+2^r-1}) = (\overline{U}, \zeta, U)$ with $\zeta = a_k$ and $U = (a_{k+1}, \dots, a_{k+2^r-1})$.

If $k \in E_{r+1}$, we consider $m \in F_{r+1}$ such that $a_m = \zeta$; we have $a_{m+n \cdot 2^{r+2}} = (-1)^n \zeta$ for each $n \in \mathbf{Z}$. If $k \in F_r$, we write $m = k$; we have $a_{m+n \cdot 2^{r+1}} = (-1)^n \zeta$ for each $n \in \mathbf{Z}$. In both cases, for each $n \in \mathbf{Z}$, we have $a_{m+n \cdot 2^{r+3}} = \zeta$ and $(a_{m+n \cdot 2^{r+3} - 2^r + 1}, \dots, a_{m+n \cdot 2^{r+3} + 2^r - 1}) = (\overline{U}, \zeta, U)$.

For each $h \in \mathbf{Z}$, there exists $n \in \mathbf{Z}$ such that $h - m + 2^r \leq n \cdot 2^{r+3} \leq h - m + 9 \cdot 2^r - 1$, which implies $h + 1 \leq m + n \cdot 2^{r+3} - 2^r + 1$ and $h + 10 \cdot 2^r - 2 \geq m + n \cdot 2^{r+3} + 2^r - 1$. Then $(a_{m+n \cdot 2^{r+3} - 2^r + 1}, \dots, a_{m+n \cdot 2^{r+3} + 2^r - 1})$ is a subword of $(a_{h+1}, \dots, a_{h+10 \cdot 2^r - 2})$, which completes the proof of the Theorem since T is a subword of $(a_{m+n \cdot 2^{r+3} - 2^r + 1}, \dots, a_{m+n \cdot 2^{r+3} + 2^r - 1})$. ■

The second part of the Theorem below is similar to results which were proved for Robinson tilings and Penrose tilings:

Theorem 1.11. 1) *There exist* 2^ω *complete folding sequences which are pairwise not locally isomorphic.*

2) *For each complete folding sequence* S*, there exist* 2^ω *isomorphism classes of sequences which are locally isomorphic to* S*.*

Proof of 1). It follows from Proposition 1.1 that each complete folding sequence $(a_h)_{h\in\mathbf{Z}}$ is completely determined by the following operations:

– successively for each $n \in \mathbf{N}$, we choose among the 2 possible values the smallest $h \in \mathbf{N} \cap F_n$, then we fix $a_h \in \{+1, -1\}$;

– for the unique $h \in \bigcap_{n\in\mathbf{N}} E_n$ if it exists, we fix $a_h \in \{+1, -1\}$.

Moreover, each possible sequence of choices determines a complete folding sequence.

Now, it follows from Corollary 1.7 that, for each complete folding sequence S and each integer m, there exist an integer $n > m$ and a complete folding sequence T such that S and T contain as subwords the same m-folding sequences, but not the same n-folding sequences.

Proof of 2). The sequence S is not periodic by Corollary 1.3, and satisfies the local isomorphism property according to Theorem 1.10. By [7, Corollary 3.7], it follows that there exist 2^ω isomorphism classes of sequences which are locally isomorphic to S. ∎

Remark. Concerning logic, we note two differences between complete folding sequences and Robinson or Penrose tilings. First, the set of all complete folding sequences is defined by a countable set of first-order sentences, and not by only one sentence. Second, it is the union of 2^ω classes for elementary equivalence, i.e. local isomorphism, instead of being a single class.

2. Paperfolding curves: self-avoiding, derivatives, exterior

In the present section, we define n-folding curves, finite folding curves, ∞-folding curves and complete folding curves associated to n-folding sequences, finite folding sequences, ∞-folding sequences and complete folding sequences. We show their classical properties: self-avoiding, existence of "derivatives."

Then we prove that any complete folding curve divides the set of all points of $\mathbf{Z}^2$ which are "exterior" to it into zero, one or two "connected components," and that these components are infinite.

As an application, we consider curves which are limits of successive antiderivatives of a complete folding curve. Any such curve is equal to the closure of its interior. We show that, except in a special case, its exterior is the union of zero, one or two connected components. In some cases, its boundary is a fractal.

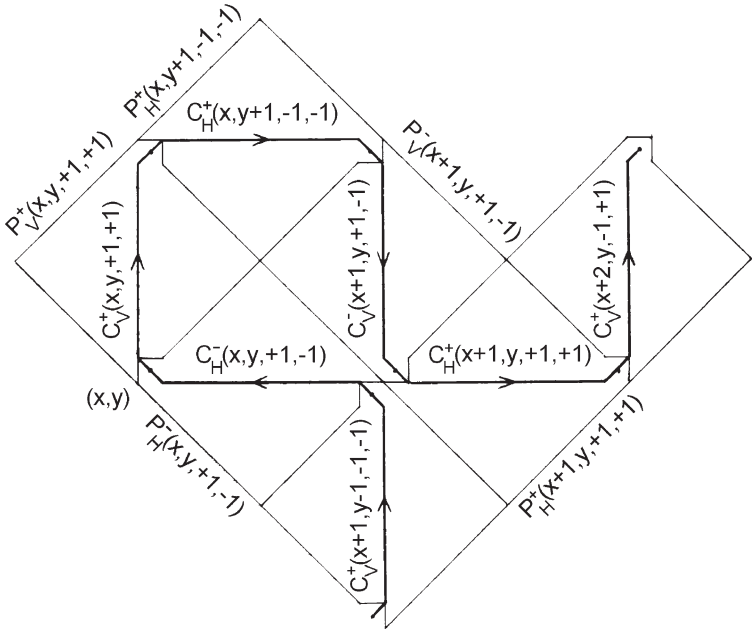

Fig. 1

Finally we prove that, for each finite subcurve F of a complete folding curve C, there exist everywhere in C some subcurves which are parallel to F.

We provide $\mathbf{R}^2$ with the euclidean distance defined as $d((x,y),(x',y')) = \sqrt{(x'-x)^2+(y'-y)^2}$ for any $x,y,x',y' \in \mathbf{R}$.

We fix $\alpha \in \mathbf{R}_+^*$ small compared to 1. For any $x,y \in \mathbf{Z}$ and any $\zeta,\eta \in \{+1,-1\}$, we consider (cf. Fig. 1) the *segments of curves*

$$C_H(x,y,\zeta,\eta) = [(x+\alpha, y+\zeta\alpha),(x+2\alpha,y)] \cup [(x+2\alpha,y),(x+1-2\alpha,y)] \cup [(x+1-2\alpha,y),(x+1-\alpha,y+\eta\alpha)]$$

and

$$C_V(x,y,\zeta,\eta) = [(x+\zeta\alpha, y+\alpha),(x,y+2\alpha)] \cup [(x,y+2\alpha),(x,y+1-2\alpha)] \cup [(x,y+1-2\alpha),(x+\eta\alpha,y+1-\alpha)].$$

We say that $[(x,y),(x+1,y)]$ is the *support* of $C_H(x,y,\zeta,\eta)$ and $[(x,y),(x,y+1)]$ is the *support* of $C_V(x,y,\zeta,\eta)$.

We denote by $C_H^+(x,y,\zeta,\eta)$ the segment $C_H(x,y,\zeta,\eta)$ oriented from left to right, and $C_H^-(x,y,\zeta,\eta)$ the segment $C_H(x,y,\zeta,\eta)$ oriented from right to left. Similarly, we denote by $C_V^+(x,y,\zeta,\eta)$ the segment $C_V(x,y,\zeta,\eta)$ oriented from bottom to top, and $C_V^-(x, y, \zeta, \eta)$ the segment $C_V(x, y, \zeta, \eta)$ oriented from top to bottom. From now on, all the segments considered are oriented.

We associate to $C_H^+(x, y, \zeta, \eta)$ the tile

$$P_H^+(x, y, \zeta, \eta) = \{(u, v) \in \mathbf{R}^2 \mid |u - (x + 1/2)| + |v - y| \leq 1/2\}$$
$$\cup \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + 1 - \alpha)|, |v - (y + \eta\alpha)|) \leq \alpha\}$$
$$- \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + \alpha)|, |v - (y + \zeta\alpha)|) < \alpha\},$$

and to $C_H^-(x, y, \zeta, \eta)$ the tile

$$P_H^-(x, y, \zeta, \eta) = \{(u, v) \in \mathbf{R}^2 \mid |u - (x + 1/2)| + |v - y| \leq 1/2\}$$
$$\cup \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + \alpha)|, |v - (y + \zeta\alpha)|) \leq \alpha\}$$
$$- \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + 1 - \alpha)|, |v - (y + \eta\alpha)|) < \alpha\}.$$

Similarly, we associate to $C_V^+(x, y, \zeta, \eta)$ the tile

$$P_V^+(x, y, \zeta, \eta) = \{(u, v) \in \mathbf{R}^2 \mid |u - x| + |v - (y + 1/2)| \leq 1/2\}$$
$$\cup \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + \eta\alpha)|, |v - (y + 1 - \alpha)|) \leq \alpha\}$$
$$- \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + \zeta\alpha)|, |v - (y + \alpha)|) < \alpha\},$$

and to $C_V^-(x, y, \zeta, \eta)$ the tile

$$P_V^-(x, y, \zeta, \eta) = \{(u, v) \in \mathbf{R}^2 \mid |u - x| + |v - (y + 1/2)| \leq 1/2\}$$
$$\cup \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + \zeta\alpha)|, |v - (y + \alpha)|) \leq \alpha\}$$
$$- \{(u, v) \in \mathbf{R}^2 \mid \sup(|u - (x + \eta\alpha)|, |v - (y + 1 - \alpha)|) < \alpha\}.$$

We note that each tile is obtained from a square, which has a diagonal of length 1 with endpoints in $\mathbf{Z}^2$, by putting "bumps" on two of its four edges.

We say that two segments C_1, C_2 are *consecutive* if they just have one common point and if the end of C_1 is the beginning of C_2. This property is true if and only if the intersection of the associated tiles consists of an edge with a bump (see Fig. 1). The supports of two consecutive segments form a right angle.

A *finite* (resp. *infinite*, *complete*) *curve* is a sequence $(C_1, \ldots, C_n)$ (resp. $(C_i)_{i \in \mathbf{N}^*}$, $(C_i)_{i \in \mathbf{Z}}$) of segments with C_i, C_{i+1} consecutive for each integer i such that C_i and C_{i+1} exist.

We identify two finite curves if they only differ in the beginning of the first segment and the end of the last one. We also identify two infinite curves if they only differ in the beginning of the first segment.

In the case of a complete curve, the tiles and the squares obtained by erasing their bumps cover the same part of $\mathbf{R}^2$.

We say that a curve $(C_1, \dots, C_n)$ (resp. $(C_i)_{i \in \mathbf{N}^*}$, $(C_i)_{i \in \mathbf{Z}}$) is *self-avoiding* if we have $C_i \cap C_j = \emptyset$ for $|j - i| \geq 2$. Such a curve defines an injective continuous function from a closed connected subset of $\mathbf{R}$ to $\mathbf{R}^2$.

The tiles associated to the segments of a self-avoiding curve are nonoverlapping, except possibly because of the bump at the end of the last segment.

We consider that two curves $(C_1, \dots, C_m)$ and $(D_1, \dots, D_n)$ can be concatenated if the end of the support of C_m and the beginning of the support of D_1 form a right angle. Then we modify the end of C_m and the beginning of D_1 in order to make them consecutive.

For each finite curve $(C_i)_{1 \leq i \leq n}$ (resp. infinite curve $(C_i)_{i \in \mathbf{N}^*}$, complete curve $(C_i)_{i \in \mathbf{Z}}$), we consider the sequence $(\eta_i)_{1 \leq i \leq n-1}$ (resp. $(\eta_i)_{i \in \mathbf{N}^*}$, $(\eta_i)_{i \in \mathbf{Z}}$) defined as follows: for each i, we write $\eta_i = +1$ (resp. $\eta_i = -1$) if we turn left (resp. right) when we pass from C_i to C_{i+1}. Two curves are associated to the same sequence if and only if they are equivalent modulo a positive isometry.

For each segment of curve D, we denote by $\overline{D}$ the segment obtained from D by changing the orientation. If a finite curve $C = (C_1, \dots, C_n)$ is associated to $S = (\eta_1, \dots, \eta_{n-1})$, then $\overline{C} = (\overline{C}_n, \dots, \overline{C}_1)$ is associated to $\overline{S} = (-\eta_{n-1}, \dots, -\eta_1)$. If a complete curve $C = (C_i)_{i \in \mathbf{Z}}$ is associated to $S = (\eta_i)_{i \in \mathbf{Z}}$, then $\overline{C} = (\overline{C}_{-i+1})_{i \in \mathbf{Z}}$ is associated to $\overline{S} = (-\eta_{-i})_{i \in \mathbf{Z}}$.

For each segment of curve D with support $[X, Y]$ and oriented from X to Y, we call X, Y the *endpoints*, X the *initial point* and Y the *terminal point* of D. The *initial point* of a curve $(C_1, \dots, C_n)$ is the initial point of C_1, and its *terminal point* is the terminal point of C_n. The *vertices* of a curve are the endpoints of its segments.

We say that two segments of curves, or two curves, are *parallel* (resp. *opposite*) if they are equivalent modulo a translation (resp. a rotation of angle π).

We have $\mathbf{Z}^2 = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$ for $M_1 = \{(x, y) \in \mathbf{Z}^2 \mid x + y \text{ odd}\}$ and $M_2 = \{(x, y) \in \mathbf{Z}^2 \mid x + y \text{ even}\}$. We denote by M one of these two sets and we consider, on the one hand the curves with supports of length 1 and vertices in $\mathbf{Z}^2$, on the other hand the curves with supports of length $\sqrt{2}$ and vertices in M.

Let C be a segment of the second system, let X be its initial point and let X' be its terminal point. Then, in the first system, there exist two curves (A_1, A_2) and (B_1, B_2), associated to the sequences (-1) and $(+1)$, such that X is the initial point of A_1 and B_1, and X' is the terminal point of A_2 and B_2 (see Fig. 2A).

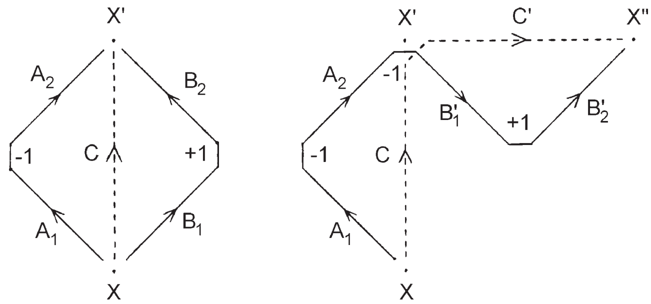

Fig. 2A **Fig. 2B**

Now, consider in the second system a segment C' such that (C, C') is a curve associated to a sequence (ε) with $\varepsilon \in \{+1, -1\}$. Let X'' be the terminal point of C'. In the first system, denote by (A_1', A_2') and (B_1', B_2') the curves associated to the sequences (-1) and $(+1)$, such that X' is the initial point of A_1' and B_1', and X'' is the terminal point of A_2' and B_2'.

Then (see Fig. 2B), (A_1, A_2, B_1', B_2') and (B_1, B_2, A_1', A_2') are curves associated to $(-1, \varepsilon, +1)$ and $(+1, \varepsilon, -1)$. Each of these curves has X, X', X'' among its vertices, and crosses the curve (C, C') near X'. Moreover (A_1, A_2, A_1', A_2') and (B_1, B_2, B_1', B_2') are not curves.

For each curve $(C_1, \dots, C_{2n})$ (resp. $(C_i)_{i \in \mathbf{N}^*}$, $(C_i)_{i \in \mathbf{Z}}$) of the first system and each curve $(D_1, \dots, D_n)$ (resp. $(D_i)_{i \in \mathbf{N}^*}$, $(D_i)_{i \in \mathbf{Z}}$) of the second system, we say that C is an *antiderivative* of D or that D is the *derivative* of C if, for each integer i:

a) if D_{i+1} exists, then C_{2i+1} and D_{i+1} have the same initial point;
b) if D_i exists, then C_{2i} and D_i have the same terminal point;
c) if D_i and D_{i+1} exist, then C crosses D near the terminal point of D_i (we say that C *alternates* around D).

Each curve of the second system has exactly two antiderivatives in the first one. Each curve C of the first system has at most one derivative in the second one. If that derivative exists, then the sequence $(\eta_1, \dots, \eta_{2n-1})$ (resp. $(\eta_i)_{i \in \mathbf{N}^*}$, $(\eta_i)_{i \in \mathbf{Z}}$) of elements of $\{-1, +1\}$ associated to C satisfies $\eta_{2i+1} = (-1)^i \eta_1$ for each integer i such that η_{2i+1} exists. Conversely, if this condition is satisfied, then the *derivative* of C is defined by taking for M the set M_1 or M_2 which contains the initial point of C_1, and replacing each pair of segments (C_{2i-1}, C_{2i}) with a segment D_i.

For the definition of the derivative of a complete curve $(C_i)_{i \in \mathbf{Z}}$, we permit ourselves to change the initial point of indexation, i.e. to replace $(C_i)_{i \in \mathbf{Z}}$ with $(C_{i+k})_{i \in \mathbf{Z}}$ for an integer k. With this convention, the derivative exists if and only if $\eta_{2i} = (-1)^i \eta_0$ for each $i \in \mathbf{Z}$ or $\eta_{2i+1} = (-1)^i \eta_1$ for each $i \in \mathbf{Z}$. If these two conditions are simultaneously satisfied, we obtain two different derivatives; in that situation, which only concerns one isometry class of curves, we consider that the derivative is not defined.

We define by induction the *n-th derivative* $C^{(n)}$ of a curve C, with $C^{(0)} = C$ and $C^{(n+1)}$ derivative of $C^{(n)}$ for each $n \in \mathbf{N}$, as well as the *n-th antiderivatives*. It is convenient to represent the successive derivatives of a curve C on the same figure in such a way that $C^{(n)}$ alternates around $C^{(n+1)}$ for each $n \in \mathbf{N}$ such that $C^{(n+1)}$ exists. This convention will be used later in the paper.

For each $n \in \mathbf{N}$, we call an *n-folding curve* any finite curve associated to an n-folding sequence. For each n-folding sequence obtained by folding n times a strip of paper, we obtain the associated n-folding curve by keeping the strip folded according to right angles instead of unfolding it completely.

We see by induction on n that the n-folding curves are the n-th antiderivatives of the curves which consist of one segment. Consequently, up to isometry and up to the orientation, there exist one 2-folding curve (cf. Fig. 3A), two 3-folding curves (cf. Fig. 3B), and four 4-folding curves (cf. Fig. 3C).

We call *∞-folding curve* (resp. *finite folding curve*, *complete folding curve*) each curve associated to an ∞-folding sequence (resp. a finite folding sequence, a complete folding sequence). Any curve $(C_i)_{i \in \mathbf{N}^*}$ (resp. $(C_i)_{i \in \mathbf{Z}}$) is an ∞-folding curve (resp. a complete folding curve) if and only if it is indefinitely derivable.

The successive antiderivatives of a paperfolding curve, as well as its successive derivatives if they exist, are also paperfolding curves.

PROPOSITION 2.1. *Antiderivatives of self-avoiding curves are self-avoiding.*

PROOF. Consider a curve C whose derivative D is self-avoiding. If C is not self-avoiding, then there exist two segments of C which have the same support. These two segments are necessarily coming from segments of D which have a common endpoint.

In order to prove that this situation is impossible, we consider the function τ which is defined on the set of all supports of segments of D with $\tau([(u,v),$

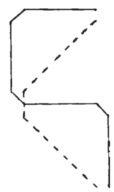

Fig. 3A. one 2-folding curve

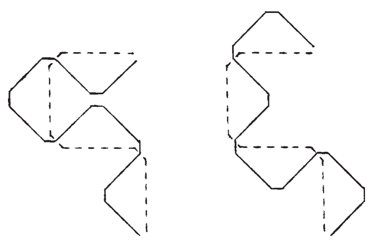

Fig. 3B. two 3-folding curves

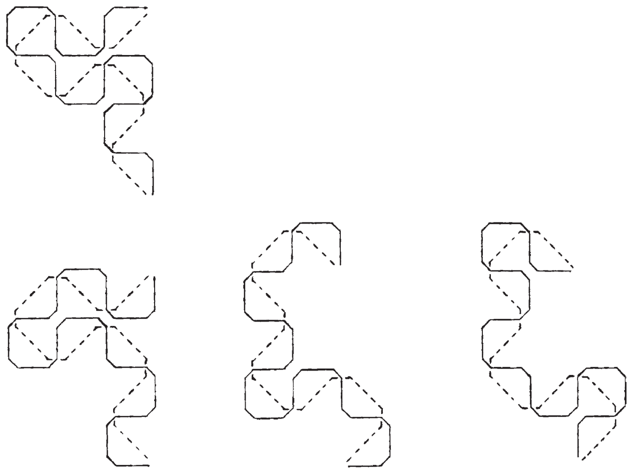

Fig. 3C. four 4-folding curves

$(u+1, v+\varepsilon)]) = +1$ (resp. -1) for each $(u, v) \in \mathbf{Z}^2$ and each $\varepsilon \in \{-1, +1\}$ such that C is above (resp. below) D on $[(u, v), (u+1, v+\varepsilon)]$. It suffices to observe that the equality $\tau([(u', v'), (u'+1, v'+\varepsilon')]) = (-1)^{u'-u}\tau([(u, v), (u+1, v+\varepsilon)])$ is true wherever τ is defined. In fact, it is true for the supports of consecutive segments of D because C alternates around D, and it is proved in the general case by induction on the number of consecutive segments between the two segments considered. ■

COROLLARY 2.2. *Paperfolding curves are self-avoiding.*

PROOF. For each integer n, each n-folding curve is self-avoiding because it is the n-th antiderivative of a self-avoiding curve which consists of one segment. Each finite folding curve is self-avoiding since it is a subcurve of an n-folding curve for an integer n. Complete folding curves and ∞-folding curves are self-avoiding because their finite subcurves are self-avoiding.

Another proof is given by [4, Observation 1.11, p. 134]. ■

For each self-avoiding curve C and any $x, y \in \mathbf{Z}$, we write:
$\rho_C([(x, y), (x+1, y)]) = +1$ (resp. -1) if C contains a segment with the initial point (x, y) (resp. $(x+1, y)$) and the terminal point $(x+1, y)$ (resp. (x, y));
$\rho_C([(x, y), (x, y+1)]) = +1$ (resp. -1) if C contains a segment with the initial point (x, y) (resp. $(x, y+1)$) and the terminal point $(x, y+1)$ (resp. (x, y)).

There exists $\varepsilon \in \{-1, +1\}$ such that $\rho_C([(x, y), (x+1, y)]) = (-1)^{y-x+\varepsilon}$ and $\rho_C([(x, y), (x, y+1)]) = (-1)^{y-x+\varepsilon+1}$ wherever ρ_C is defined. In fact, ε is the same for the supports of two consecutive segments, and we see that it is the same for the supports of any two segments by induction on the number of consecutive segments between them. We extend the definition of ρ_C, according to this property, to the set of all intervals $[(x, y), (x+1, y)]$ or $[(x, y), (x, y+1)]$ with $x, y \in \mathbf{Z}$.

For each self-avoiding curve $C = (C_i)_{i \in \mathbf{Z}}$ and any pairs (C_i, C_{i+1}), (C_j, C_{j+1}) of consecutive segments, if C_i and C_j have the same terminal point, then, by the property of ρ_C stated above, we turn left when passing from C_i to C_{i+1} if and only if we turn left when passing from C_j to C_{j+1}. For each $X \in \mathbf{Z}^2$, we write $\sigma_C(X) = +1$ (resp. -1) if X is the common endpoint of two consecutive segments C_i, C_{i+1} of C and if we turn left (resp. right) when passing from C_i to C_{i+1}.

It follows from the definition of derivatives that, for each $k \in \mathbf{N}$:

a) if $C^{(2k)}$ exists, then there exists $X_{2k} \in \mathbf{Z}^2$ such that the set of all vertices of $C^{(2k)}$ is contained in $E_{2k}(C) = X_{2k} + 2^k\mathbf{Z}^2$;

b) if $C^{(2k+1)}$ exists, then there exists $X_{2k+1} \in \mathbf{Z}^2$ such that the set of all vertices of $C^{(2k+1)}$ is contained in $E_{2k+1}(C) = X_{2k+1} + (2^k, 2^k)\mathbf{Z} + (2^k, -2^k)\mathbf{Z}$.

The set $E_n(C)$ is defined for each $n \in \mathbf{N}$ such that $C^{(n)}$ exists. If $C^{(n+1)}$ exists, then we have $E_{n+1}(C) \subset E_n(C)$; we write $F_n(C) = E_n(C) - E_{n+1}(C)$.

If $S = (\eta_i)_{i \in \mathbf{Z}}$ is the sequence associated to a complete folding curve $C = (C_i)_{i \in \mathbf{Z}}$, then, for each $i \in \mathbf{Z}$ and each $n \in \mathbf{N}$, the terminal point of C_i belongs to $E_n(C)$ if and only if i belongs to $E_n(S)$.

The following lemma applies, in particular, to complete folding curves:

LEMMA 2.3. *Let C be a derivable self-avoiding complete curve. Consider a square $Q = [x, x+1] \times [y, y+1]$ with $x, y \in \mathbf{Z}$. If four vertices of Q are endpoints of segments of C, then at least three segments of C, with two of them consecutive, have supports which are edges of Q. If three vertices of Q are endpoints, then the two edges determined by these vertices are supports of segments of C, or neither of them is a support. If C is derivable twice and if two vertices of Q are endpoints, then they are necessarily adjacent.*

PROOF. We denote by W, X, Y, Z the vertices of Q taken consecutively, and we show that the cases excluded by the Lemma are impossible.

First suppose that an edge of Q, for instance WX, is the support of a segment of C, that a vertex of Q which does not belong to this edge, for instance Z, is a vertex of C, and that the edges of Q which contain this vertex are not supports of segments of C. Consider the two pairs of consecutive segments of C which respectively have W and Z as a common endpoint. Then the property of ρ_C implies that these two pairs both have the orientation shown by Figure 4A, or both have the contrary orientation, which contradicts the connectedness of C (see Fig. 4A).

Then suppose that two opposite edges of Q, for instance WX and YZ, are supports of segments of C, and that the two preceding segments, as well as the

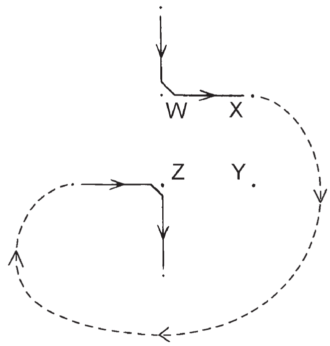

Fig. 4A

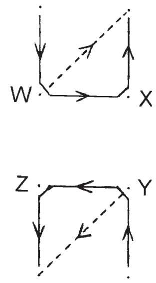

Fig. 4B

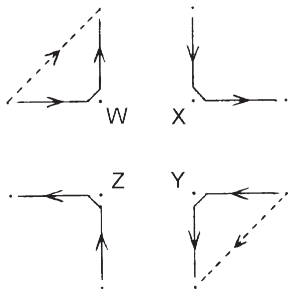

Fig. 4C

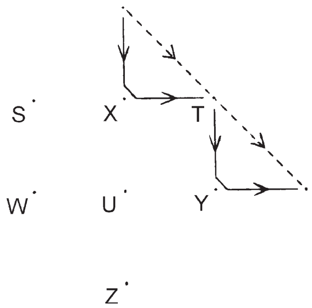

Fig. 4D

two following segments, have supports which are not edges of Q. Then the property of ρ_C implies that the two sequences of three consecutive segments formed from these six segments both have the orientation shown by Figure 4B, or both have the contrary orientation. Suppose for instance that X and Z belong to $F_0(C)$. Then the two pairs of segments of C, extracted from the two sequences, which respectively have X and Z as a common endpoint, give opposite segments of D, which contradicts the property of ρ_D (see Fig. 4B).

Now suppose that W, X, Y, Z are vertices of C and that no edge of Q is the support of a segment of C. Then the property of ρ_C implies that the pairs of segments of C which respectively have W, X, Y, Z as a common endpoint all have the orientation shown by Figure 4C, or all have the contrary orientation. Suppose for instance that W and Y belong to $F_0(C)$. Then the pairs of segments of C which respectively have W and Y as a common endpoint give opposite segments of D, which contradicts the property of ρ_D (see Fig. 4C).

Finally suppose that C is derivable twice and that only two opposite vertices of Q, for instance W and Y, are vertices of C. If W and Y belong to $F_0(C)$, then, as in the previous case, the pairs of segments of C which respectively have W and Y as a common endpoint give opposite segments of D, which contradicts the property of ρ_D.

If W and Y belong to $E_1(C)$, we consider the two squares of width $\sqrt{2}$ which have WY as their common edge. As W and Y are vertices of D, one of the squares has an edge adjacent to WY which is the support of a segment of D. On the other hand, as the center X or Z of that square is not a vertex of C, the edge WY and the opposite edge are not supports of segments of D, which contradicts the two first statements of the Lemma applied to D. ∎

DEFINITIONS. For any $X, Y \in \mathbf{Z}^2$, a *path* from X to Y is a sequence $(X_0, \dots, X_n) \subset \mathbf{Z}^2$ with $n \in \mathbf{N}$, $X_0 = X$, $X_n = Y$ and $d(X_{i-1}, X_i) = 1$ for $1 \leq i \leq n$. For each complete curve C, we call *exterior* of C and we denote by $\mathrm{Ext}(C)$ the set of all points of $\mathbf{Z}^2$ which are not vertices of C. A *connected component* of $\mathrm{Ext}(C)$ is a subset K which is maximal for the following property: any two points of K are connected by a path which only contains points of K.

THEOREM 2.4. *The exterior* $\mathrm{Ext}(C)$ *of a complete folding curve* C *is the union of* 0, 1 *or* 2 *infinite connected components, and each of these components is the intersection of* $\mathrm{Ext}(C)$ *with one of the* 2 *connected components of* $\mathbf{R}^2 - C$.

LEMMA 2.4.1. *The connected components of* $\mathrm{Ext}(C)$ *are infinite.*

PROOF OF LEMMA 2.4.1. For each $n \in \mathbf{N}$, we have $\mathrm{Ext}(C^{(n)}) = \mathrm{Ext}(C) \cap E_n(C)$ since each point of $E_n(C)$ is a vertex of $C^{(n)}$ if and only if it is a vertex of C.

If K is a connected component of $\mathrm{Ext}(C)$, then $K \cap E_n(C)$ is a union of connected components of $\mathrm{Ext}(C^{(n)})$ for each $n \in \mathbf{N}$. Otherwise, the smallest integer n such that this property is false satisfies $n \geq 1$, and $E_{n-1}(C)$ contains the consecutive vertices W, X, Y, Z of a square of width $(\sqrt{2})^{n-1}$ with $W, Y \in \mathrm{Ext}(C^{(n)})$, $W \in K$, $Y \notin K$ and X, Z vertices of $C^{(n-1)}$, which contradicts Lemma 2.3 applied to $C^{(n-1)}$.

For each connected component K of $\mathrm{Ext}(C)$ and each $n \in \mathbf{N}$, we have $\varnothing \subsetneq K \cap E_{n+1}(C) \subsetneq K \cap E_n(C)$, or $K \cap E_n(C)$ is a union of connected components which consist of one point; in fact, for any $X, Y \in E_n(C)$ with $d(x, y) = (\sqrt{2})^n$, we have $X \in F_n(C)$ and $Y \in E_{n+1}(C)$, or $Y \in F_n(C)$ and $X \in E_{n+1}(C)$. Consequently, in order to prove that $\mathrm{Ext}(C)$ has no finite connected component, it suffices to show that each $\mathrm{Ext}(C^{(n)})$ has no connected component which consists of one point.

Suppose that there exist a twice derivable complete curve D and a point $U = (u, v) \in \mathbf{Z}^2$ such that $\{U\}$ is a connected component of $\mathrm{Ext}(D)$. Write $W = (u - 1, v)$, $X = (u, v + 1)$, $Y = (u + 1, v)$ and $Z = (u, v - 1)$.

If U belongs to $F_0(D)$, then W, X, Y, Z belong to $E_1(D)$ and they are vertices of $D^{(1)}$. By Lemma 2.3, two consecutive segments of $D^{(1)}$ have supports which are edges of $WXYZ$. As D alternates around $D^{(1)}$, it follows that U is a vertex of D, whence a contradiction.

If U belongs to $E_1(D)$, consider the point S (resp. T) which forms a square with X, U, W (resp. X, U, Y). Then SX or XT is the support of a segment of D since X is a vertex of D.

Suppose for instance that XT is the support of a segment of D. As Y is a vertex of D contrary to U, Lemma 2.3 implies that TY is also the support of a segment of D. As X and Y belong to $F_0(D)$, it follows from the property of ρ_D (see Fig. 4D) that there exist two parallel segments of $D^{(1)}$ such that T is the terminal point of one of them and the initial point of the other one, whence a contradiction. ∎

Proof of Theorem 2.4. Write $C = (C_i)_{i \in \mathbf{Z}}$. Consider a connected component K of $\mathrm{Ext}(C)$ and write $M = \{X \in \mathbf{Z}^2 - K \mid d(X, K) = 1\}$.

Denote by Ω the set of all squares $S = [x, x+1] \times [y, y+1]$ with $x, y \in \mathbf{Z}$ such that K contains one or two vertices of S. For each $S \in \Omega$, if K contains one vertex X of S, consider the segment E_S of length $\sqrt{2}$ joining the vertices of S adjacent to X. If K contains two vertices of S, consider the segment E_S of length 1 joining the two other vertices, which are adjacent by Lemma 2.3.

The endpoints of the segments E_S for $S \in \Omega$ belong to M. We are going to prove that each $U \in M$ is an endpoint of exactly two such segments. We consider the points $V, W, X, Y \in \mathbf{Z}^2$ with $d(U, V) = d(U, W) = d(U, X) = d(U, Y) = 1$ such that $VWXY$ is a square, V, W are vertices of C, and $X \in K$. We denote by P, Q, R, S the squares determined by the pairs of edges (UV, UW), (UW, UX), (UX, UY), (UY, UV).

As C is connected, the fourth vertex of P does not belong to K, and P does not belong to Ω. On the other hand, Q belongs to Ω since X belongs to K contrary to U and W.

If Y is a vertex of C, then R belongs to Ω since X belongs to K and U, Y do not belong to K. Moreover, Lemma 2.3 implies that the fourth vertex of S is a vertex of C, since UV is the support of a segment of C contrary to UY. Consequently, S does not belong to Ω.

If Y is not a vertex of C, then, by Lemma 2.3, the fourth vertex of R is not a vertex of C, and therefore belongs to K. Consequently, Y belongs to K, S belongs to Ω and R does not belong to Ω.

Moreover, U is the common endpoint of E_Q and E_R if Y is a vertex of C, and the common endpoint of E_Q and E_S if Y is not a vertex of C.

As K is infinite by Lemma 2.4.1, it follows that the segments E_S for $S \in \Omega$ form an unbounded self-avoiding curve E. The vertices of E are the points of M. One connected component of $\mathbf{R}^2 - E$ contains K, but contains no point of C and no point of $\mathbf{Z}^2$ which does not belong to K.

The points of M taken along E form a sequence $(X_i)_{i \in \mathbf{Z}}$. For each $X \in M$, denote by $r(X)$ the unique integer j such that X is the terminal point of C_j and the initial point of C_{j+1}. Suppose for instance $r(X_0) < r(X_1)$. Then, using the connectedness of C, we see by induction on i that $r(X_i) < r(X_{i+1})$ for each $i \in \mathbf{Z}$.

Now suppose that there exists a connected component $L \neq K$ of $\mathrm{Ext}(C)$ such that K and L are contained in the same connected component of $\mathbf{R}^2 - C$. Then there exists $i \in \mathbf{Z}$ such that L is contained in the loop formed by C between X_i and X_{i+1}, and L is finite contrary to Lemma 2.4.1. ■

Now, we apply Theorem 2.4 to limits of complete folding curves. We give less details in this part, which will not be used in the remainder of the paper.

We consider some complete folding curves $C_n = (C_{n,p})_{p \in \mathbf{Z}}$ with $C_n = C_{n+1}^{(1)}$ for each $n \in \mathbf{N}$. We suppose that the curves C_n are represented on the same figure in such a way that:
- C_0 has vertices in $\mathbf{Z}^2$ and supports of length 1;
- all the segments $C_{n,1}$ have the same initial point;
- C_{n+1} alternates around C_n for each $n \in \mathbf{N}$.

We denote by L the limit of the curves C_n considered as representations of functions from $\mathbf{R}$ to $\mathbf{R}^2$.

The curve L is associated to a function from $\mathbf{R}$ to $\mathbf{R}^2$ which is continuous everywhere, but derivable nowhere. Moreover, L is closed as a subset of $\mathbf{R}^2$. By Theorem 3.1 below, L contains arbitrarily large open balls. It follows from Proposition 2.6 and Theorem 3.1 that L is equal to the closure of its interior.

Now, we consider the complete folding sequences $(\eta_{n,p})_{p \in \mathbf{Z}}$ associated to the curves C_n. We say that $(\eta_{n,1})_{n \in \mathbf{N}}$ is *ultimately alternating* if we have $\eta_{n,1} = (-1)^n$ for n large enough, or $\eta_{n,1} = (-1)^{n+1}$ for n large enough.

COROLLARY 2.5. *If $(\eta_{n,1})_{n \in \mathbf{N}}$ is not ultimately alternating, then $\mathbf{R}^2 - L$ is the union of* 0, 1 *or* 2 *unbounded connected components.*

PROOF. For each $n \in \mathbf{N}$, we have $\mathrm{Ext}(C_n) \subset \mathbf{R}^2 - L$ since $(\eta_{m,1})_{m \in \mathbf{N}}$ is not ultimately alternating. By Theorem 2.4, it suffices to show that, for each $n \in \mathbf{N}$, any $X, X' \in \mathrm{Ext}(C_n)$ belong to the same connected component of $\mathbf{R}^2 - L$ if they belong to the same connected component of $\mathrm{Ext}(C_n)$. Moreover, it is enough to prove this property when $d(X, X') = (1/\sqrt{2})^n$.

We consider some distinct points Y, Y', Z, Z', U, V with YY' and ZZ' parallel to XX' such that $XX'Y'Y$ (resp. $XX'Z'Z$) is a square of center U

(resp. V). Then XX', XY, XZ, $X'Y'$, $X'Z'$ are not supports of segments of C_n. Consequently, at least one of the points U, V is not a vertex of C_{n+1}.

If U (resp. V) is not a vertex of C_{n+1}, then the open triangle XUX' (resp. XVX') is contained in $\mathbf{R}^2 - L$. It follows that X and X' belong to the same connected component of $\mathbf{R}^2 - L$. ■

Remark. Let L be the limit of the curves C_n, where C_0 is the curve C of Example 3.13 below and $\eta_{n,1} = (-1)^{n+1}$ for $n \geq 1$. Then we have $[(0,1), (2,1)] \subset L$ even though $(1,1)$ is not a vertex of C_0. It follows that $(1,0)$ belongs to a bounded connected component of $\mathbf{R}^2 - L$. It can be proved that $\mathbf{R}^2 - L$ has infinitely many such components.

Examples. The limit curve L obtained from the curve C of Example 3.13 by writing $\eta_{n,1} = +1$ for $n \geq 1$ is called a *dragon curve*. It follows from [6] that the interior of L is a union of countably many bounded connected components. According to [2], the boundary of L is a fractal.

On the other hand, Example 3.8 gives a case with $L = \mathbf{R}^2$. Using a similar construction, we can obtain L such that its boundary is a line. If L is one of the limit curves obtained from the curves of Example 3.14 by writing $\eta_{n,1} = (-1)^{n+1}$ for $n \geq 1$, then its boundary is the union of two or four half-lines with the same origin.

The Proposition below is not a priori obvious since two curves associated to the same folding sequence are not necessarily parallel.

Proposition 2.6. *For each complete folding curve $C = (C_h)_{h \in \mathbf{Z}}$, for each $n \in \mathbf{N}$ and for any $i, j \in \mathbf{Z}$, $(C_{j+1}, \ldots, C_{j+88n})$ contains a curve which is parallel and another curve which is opposite to $(C_{i+1}, \ldots, C_{i+n})$.*

For the proof of Proposition 2.6, we fix C, n, i, j and we write $A = (C_{i+1}, \ldots, C_{i+n})$. We consider the integer m such that $2^{m-1} < n \leq 2^m$, and the sequence $S = (\eta_h)_{h \in \mathbf{Z}}$ associated to C.

Lemma 2.6.1. *There exists $k \in E_{m+2}(S)$ such that $(C_{k+1}, \ldots, C_{k+2^{m+2}})$ contains a subcurve which is parallel or opposite to A.*

Proof of Lemma 2.6.1. We can suppose that there exists $l \in E_{m+2}(S)$ such that $i+1 \leq l \leq i+n-1$ since, otherwise, there exists $k \in E_{m+2}(S)$ such that $A \subset (C_{k+1}, \ldots, C_{k+2^{m+2}})$.

Then, for each $h \in \{i+1, \ldots, i+n-1\} - \{l\}$, we have $h \in \mathbf{Z} - E_n$, and therefore $\eta_h = \eta_{h+2^{m+1}+r \cdot 2^{m+2}}$ for each $r \in \mathbf{Z}$. We also have $\eta_{l+2^{m+1}+r \cdot 2^{m+2}} = (-1)^r \eta_{l+2^{m+1}}$ for each $r \in \mathbf{Z}$. Consequently, there exists $r \in \mathbf{Z}$ such that $\eta_l = \eta_{l+2^{m+1}+r \cdot 2^{m+2}}$, and therefore $\eta_h = \eta_{h+2^{m+1}+r \cdot 2^{m+2}}$ for $i+1 \leq h \leq i+n-1$.

Now the supports of C_{i+1} and $C_{i+1+2^{m+1}+r\cdot 2^{m+2}}$ are parallel or opposite since $(i+1+2^{m+1}+r\cdot 2^{m+2})-(i+1)$ is even. It follows that $(C_{i+1+2^{m+1}+r\cdot 2^{m+2}},\dots,C_{i+n+2^{m+1}+r\cdot 2^{m+2}})$ is parallel or opposite to A. ■

PROOF OF PROPOSITION 2.6. By Lemma 2.6.1, we can suppose that there exists $k\in E_{m+2}(S)$ such that $k\le i$ and $k+2^{m+2}\ge i+n$. For each $r\in\mathbf{Z}$, we write $C_r^*=(C_{k+1+r\cdot 2^{m+2}},\dots,C_{k+(r+1)\cdot 2^{m+2}})$. We consider the $(m+2)$-th derivative $D=(D_r)_{r\in\mathbf{Z}}$ of C, indexed in such a way that, for each $r\in\mathbf{Z}$, the initial point of D_r is the initial point of $C_{k+1+r\cdot 2^{m+2}}$.

For any $r,s\in\mathbf{Z}$, we have

$$(\eta_{k+1+r\cdot 2^{m+3}},\dots,\eta_{k+2^{m+2}-1+r\cdot 2^{m+3}})=(\eta_{k+1+s\cdot 2^{m+3}},\dots,\eta_{k+2^{m+2}-1+s\cdot 2^{m+3}}).$$

Consequently, C_{2r}^* and C_{2s}^* are parallel (resp. opposite) if and only if D_{2r} and D_{2s} are parallel (resp. opposite). The same property is true for the copies of A contained in C_{2r}^* and C_{2s}^*.

As we have $88n>11\cdot 2^{m+2}$, there exists $r\in\mathbf{Z}$ such that $(C_{j+1},\dots,C_{j+88n})$ contains $(C_{2r}^*,\dots,C_{2r+8}^*)$. Moreover, $(D_{2r},\dots,D_{2r+8})$ is necessarily contained in a 4-folding curve. It follows (see Fig. 3C) that there exist $p,q\in\{0,1,2,3,4\}$ such that D_{2r+p} and D_{2r+q} are opposite. Then C_{2r+p}^* and C_{2r+q}^* are also opposite, as well as the copies of A that they contain. Consequently, one of these copies is parallel to A. ■

3. Coverings of $\mathbf{R}^2$ by sets of disjoint complete folding curves

Consider $\Omega=\mathbf{R}^2$ or $\Omega=[a,b]\times[c,d]$ with $a,b,c,d\in\mathbf{Z}$, $b\ge a+1$ and $d\ge c+1$. Let $\mathscr{E}$ be a set of segments with supports contained in Ω. We say that $\mathscr{E}$ is a *covering* of Ω if it satisfies the following conditions:

1) each interval $[(x,y),(x+1,y)]$ or $[(x,y),(x,y+1)]$ contained in Ω, with $x,y\in\mathbf{Z}$, is the support of a unique segment of $\mathscr{E}$;
2) if two distinct non consecutive segments of $\mathscr{E}$ have a common endpoint X, then they can be completed into pairs of consecutive segments, with all four segments having distinct supports which contain X.

The set $\mathscr{E}$ is a covering of $\mathbf{R}^2$ (resp. $[a,b]\times[c,d]$) if and only if the tiles associated to the segments belonging to $\mathscr{E}$ form a tiling of $\mathbf{R}^2$ (resp. a patch which covers $[a,b]\times[c,d]$).

For each covering $\mathscr{E}$ of $\mathbf{R}^2$, if each finite sequence of consecutive segments belonging to $\mathscr{E}$ is a folding curve, then $\mathscr{E}$ is a *covering of* $\mathbf{R}^2$ *by complete folding curves*, in the sense that $\mathscr{E}$ is a union of disjoint complete folding curves.

We say that a covering of $\mathbf{R}^2$ by complete folding curves *satisfies the local isomorphism property* if the associated tiling satisfies the local isomorphism property. Two such coverings are said to be *locally isomorphic* if the associated tilings are locally isomorphic.

It often happens that a covering of $\mathbf{R}^2$ by complete folding curves satisfies the local isomorphism property. In particular, we show that this property is satisfied by any covering of $\mathbf{R}^2$ by 1 complete folding curve, or by 2 complete folding curves associated to the "positive" folding sequence. Two complete folding curves associated to the "alternating" folding sequence do not give a covering of $\mathbf{R}^2$, but we prove that a covering of $\mathbf{R}^2$ which satisfies the local isomorphism property is obtained naturally from these 2 curves by adding 4 other complete folding curves.

We characterize the coverings of $\mathbf{R}^2$ by sets of complete folding curves which satisfy the local isomorphism property, and the pairs of locally isomorphic such coverings. We show that each complete folding curve can be completed in a quasi-unique way into such a covering and that, for each complete folding sequence S, there exists a covering of $\mathbf{R}^2$ by a complete folding curve associated to a sequence which is locally isomorphic to S.

Finally, we prove that each complete folding curve covers a "significant" part of $\mathbf{R}^2$. In that way, we show that the maximum number of disjoint complete folding curves in $\mathbf{R}^2$, and therefore the maximum number of complete folding curves in a covering of $\mathbf{R}^2$, is at most 24. We also prove that such a covering cannot contain more than 6 curves if it satisfies the local isomorphism property.

The following result will be used frequently in the proofs:

THEOREM 3.1. *There exists a function $f : \mathbf{N} \to \mathbf{N}$ with exponential growth such that, for each integer n, each n-folding curve contains a covering of a square $[x, x + f(n)] \times [y, y + f(n)]$ with $x, y \in \mathbf{Z}$.*

PROOF. By Figure 3C, each 4-folding curve contains a copy, up to isometry and modulo the orientation, of the curve given by Figure 5A. Consequently, each 5-folding curve contains a copy of the curve given by Figure 5B, and each 6-folding curve contains a copy of one of the two curves given by Figures 5C and 5D.

For each $X = (x, y) \in \mathbf{Z}^2$ and each $k \in \mathbf{N}^*$, we write $L(X, k) = \{(u, v) \in \mathbf{R}^2 \mid |u - x| + |v - y| \leq k\}$. We say that a folding curve C covers $L(X, k)$ if

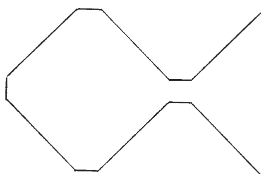

Fig. 5A

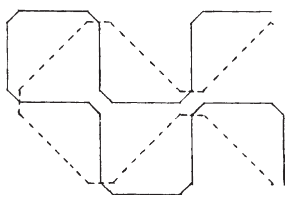

Fig. 5B

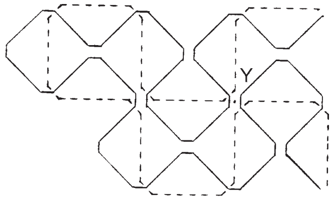

Fig. 5C

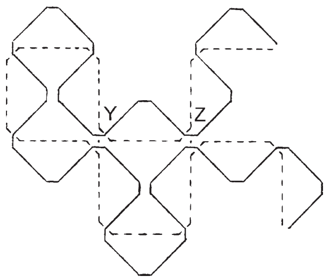

Fig. 5D

each interval $[(u,v),(u+1,v)]$ or $[(u,v),(u,v+1)]$ with $u,v \in \mathbf{Z}$ contained in $L(X,k)$ is the support of a segment of C.

The curve in Figure 5C covers $L(Y,2)$, and each of its two antiderivatives covers $L(X,2)$ where X is the point corresponding to Y. Each antiderivative of the curve in Figure 5D covers $L(W,2)$ or $L(X,2)$ where W and X are the points corresponding to Y and Z. Consequently, each 7-folding curve covers an $L(U,2)$.

For each $k \in \mathbf{N}^*$, if a folding curve C covers an $L(X,k)$, then each second antiderivative D of C covers an $L(Y,2k-1)$. More precisely, if we put D on the figure containing C, then D covers $L(X,k-1/2)$, and we obtain a curve which covers an $L(Y,2k-1)$ when we apply a homothety of ratio 2 in order to give the length 1 to the supports of the segments of D.

We see by induction on n that each $(2n+7)$-folding curve covers an $L(X,2^n+1)$. ■

For each complete folding curve $C=(C_i)_{i\in\mathbf{Z}}$, there are two possibilities:
– either all the segments C_i for $i \in E_1$ have horizontal supports, and we say that C has the *type H*;
– or all the segments C_i for $i \in E_1$ have vertical supports, and we say that C has the *type V*.

Theorem 3.2. *Let $\mathscr{C}$, $\mathscr{D}$ be coverings of $\mathbf{R}^2$ by sets of complete folding curves which satisfy the local isomorphism property. Then $\mathscr{C}$ and $\mathscr{D}$ are locally isomorphic if and only if each curve of $\mathscr{C}$ and each curve of $\mathscr{D}$ have the same type and have locally isomorphic associated sequences.*

Remark. In particular, if a covering of $\mathbf{R}^2$ by a set of complete folding curves satisfies the local isomorphism property, then all the curves have the same type and have locally isomorphic associated sequences.

Proof of Theorem 3.2. First we show that the condition is necessary. Let $C=(C_i)_{i\in\mathbf{Z}}$ be a curve of $\mathscr{C}$ and let $D=(D_i)_{i\in\mathbf{Z}}$ be a curve of $\mathscr{D}$. Consider a finite subcurve F of C.

As $\mathscr{C}$ satisfies the local isomorphism property, there exists an integer n such that each square $[x, x+n] \times [y, y+n]$ with $x, y \in \mathbf{Z}$ contains a subcurve of a curve of $\mathscr{C}$ which is parallel to F. As $\mathscr{C}$ is locally isomorphic to $\mathscr{D}$, each square $[x, x+n] \times [y, y+n]$ with $x, y \in \mathbf{Z}$ also contains a subcurve of a curve of $\mathscr{D}$ which is parallel to F. According to Theorem 3.1, there exist $x, y \in \mathbf{Z}$ such that $[x, x+n] \times [y, y+n]$ is covered by D. It follows that D contains a subcurve which is parallel to F. In particular, the folding sequence associated to F is a subword of the folding sequence associated to D.

Now, take for F a 3-folding subcurve $(C_{i+1}, \dots, C_{i+8})$ of C, and consider $j \in \mathbf{Z}$ such that $(D_{j+1}, \dots, D_{j+8})$ is parallel to F. Then we have $i \in E_1(C)$ and $j \in E_1(D)$. It follows that C and D have the same type.

It remains to be proved that the condition is sufficient. We show that, for each finite set $\mathscr{E}$ of tiles, if $\mathscr{C}$ contains the image of $\mathscr{E}$ under a translation, then $\mathscr{D}$ contains the image of $\mathscr{E}$ under a translation; the converse can be proved in the same way.

As $\mathscr{C}$ satisfies the local isomorphism property, there exists an integer m such that each square $[a, a+m] \times [b, b+m]$ contains a set of tiles of $\mathscr{C}$ which is the image of $\mathscr{E}$ under a translation. By Theorem 3.1, there exists an integer n such that each n-folding curve contains a covering of a square $[a, a+m] \times [b, b+m]$. For such an n, each n-folding subcurve of a curve of $\mathscr{C}$ contains the image of $\mathscr{E}$ under a translation.

We can take $n \geq 3$. Then each n-folding subcurve F of a curve of $\mathscr{D}$ is parallel or opposite to a subcurve of a curve of $\mathscr{C}$ since each curve of $\mathscr{C}$ and each curve of $\mathscr{D}$ have the same type and have locally isomorphic associated sequences. Now, it follows from Proposition 2.6 applied to $\mathscr{C}$ that such an F is parallel to a subcurve of a curve of $\mathscr{C}$, and therefore contains the image of $\mathscr{E}$ under a translation. ■

For any disjoint complete folding curves C, D, we call *boundary between C and D* the set of all points of $\mathbf{Z}^2$ which are vertices of two segments of C and two segments of D.

Proposition 3.3. *Let $n \geq 1$ be an integer and let $\mathscr{C}$ be a covering of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. Then the curves of $\mathscr{C}$ define the same E_n, and their n-th derivatives form a covering of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. If the boundary between two curves $C, D \in \mathscr{C}$ is nonempty, then the boundary between $C^{(n)}$ and $D^{(n)}$ is nonempty.*

Proof. By induction, it suffices to show the Proposition for $n = 1$. Consequently, it suffices to prove that, for each covering $\mathscr{C}$ of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property, if the boundary between two curves $C, D \in \mathscr{C}$ is nonempty, then we have $E_1(C) = E_1(D)$, the curves $C^{(1)}$ and $D^{(1)}$ are disjoint and the boundary between $C^{(1)}$ and $D^{(1)}$ is nonempty. Then each point of $E_1(\mathscr{C})$ will be an endpoint of 4 segments of $\mathscr{C}^{(1)} = \{C^{(1)} \mid C \in \mathscr{C}\}$ since it is an endpoint of 4 segments of $\mathscr{C}$, and $\mathscr{C}^{(1)}$ will satisfy the local isomorphism property like $\mathscr{C}$.

We write $C = (C_i)_{i \in \mathbf{Z}}$ and $D = (D_i)_{i \in \mathbf{Z}}$. We denote by $S = (\zeta_i)_{i \in \mathbf{Z}}$ and $T = (\eta_i)_{i \in \mathbf{Z}}$ the associated sequences. We consider a point X which belongs to the boundary between C and D. We write $A = (C_{i-4}, \dots, C_{i+3})$ and $B = (D_{j-4}, \dots, D_{j+3})$, where i (resp. j) is the integer such that X is the common endpoint of C_i and C_{i+1} (resp. D_j and D_{j+1}). As $\mathscr{C}$ satisfies the local isomorphism property, it follows from Theorem 3.1 applied to C that there exist a translation τ of $\mathbf{R}^2$ and two sequences $A' = (C_{k-4}, \dots, C_{k+3})$ and $B' = (C_{l-4}, \dots, C_{l+3})$ such that $\tau(A) = A'$ and $\tau(B) = B'$.

If X belongs to $F_0(C)$, then we have $i \in F_0(S)$, and therefore $\zeta_{i-4} = -\zeta_{i-2} = \zeta_i = -\zeta_{i+2}$, which implies $\zeta_{k-4} = -\zeta_{k-2} = \zeta_k = -\zeta_{k+2}$ and $k \in F_0(S)$. Consequently, we have $\tau(X) \in F_0(C)$ and $l \in F_0(S)$, which implies $\zeta_{l-4} = -\zeta_{l-2} = \zeta_l = -\zeta_{l+2}$ and $\eta_{j-4} = -\eta_{j-2} = \eta_j = -\eta_{j+2}$. It follows that X belongs to $F_0(D)$. We show in the same way that X belongs to $F_0(C)$ if it belongs to $F_0(D)$. Consequently, we have $F_0(C) = F_0(D)$.

If X belongs to $F_0(C) = F_0(D)$, then the segment of $C^{(1)}$ obtained from (C_i, C_{i+1}) and the segment of $D^{(1)}$ obtained from (D_j, D_{j+1}) have supports which are opposite edges of a square of center X and width $\sqrt{2}$. Moreover, the segments of $C^{(1)}$ obtained from (C_k, C_{k+1}) and (C_l, C_{l+1}) have supports which are opposite edges of a square of center $\tau(X)$ and width $\sqrt{2}$. By Lemma 2.3, a third edge of the second square is the support of a segment of $C^{(1)}$ obtained from one of the pairs (C_{k-2}, C_{k-1}), (C_{k+2}, C_{k+3}), (C_{l-2}, C_{l-1}), (C_{l+2}, C_{l+3}). Consequently, a third edge of the first square is the support of a segment of $C^{(1)}$ obtained from one of the pairs (C_{i-2}, C_{i-1}), (C_{i+2}, C_{i+3}), or the support of a segment of $D^{(1)}$ obtained from one of the pairs (D_{j-2}, D_{j-1}), (D_{j+2}, D_{j+3}). In both cases, $C^{(1)}$ and $D^{(1)}$ have a common vertex.

If X belongs to $E_1(C) = E_1(D)$, then X is a common vertex of $C^{(1)}$ and $D^{(1)}$. Moreover, the two segments of $C^{(1)}$ and the two segments of $D^{(1)}$ which have X as an endpoint all have different supports since they are the images under τ^{-1} of the four segments of $C^{(1)}$ which have $\tau(X)$ as an endpoint. As this property is true for each point of the boundary between C and D which belongs to $E_1(C) = E_1(D)$, the curves $C^{(1)}$ and $D^{(1)}$ are disjoint. ∎

Remark. If the boundary between two curves $C, D \in \mathscr{C}$ is finite, then it contains a point of E_∞. Otherwise, for n large enough, it would contain no point of E_n, and the boundary between $C^{(n)}$ and $D^{(n)}$ would be empty, contrary to Proposition 3.3.

If two disjoint complete folding curves C, D have the same type and if $E_1(C) = E_1(D)$, then we have $\sigma_C(X) = \sigma_D(X)$ for each point X of the boundary between C and D. Consequently, for each covering $\mathscr{C}$ of $\mathbf{R}^2$ by a set of complete folding curves which have the same type and define the same set E_1, there exists a unique function $\sigma_{\mathscr{C}} : \mathbf{Z}^2 \to \{-1, +1\}$ such that $\sigma_{\mathscr{C}}(X) = \sigma_C(X)$ for each curve $C \in \mathscr{C}$ and each vertex X of C.

Lemma 3.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be coverings of $\mathbf{R}^2$ by sets of complete folding curves. Suppose that all the curves have the same type, define the same sets E_k and have locally isomorphic associated sequences. Then, for each $n \in \mathbf{N}^*$, each $X \in \mathbf{Z}^2 - E_{2n-1}$ and each $U \in 2^n\mathbf{Z}^2$, we have $\sigma_{\mathscr{C}}(X) = \sigma_{\mathscr{D}}(U + X)$.*

Proof. For each integer $k \leq 2n - 1$, we have $F_k + 2^n\mathbf{Z}^2 = F_k$ and therefore $U + F_k = F_k$. In particular, X and $U + X$ belong to F_m for the same integer $m \leq 2n - 2$. As each curve of $\mathscr{C}$ and each curve of $\mathscr{D}$ have the same type and define the same E_1, the map $Z \to U + Z$ induces a bijection from the set of all supports of segments of $\mathscr{C}$ to the set of all supports of segments of $\mathscr{D}$ which respects the orientation of the segments.

We consider a curve $C = (C_i)_{i \in \mathbf{Z}} \in \mathscr{C}$ indexed in such a way that the terminal point of C_{2^m} is X, and a curve $D = (D_i)_{i \in \mathbf{Z}} \in \mathscr{D}$ indexed in such a way that the support of D_{2^m} is $U + S$, where S is the support of C_{2^m}. The initial point Y of C_1 and the initial point Z of D_1 belong to E_{m+1} since X and $U + X$ belong to F_m.

The sequences $(\zeta_i)_{i \in \mathbf{Z}}$ and $(\eta_i)_{i \in \mathbf{Z}}$ associated to C and D are locally isomorphic. Consequently, we have $(\zeta_1, \ldots, \zeta_{2^m-1}) = (\eta_1, \ldots, \eta_{2^m-1})$ since X and $U + X$ belong to F_m, and therefore $(D_1, \ldots, D_{2^m}) = U + (C_1, \ldots, C_{2^m})$ and $Z = U + Y$. As $U + F_{m+1} = F_{m+1}$, it follows that Y and Z both belong to F_{m+1}, or both belong to E_{m+2}. In both cases, we have $(\zeta_1, \ldots, \zeta_{2^{m+1}-1}) = (\eta_1, \ldots, \eta_{2^{m+1}-1})$ since C is locally isomorphic to D, and therefore $\sigma_{\mathscr{C}}(X) = \zeta_{2^m} = \eta_{2^m} = \sigma_{\mathscr{D}}(U + X)$. ■

Theorem 3.5. *Let $\mathscr{C}$ be a covering of $\mathbf{R}^2$ by a set of complete folding curves. Then $\mathscr{C}$ satisfies the local isomorphism property if and only if all the curves have the same type, define the same sets E_k and have locally isomorphic associated sequences.*

Proof. The condition is necessary according to Proposition 3.3 and the remark after Theorem 3.2. Now we show that it is sufficient.

For each $X = (x, y) \in \mathbf{Z}^2$ and each $k \in \mathbf{N}^*$, we write $S_{X,k} = [x, x + k] \times [y, y + k]$, and we denote by $\mathscr{E}_{X,k}$ the set of all segments of $\mathscr{C}$ whose supports are contained in $S_{X,k}$. It suffices to prove that, for each $X \in \mathbf{Z}^2$ and each $k \in \mathbf{N}^*$, there exists $l \in \mathbf{N}^*$ such that each $S_{Y,l}$ contains some $Z \in \mathbf{Z}^2$ with $\mathscr{E}_{Z,k} = (Z - X) + \mathscr{E}_{X,k}$.

We consider the largest integer m such that $S_{X,k}$ contains a point of F_m, and an integer n such that $m \leq 2n - 2$. For each $U \in 2^n\mathbf{Z}^2$, we have, according to Lemma 3.4, $\sigma_{\mathscr{C}}(U + Y) = \sigma_{\mathscr{C}}(Y)$ for each $Y \in \mathbf{Z}^2 - E_{2n-1}$, and in particular for each $Y \in S_{X,k}$ which does not belong to E_∞.

If $S_{X,k} \cap E_\infty = \emptyset$, it follows that $\mathscr{E}_{U+X,k} = U + \mathscr{E}_{X,k}$ for each $U \in 2^n\mathbf{Z}^2$, since the curves of $\mathscr{C}$ have the same type. Then we have the required property for $l = 2^n$.

If $S_{X,k}$ contains the unique point W of E_∞, then we still have $\mathscr{E}_{U+X,k} = U + \mathscr{E}_{X,k}$ for each $U \in 2^n\mathbf{Z}^2$ such that $\sigma_{\mathscr{C}}(U + W) = \sigma_{\mathscr{C}}(W)$, since the curves of $\mathscr{C}$ have the same type. Moreover, we have $E_{2n} = W + 2^n\mathbf{Z}^2$ since W belongs to E_{2n}. We consider $V \in 2^n\mathbf{Z}^2$ such that $V + W \in F_{2n}$ and $\sigma_{\mathscr{C}}(V + W) = \sigma_{\mathscr{C}}(W)$. We have $V + W + 2^{n+1}\mathbf{Z}^2 = \{Y \in \mathbf{Z}^2 \mid Y \in F_{2n} \text{ and } \sigma_{\mathscr{C}}(Y) = \sigma_{\mathscr{C}}(W)\}$, and therefore $\mathscr{E}_{U+V+X,k} = U + V + \mathscr{E}_{X,k}$ for each $U \in 2^{n+1}\mathbf{Z}^2$. Consequently we have the required property for $l = 2^{n+1}$. ■

Now we consider the coverings which consist of one complete folding curve. The following result is a particular case of Theorem 3.5. It applies to the folding curves considered in Theorem 3.7 and Example 3.8 below.

COROLLARY 3.6. *Any covering of $\mathbf{R}^2$ by a complete folding curve satisfies the local isomorphism property.*

THEOREM 3.7. *For each complete folding sequence S, there exists a covering of $\mathbf{R}^2$ by a complete folding curve associated to a sequence which is locally isomorphic to S.*

PROOF. Consider a curve $C = (C_i)_{i \in \mathbf{Z}}$ associated to S. Let Ω consist of the finite curves parallel to subcurves of C, such that $(0,0)$ is one of their vertices. For each $F \in \Omega$, denote by $N(F)$ the largest integer n such that F contains a covering of $[-n, +n]^2$.

For any $F, G \in \Omega$, write $F < G$ if $F \subset G$ and $N(F) < N(G)$. As S satisfies the local isomorphism property, the union of any strictly increasing sequence in Ω is a covering of $\mathbf{R}^2$ by a complete folding curve associated to a sequence which is locally isomorphic to S.

It remains to be proved that, for each $F \in \Omega$, there exists $G > F$ in Ω. According to Proposition 2.6, there is an integer m such that each $(C_{i+1}, \dots, C_{i+m})$ contains a curve parallel to F. By Theorem 3.1, there exists a finite subcurve K of C which contains a covering of a square $X + [-n, +n]^2$ with $X \in \mathbf{Z}^2$ and $n = m + N(F) + 1$.

Let i be an integer such that X is the terminal point of C_i. Consider a curve H parallel to F and contained in $(C_{i+1}, \dots, C_{i+m})$. Denote by τ the translation such that $\tau(F) = H$. Then $\tau^{-1}(X)$ belongs to $[-m, +m]^2$ since

$\tau((0,0))$ belongs to $X+[-m,+m]^2$. For $G=\tau^{-1}(K)$, we have $F\subset G$ and G covers $\tau^{-1}(X)+[-n,+n]^2$. In particular, G covers $[-N(F)-1,+N(F)+1]^2$. ■

REMARK. There is no covering of $\mathbf{R}^2$ by a curve associated to $(\overline{S},+1,S)$ or to $(\overline{S},-1,S)$, where S is an ∞-folding sequence. In fact, such a curve C would contain 4 segments having the point of $E_\infty(C)$ as an endpoint, and $E_\infty(\overline{S},+1,S)$ (resp. $E_\infty(\overline{S},-1,S)$) would contain 2 integers.

EXAMPLE 3.8. There exists a covering of $\mathbf{R}^2$ by a complete folding curve defined in an effective way.

PROOF. For each $(x,y)\in\mathbf{Z}^2$ and each $n\in\mathbf{N}^*$, we say that a $(2n)$-folding (resp. $(2n+1)$-folding) curve C covers the isosceles right triangle $T=\langle(x,y),(x+2^n,y),(x,y+2^n)\rangle$ (resp. $T=\langle(x,y),(x+2^n,y+2^n),(x+2^{n+1},y)\rangle$) if it satisfies the following conditions (cf. Fig. 6A, 6B, 6C):
- each interval $[(u,v),(u+1,v)]$ or $[(u,v),(u,v+1)]$ with $u,v\in\mathbf{Z}$, contained in the interior of T, is the support of a segment of C;
- among the intervals $[(u,v),(u+1,v)]$ or $[(u,v),(u,v+1)]$ with $u,v\in\mathbf{Z}$, contained in the same vertical or horizontal edge of T, alternatively one over two is the support of a segment of C;
- no interval $[(u,v),(u+1,v)]$ or $[(u,v),(u,v+1)]$ with $u,v\in\mathbf{Z}$, contained in the exterior of T, is the support of a segment of C;

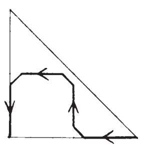

Fig. 6A

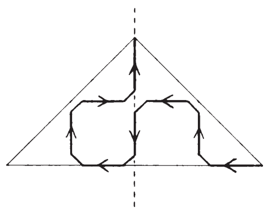

Fig. 6B

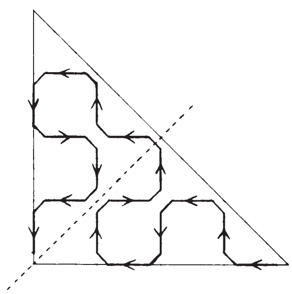

Fig. 6C

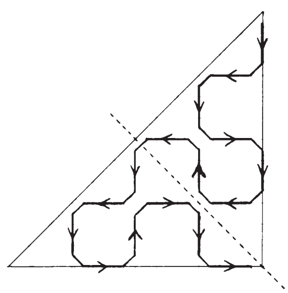

Fig. 6D

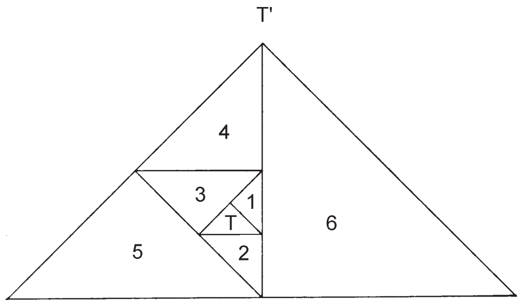

Fig. 6E

- the vertex of the right angle of T is the initial or the terminal point of C;
- the vertex of one of the acute angles of T is the initial or the terminal point of C.

We extend this definition to the isosceles right triangles with vertices in $\mathbf{Z}^2$ obtained from T by rotations of angles $\pi/2$, π, $3\pi/2$ (cf. Fig. 6D).

Now, let T be one of the isosceles right triangles considered, let k be an integer, let C be a k-folding curve which covers T, and let S be the sequence associated to C. If the initial (resp. terminal) point of C is the vertex of the right angle of T, we associate to $(\overline{S}, +1, S)$ and $(\overline{S}, -1, S)$ (resp. $(S, +1, \overline{S})$ and $(S, -1, \overline{S})$) two $(k+1)$-folding curves C_1 and C_2 which contain C. In both cases, the parts of C_1 and C_2 which correspond to $\overline{S}$ respectively cover the isosceles right triangles T_1 and T_2 which have one edge of their right angle in common with T, in such a way that C_1 and C_2 respectively cover the isosceles right triangles $T \cup T_1$ and $T \cup T_2$ (cf. Fig. 6B, 6C and 6D).

For each $n \in \mathbf{N}^*$ and each triangle $T = \langle (x, y), (x + 2^n, y + 2^n), (x + 2^{n+1}, y) \rangle$ with $(x, y) \in \mathbf{Z}^2$, repeat six times the operation above according to Figure 6E, beginning with a $(2n+1)$-folding curve C which covers T. Then we obtain a $(2n+7)$-folding curve C' which contains C. Moreover, C' covers a triangle $T' = \langle (x', y'), (x' + 2^{n+3}, y' + 2^{n+3}), (x' + 2^{n+4}, y') \rangle$ with $(x', y') \in \mathbf{Z}^2$ and T contained in the interior of T'. By iterating this process, we obtain a covering of $\mathbf{R}^2$ by a complete folding curve. ■

PROPOSITION 3.9. *Let $\mathscr{C}$ be a covering of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. For each covering $\mathscr{D}$ of $\mathbf{R}^2$, the following properties are equivalent:*

1) *$\mathscr{D}$ consists of complete folding curves, and the curves of $\mathscr{C} \cup \mathscr{D}$ have the same type, define the same sets E_k and have locally isomorphic associated sequences.*
2) *$\mathscr{C} = \mathscr{D}$, or $E_\infty(\mathscr{C}) \neq \emptyset$ and $\mathscr{C}$, $\mathscr{D}$ only differ in the way to connect the four segments which have the unique point of $E_\infty(\mathscr{C})$ as an endpoint.*

PROOF. If 1) is true, then 2) is also true since Lemma 3.4 implies $\sigma_{\mathscr{C}}(X) = \sigma_{\mathscr{D}}(X)$ for each $X \in \mathbf{Z}^2 - E_\infty$. Conversely, if 2) is true, then 1) follows from the remark after Corollary 1.9. ■

REMARK. If 1) and 2) are true, then $\mathscr{D}$ satisfies the local isomorphism property by Theorem 3.5, and $\mathscr{D}$ is locally isomorphic to $\mathscr{C}$ by Theorem 3.2.

THEOREM 3.10. *For each complete folding curve C, if $E_\infty(C)$ is empty or if the unique point of $E_\infty(C)$ is a vertex of C, then C is contained in a unique covering of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. Otherwise, C is contained in exactly two such coverings, which only differ in the way to connect the four segments having the unique point of E_∞ as an endpoint.*

PROOF. It suffices to show that C is contained in a covering of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. In fact, for any such coverings $\mathscr{C}$, $\mathscr{D}$, Proposition 3.3 and the remark after Theorem 3.2 imply that each curve of $\mathscr{C}$ and each curve of $\mathscr{D}$ have the same type, define the same sets E_k and have locally isomorphic associated sequences. Then the Theorem is a consequence of Proposition 3.9 and the Remark just after.

For each $m \in \mathbf{N}^*$, denote by $\mathscr{C}_m$ the set of all segments of C with supports in $[-m, +m]^2$. Let Ω consist of the pairs $(\mathscr{E}, m)$, where $m \in \mathbf{N}^*$ and $\mathscr{E}$ is a covering of $[-m, +m]^2$ containing $\mathscr{C}_m$, for which there exists $X \in \mathbf{Z}^2$ such that $X + \mathscr{E} \subset C$.

For any $(\mathscr{E}, m), (\mathscr{F}, n) \in \Omega$, write $(\mathscr{E}, m) < (\mathscr{F}, n)$ if $\mathscr{E} \subset \mathscr{F}$ and $m < n$. If $(\mathscr{F}, n) \in \Omega$ and $m \in \{1, \dots, n-1\}$, then we have $(\mathscr{E}, m) \in \Omega$ and $(\mathscr{E}, m) < (\mathscr{F}, n)$ for the set $\mathscr{E}$ of all segments of $\mathscr{F}$ with supports in $[-m, +m]^2$.

First we show that, for each $m \in \mathbf{N}^*$, there exists $(\mathscr{E}, m) \in \Omega$. We can take m large enough so that C contains some segments with supports in $[-m, +m]^2$. We consider a finite curve $A \subset C$ which contains all these segments. According to Proposition 2.6, there exists an integer k such that each subcurve of length k of C contains a curve parallel to A.

By Theorem 3.1, C contains a covering of a square $X + [-k - 2m, +k + 2m]^2$ with $X \in \mathbf{Z}^2$. The covering of $X + [-k, +k]^2$ extracted from C contains a curve of length $\geq k$, which itself contains a curve B parallel to A. We consider $Y \in \mathbf{Z}^2$ such that $Y + A = B$. We have $Y \in X + [-k - m, +k + m]^2$, because A contains a point of $[-m, +m]^2$ and B is contained in $X + [-k, +k]^2$. Consequently, C contains a covering $\mathscr{F}$ of $Y + [-m, +m]^2$. We have $(\mathscr{E}, m) \in \Omega$ for $\mathscr{E} = -Y + \mathscr{F}$ since $\mathscr{F}$ contains the set $Y + \mathscr{C}_m$ of all segments of B with supports in $Y + [-m, +m]^2$.

Now, according to König's Lemma, Ω contains a strictly increasing sequence $(\mathscr{E}_m, m)_{m \in \mathbf{N}^*}$. The union $\mathscr{C}$ of the sets $\mathscr{E}_m$ is a covering of $\mathbf{R}^2$ which contains C. Any finite curve $A \subset \mathscr{C}$ is parallel to a subcurve of C since it is contained in one of the sets $\mathscr{E}_m$. In particular, $\mathscr{C}$ is a covering of $\mathbf{R}^2$ by a set of complete folding curves. It remains to be proved that $\mathscr{C}$ satisfies the local isomorphism property.

It suffices to show that, for each $m \in \mathbf{N}^*$, there exists $n \in \mathbf{N}^*$ such that each square $X + [-n, +n]^2$ contains the image of $\mathscr{E}_m$ under a translation. We consider a point $Y \in \mathbf{Z}^2$ such that $Y + \mathscr{E}_m \subset C$, and a finite curve $A \subset C$ which contains $Y + \mathscr{E}_m$. By Proposition 2.6, there exists $n \in \mathbf{N}^*$ such that each subcurve of length n of C, and therefore each subcurve of length n of a curve of $\mathscr{E}$, contains a curve parallel to A. Each square $X + [-n, +n]^2$ contains a subcurve of length n of a curve of $\mathscr{E}$, and therefore contains a curve parallel to A and the image of $\mathscr{E}_m$ under a translation. ■

REMARK. It follows that any ∞-folding curve C is contained in exactly two coverings of $\mathbf{R}^2$ by sets of complete folding curves which satisfy the local isomorphism property. These coverings only differ in the way to connect the four segments whose supports contain the origin of C.

REMARK. Let C be a complete folding curve and let S be the associated sequence. By Theorem 3.10, C is contained in a covering $\mathscr{C}$ of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. According to Theorem 3.7, there is also a covering $\mathscr{D}$ of $\mathbf{R}^2$ by a complete folding curve D associated to a sequence which is locally isomorphic to S. Moreover, $\mathscr{D}$ satisfies the local isomorphism property by Corollary 3.6. If we choose D with the same type as C, then $\mathscr{C}$ and $\mathscr{D}$ are locally isomorphic according to Theorem 3.2. On the other hand, $\mathscr{C}$ and $\mathscr{D}$ do not contain the same number of curves if $\{C\}$ is not a covering of $\mathbf{R}^2$.

COROLLARY 3.11. 1) *There exist* 2^ω *pairwise not locally isomorphic coverings of* $\mathbf{R}^2$ *by sets of complete folding curves which satisfy the local isomorphism property.*

2) *If* $\mathscr{C}$ *is such a covering, then there exist* 2^ω *isomorphism classes of coverings of* $\mathbf{R}^2$ *which are locally isomorphic to* $\mathscr{C}$.

PROOF. For any complete folding sequences S, T, consider two curves C, D associated to S, T which have the same type. By Theorem 3.10, there exist two coverings $\mathscr{C}$, $\mathscr{D}$ of $\mathbf{R}^2$, respectively containing C, D, by sets of complete folding curves which satisfy the local isomorphism property.

By Theorem 3.2, $\mathscr{C}$ and $\mathscr{D}$ are locally isomorphic if and only if S and T are locally isomorphic. In particular, the property 1) above follows from the property 1) of Theorem 1.11.

Moreover, if $\mathscr{C}$ and $\mathscr{D}$ are isomorphic, then C is isomorphic to one of the curves of $\mathscr{D}$, and S is isomorphic to one of their associated sequences. Consequently, the property 2) above follows from the property 2) of Theorem 1.11, since any covering of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property contains at most countably many curves (and in fact at most 6 curves by Theorem 3.12 below). ■

Now we investigate the number of curves in a covering of $\mathbf{R}^2$ by complete folding curves.

Theorem 3.12. *If a covering of $\mathbf{R}^2$ by a set of complete folding curves satisfies the local isomorphism property, then it contains at most* 6 *curves.*

Proof. For each $X \in \mathbf{R}^2$, each complete folding curve C and each $n \in \mathbf{N}$, write $\delta_n(X, C) = 4^{-n} \cdot \inf_{S \in E_{4n}(C)} d(X, S)$. We have $\delta_n(X, C) = \delta_0(X_n, C^{(4n)})$, where X_n is the image of X in a representation of $C^{(4n)}$ which gives the length 1 to the supports of the segments.

Now consider $R \in E_0(C)$ and $S, T \in E_4(C)$ which are joined by a 4-folding subcurve of C having R as a vertex. Then, according to Figure 3C, we have $\inf(d(X, S), d(X, T)) \leq \sqrt{(d(X, R) + 3)^2 + 2^2} = \sqrt{d(X, R)^2 + 6d(X, R) + 13}$; this maximum is reached with the second of the four 4-folding curves of Figure 3C, for each point X which is at the left and at a distance ≥ 3 from the middle of the segment ST.

Consequently we have $\delta_1(X, C) \leq (1/4)\sqrt{\delta_0(X, C)^2 + 6\delta_0(X, C) + 13}$. For $\delta_0(X, C) \leq 1.16$, it follows

$$\delta_1(X, C) \leq (1/4)\sqrt{(1.16)^2 + 6(1.16) + 13} < 1.16.$$

For $\delta_0(X, C) \geq 1.16$, it follows

$$\begin{aligned}\delta_1(X, C)/\delta_0(X, C) &\leq (1/4)\sqrt{1 + 6/\delta_0(X, C) + 13/\delta_0(X, C)^2} \\ &\leq (1/4)\sqrt{1 + 6/(1.16) + 13/(1.16)^2} < 0.995.\end{aligned}$$

For each complete folding curve C, each $X \in \mathbf{R}^2$ and each $n \in \mathbf{N}$, the argument above applied to $C^{(n)}$ gives $\delta_{n+1}(X, C) < 1.16$ if $\delta_n(X, C) < 1.16$ and $\delta_{n+1}(X, C) < (0.995)\delta_n(X, C)$ if $\delta_n(X, C) \geq 1.16$. In particular, we have $\delta_n(X, C) < 1.16$ for n large enough.

For each covering $\{C_1, \dots, C_k\}$ of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property, consider $X \in \mathbf{R}^2$ and $n \in \mathbf{N}^*$ such that $\delta_n(X, C_i) < 1.16$ for $1 \leq i \leq k$. Let Y be the image of X in a representation of $C_1^{(4n)}, \dots, C_k^{(4n)}$ which gives the length 1 to the supports of the segments. Then we have $\delta_0(Y, C_i^{(4n)}) < 1.16$ for $1 \leq i \leq k$.

Write $Y = (y, z)$ and consider $u, v \in \mathbf{Z}$ such that $|y - u| \leq 1/2$ and $|z - v| \leq 1/2$. Then each $C_i^{(4n)}$ has a vertex among the points listed below, since no other point $(w, x) \in \mathbf{Z}^2$ satisfies $d((w, x), (y, z)) < 1.16$:

$$(u, v), (u-1, v), (u+1, v), (u, v-1), (u, v+1)$$
$$\text{if } |y - u| \leq 0.16 \text{ and } |z - v| \leq 0.16;$$

$$(u-1, v-1), (u-1, v), (u-1, v+1), (u, v-1), (u, v), (u, v+1)$$
$$\text{if } y < u - 0.16;$$

$$(u, v-1), (u, v), (u, v+1), (u+1, v-1), (u+1, v), (u+1, v+1)$$
$$\text{if } y > u + 0.16;$$

$$(u-1, v-1), (u, v-1), (u+1, v-1), (u-1, v), (u, v), (u+1, v)$$
$$\text{if } z < v - 0.16;$$

$$(u-1, v), (u, v), (u+1, v), (u-1, v+1), (u, v+1), (u+1, v+1),$$
$$\text{if } z > v + 0.16.$$

In the first case, there exist 12 intervals of length 1 with endpoints in $\mathbf{Z}^2$ which have exactly one endpoint among the 5 points considered. In each of the four other cases, there exist 10 intervals of length 1 with endpoints in $\mathbf{Z}^2$ which have exactly one endpoint among the 6 points considered.

In all cases, each $C_i^{(4n)}$ has at least 2 segments with supports among these intervals. It follows $k \leq 6$ in the first case and $k \leq 5$ in each of the four other cases since, by Proposition 3.3, $C_1^{(4n)}, \dots, C_k^{(4n)}$ are disjoint. ■

Example 3.13 (curves associated to the positive folding sequence). The *positive folding sequence* mentioned in [4, p. 192] is the ∞-folding sequence R obtained as the direct limit of the n-folding sequences R_n defined with $R_{n+1} = (R_n, +1, \overline{R_n})$ for each $n \in \mathbf{N}$. According to [3, Th. 4, p. 78], or by Theorem 3.15 below, there exists a covering $\mathscr{C}$ of $\mathbf{R}^2$ by 2 complete folding curves C, D associated to $S = (\overline{R}, +1, R)$ and having the same type (see Fig. 7). We have $E_\infty(C) = E_\infty(D) = \{(0,0)\}$, and therefore $E_k(C) = E_k(D)$ for each $k \in \mathbf{N}$. It follows from Theorem 3.5 that $\mathscr{C}$ satisfies the local isomorphism property.

Example 3.14 (curves associated to the alternating folding sequence). The *alternating folding sequence* described in [4, p. 134] is the ∞-folding sequence R obtained as the direct limit of the n-folding sequences R_n defined with $R_{n+1} = (R_n, (-1)^{n+1}, \overline{R_n})$ for each $n \in \mathbf{N}$. Contrary to Example 3.13, there exists no covering of $\mathbf{R}^2$ by 2 complete folding curves associated to $S = (\overline{R}, +1, R)$.

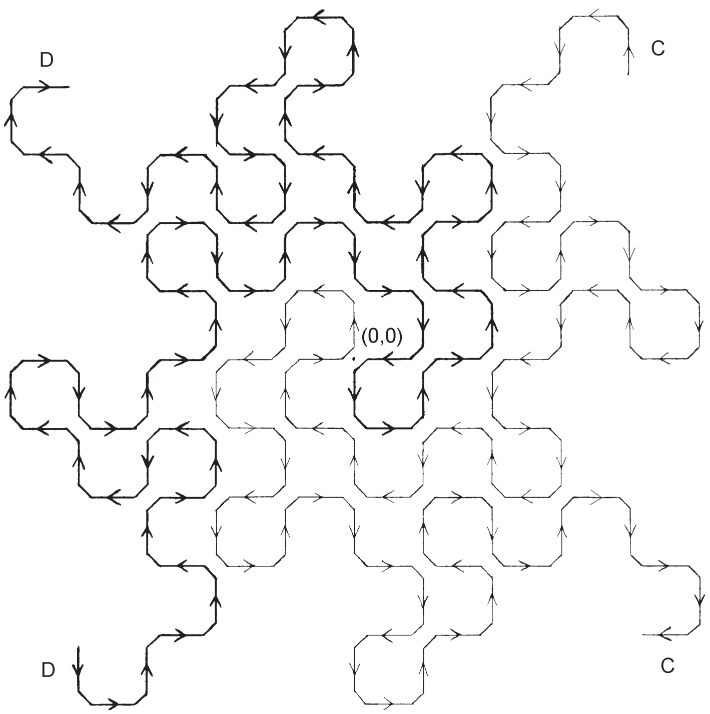

Fig. 7

Now, let T be the complete folding sequence obtained as the direct limit of the sequences R_{2n}, where each R_{2n} is identified to its second copy in $R_{2n+2} = (R_{2n}, -1, \overline{R_{2n}}, +1, R_{2n}, +1, \overline{R_{2n}})$. Then there exists (cf. Fig. 8) a covering $\mathscr{C}$ of $\mathbf{R}^2$ by 2 curves associated to S, 2 curves associated to T and 2 curves associated to $\overline{T}$, with all the curves having the same type.

The 6 curves are associated to locally isomorphic sequences. The point $(0,0)$ belongs to E_∞ in the 2 curves which contain it. For each $n \in \mathbf{N}$, and in each of the 2 curves which contain it, the point $(2^n, 0)$ (resp. $(-2^n, 0)$, $(0, 2^n)$, $(0, -2^n)$) belongs to F_{2n}, while the point $(2^n, 2^n)$ (resp. $(-2^n, 2^n)$, $(2^n, -2^n)$, $(-2^n, -2^n)$) belongs to F_{2n+1}. Consequently, the 6 curves define the same sets E_k, and $\mathscr{C}$ satisfies the local isomorphism property by Theorem 3.5.

We do not know presently if a covering $\mathscr{C}$ of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property can consist of 3, 4 or 5 curves. If $E_\infty(\mathscr{C}) \neq \varnothing$, then $\mathscr{C}$ consists of 2 or 6 curves according to the Theorem below:

Theorem 3.15. *Let C be a curve associated to $S = (\overline{R}, +1, R)$ or $S = (\overline{R}, -1, R)$ where $R = (a_h)_{h \in \mathbf{N}^*}$ is an ∞-folding sequence. Consider the unique*

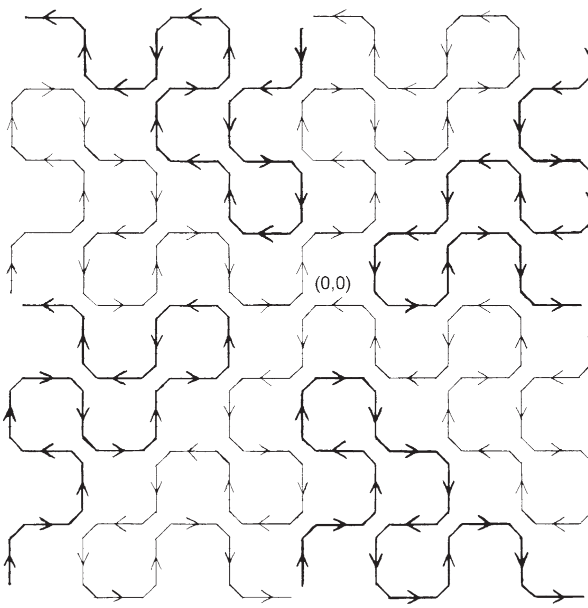

Fig. 8

covering $\mathscr{C} \supset C$ of $\mathbf{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property.

1) *If $\{n \in \mathbf{N} \mid a_{2^n} = (-1)^n\}$ is finite or cofinite, then $\mathscr{C}$ consists of 2 curves associated to S and 4 other curves.*
2) *Otherwise, $\mathscr{C}$ consists of 2 curves associated to S.*

PROOF. As the other cases can be treated in the same way, we suppose that $C = (C_h)_{h \in \mathbf{Z}}$ is a curve associated to $S = (\bar{R}, +1, R) = (a_h)_{h \in \mathbf{Z}}$, and that $E_\infty(C) = \{(0,0)\}$. Then $\mathscr{C}$ contains a curve $D \neq C$ such that $(0,0)$ is a vertex of D. We write $D = (D_h)_{h \in \mathbf{Z}}$ with $(0,0)$ terminal point of D_0 and initial point of D_1. The curve D is associated to S, since it is associated to a sequence $T = (b_h)_{h \in \mathbf{Z}}$ with T locally isomorphic to S, $E_\infty(T) = \{0\}$ and $b_0 = +1$.

If $\{n \in \mathbf{N} \mid a_{2^n} = (-1)^n\}$ is cofinite (resp. finite), we consider an odd (resp. even) integer k such that $a_{2^n} = (-1)^n$ (resp. $a_{2^n} = (-1)^{n+1}$) for $n \geq k$. In order to prove that $\mathscr{C}$ contains 6 curves, it suffices to show that $C^{(k)}$ is contained in a covering of $\mathbf{R}^2$ which satisfies the local isomorphism property and which consists of 6 complete folding curves. Consequently, it suffices to consider the case where $a_{2^n} = (-1)^{n+1}$ for each $n \in \mathbf{N}$. But this case is treated in Example 3.14 (see Fig. 8).

The proof of 2) uses arguments similar to those in the proof of Theorem 3.1. For each $k \in \mathbf{N}^*$, we say that a set $\mathscr{E}$ of disjoint curves covers $L_k =$

$\{(u,v) \in \mathbf{R}^2 \mid |u| + |v| \leq k\}$ if each interval $[(u,v),(u+1,v)]$ or $[(u,v),(u,v+1)]$ with $u,v \in \mathbf{Z}$, contained in L_k, is the support of a segment of one of the curves.

For each set $\mathscr{E}$ of disjoint complete folding curves which have the same type, define the same sets E_n and are associated to locally isomorphic sequences, if $\mathscr{E}^{(2)}$ covers L_k for an integer $k \geq 2$, then $\mathscr{E}$ covers L_{2k-1}, and therefore covers L_{k+1}. By induction, it follows that, if $\mathscr{E}^{(2k)}$ covers L_2 for an integer $k \geq 2$, then $\mathscr{E}$ covers L_{k+2}.

For each $k \in \mathbf{N}$, consider $l \geq 2k$ such that $a_{2^l} = a_{2^{l+1}}$. Then $\{(C_{-3}^{(l)}, \ldots, C_4^{(l)}), (D_{-3}^{(l)}, \ldots, D_4^{(l)})\}$ covers L_2 (see Fig. 7 for the case $a_{2^l} = a_{2^{l+1}} = +1$). Consequently, $\{C^{(l-2k)}, D^{(l-2k)}\}$ covers L_{k+2}, and the same property is true for $\{C, D\}$. ■

A covering of $\mathbf{R}^2$ by a set of complete folding curves can contain more than 6 curves if it does not satisfy the local isomorphism property. For the complete folding sequence T of Example 3.14, Figure 9 gives an example of a covering of $\mathbf{R}^2$ by 4 curves associated to T and 4 curves associated to $-\overline{T}$ (T and $-\overline{T}$ are not locally isomorphic by Corollary 1.9). Anyway, the Theorem below implies that the number of complete folding curves in a covering of $\mathbf{R}^2$ is at most 24:

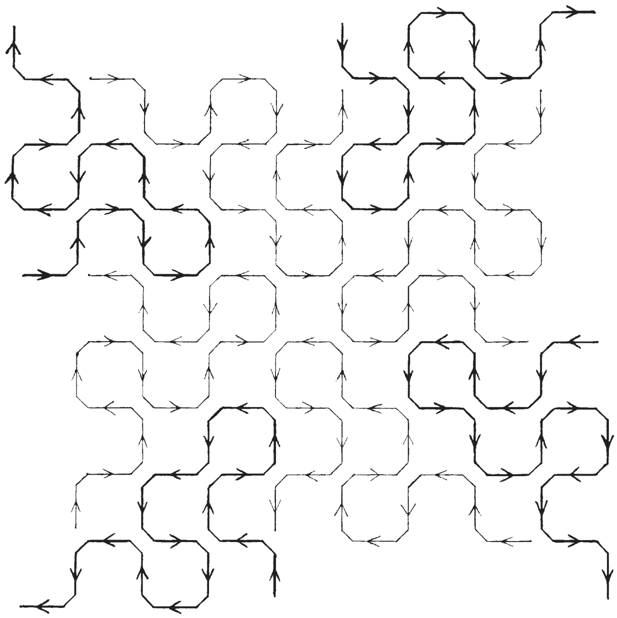

Fig. 9

THEOREM 3.16. *There exist no more than* 24 *disjoint complete folding curves in* $\mathbf{R}^2$.

In the proof of Theorem 3.16, we write $\delta((x, y), (x', y')) = \sup(|x' - x|, |y' - y|)$ for any $(x, y), (x', y') \in \mathbf{R}^2$.

LEMMA 3.16.1. *We have* $\delta(X, Y) \leq 7 \cdot 2^{n-2} - 2$ *for each integer* $n \geq 2$ *and for any vertices* X, Y *of a* $(2n)$*-folding curve.*

PROOF OF LEMMA 3.16.1. The Lemma is true for $n = 2$ according to Figure 3C. Now we show that, if it is true for $n \geq 2$, then it is true for $n + 1$.

Let C be a $(2n + 2)$-folding curve, and let $Z_0, \ldots, Z_{2^{2n+2}}$ be its vertices taken consecutively. Then $(Z_{4i})_{0 \leq i \leq 2^{2n}}$ is the sequence of all vertices of the $(2n)$-folding curve $C^{(2)}$ represented on the same figure as C. It follows from the induction hypothesis applied to $C^{(2)}$ that we have $\delta(X, Y) \leq 2(7 \cdot 2^{n-2} - 2)$ for any $X, Y \in (Z_{4i})_{0 \leq i \leq 2^{2n}}$. Moreover, for each vertex U of C, there exists $V \in (Z_{4i})_{0 \leq i \leq 2^{2n}}$ such that $\delta(U, V) \leq 1$. Consequently, we have $\delta(X, Y) \leq 2(7 \cdot 2^{n-2} - 2) + 2 = 7 \cdot 2^{n-1} - 2$ for any vertices X, Y of C. ■

LEMMA 3.16.2. *Let* $n \geq 2$ *be an integer and let* C *be a finite folding curve with two vertices* X, Y *such that* $\delta(X, Y) \geq 7 \cdot 2^{n-2} - 1$. *Then* C *contains at least* 2^{2n-1} *segments.*

PROOF OF LEMMA 3.16.2. By Lemma 3.16.1, we have $k \geq 2n + 1$ for the smallest integer k such that C is contained in a k-folding curve D. We consider the $(k - 2)$-folding curves D_1, D_2, D_3, D_4 such that $D = (D_1, D_2, D_3, D_4)$. The curve C contains one of the curves D_2, D_3 since the $(k - 1)$-folding curves (D_1, D_2), (D_2, D_3), (D_3, D_4) do not contain C. Consequently, C contains a $(2n - 1)$-folding curve, which consists of 2^{2n-1} segments. ■

PROOF OF THEOREM 3.16. Let $r \geq 2$ be an integer and let $C_1, \ldots, C_r$ be disjoint complete folding curves. Consider an integer k and some vertices $X_1, \ldots, X_r$ of $C_1, \ldots, C_r$ belonging to $[-k, +k]^2$.

Now consider an integer $n \geq 2$ and write $N = 7 \cdot 2^{n-2} + k$. For each $i \in \{1, \ldots, r\}$, there exist some vertices Y_i, Z_i of C_i, with $\delta(0, Y_i) = N$ and $\delta(0, Z_i) = N$, such that $]-N, +N[^2$ contains a subcurve of C_i which has X_i as a vertex and Y_i, Z_i as endpoints.

For each $i \in \{1, \ldots, r\}$, we have $\delta(X_i, Y_i) \geq N - k = 7 \cdot 2^{n-2}$ and $\delta(X_i, Z_i) \geq N - k = 7 \cdot 2^{n-2}$. By Lemma 3.16.2, the part of the subcurve of C_i between X_i and Y_i (resp. between X_i and Z_i) contains at least 2^{2n-1} segments.

As $C_1, \ldots, C_r$ are disjoint, it follows that $]-N, +N[^2$ contains at least $2r \cdot 2^{2n-1} = 2^{2n}r$ supports of segments of $C_1 \cup \cdots \cup C_r$. But $]-N, +N[^2$ only contains $2(2N)(2N - 1) < 8N^2$ intervals of the form $](u, v), (u + 1, v)[$ or $](u, v),$

$(u, v+1)[$ with $u, v \in \mathbf{Z}$. Consequently, we have $2^{2n}r < 8N^2$ and $r < N^2/2^{2n-3} = (7 \cdot 2^{n-2} + k)^2/2^{2n-3}$.

As the last inequality is true for each $n \geq 2$, it follows $r \leq 49/2 < 25$. ∎

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