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Stability of travelling wave solutions for the Landau-Lifshitz equation

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ABSTRACT. We prove that the one dimensional travelling wave solutions corresponding to the Walker wall for the Landau-Lifshitz equation are asymptotically stable for small external magnetic field.

1. Introduction

The Landau-Lifshitz equation was introduced by Landau and Lifshitz in 1935 [8] to describe the motion of the magnetization vectors in ferromagnetic bodies. As the particularly interesting object of study, thin ferromagnetic films have been studied by a number of physicists and engineers and have found various applications in ubiquitous magnetic storage media.

It is well known that a variety of patterns of magnetization vectors appears on thin ferromagnetic films [6]. Generally, the magnetization vector pattern has a line called domain wall. The magnetization vectors change sharply on the neighborhood of the domain wall, and the vectors face to almost opposite directions on the two side of the wall. When there is no external magnetic field, a pattern called the Bloch wall on the thin ferromagnetic film arises which corresponds to the one-dimensional stationary solution of the Landau-Lifshitz equation.

When some external magnetic field h is present, the Landau-Lifshitz equation has a travelling wave solution, which moves at a constant velocity, called the Walker wall. It is well-known [6] that h cannot be arbitrary for the existence of such travelling wave; there is a finite limit (denoted by h_w in this paper) called the Walker limit for the existence of the Walker wall. The explicit formula for the relation of h and velocity is recalled in Section 2. While the velocity of the Walker wall is a monotone increasing function of h for small h, there exists a saturation point of velocity beyond which the velocity is monotone decreasing function of h until the Walker limit. Since stronger external field should produce faster wall motion, the Walker walls beyond

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saturation point are expected to be 'unstable' in some appropriate sense. In physics literature Magyari and Thomas [10] studied the linear stability of the Walker wall and suggested that it is stable for small h while it is unstable for a range of h close to the value corresponding to the maximum velocity. Later, however, the numerical study in [12] indicates that the instability appear to set in even with much smaller h.

Under this circum-stance, we find it worthwhile to rigorously prove the stability of the Walker wall. In this paper we show that the Walker wall is asymptotically stable for any Gilbert loss parameter $\eta > 0$ (see Theorem 1 for the precise statement) when the external magnetic field is sufficiently small.

Regarding the known results with close relevance, Guo and Fengqiu proved the existence of the periodic global solution for the Landau-Lifshitz equation [4, 5]. Tsutsumi studied the Cauchy problem for the noncompact Landau-Lifshitz equation [11]. Carbou and Labbé proved the stability of the Bloch wall with $a = h = 0$ [1] (where these constants are defined in Section 2). They used the semigroup theory for the proof. Furthermore Carbou, Labbé and Trélat proved that the Walker wall can be controlled by some time dependent external field [2]. Our result is closely related to [1], while we point out that our proof is solely based on the energy method and we require the initial data is close in H^1 norm instead of H^2 in [1].

The organization of the paper is as follows. In Section 2 we discuss the derivation of the Walker wall solution from the Landau-Lifshitz equation and state our main stability theorem. In Section 3 we linearize the Landau-Lifshitz equation around the Walker wall solution by using a moving frame. We derive the second order differential operator $\mathscr{L}$ and study the basic properties. In Section 4 we give a proof of our main theorem.

2. Walker wall and its stability result

We regard $\mathbf{R}^2$ as a thin ferromagnetic film. Below the Curie temperature the length of the magnetization vectors are constant which we normalize as 1. Let $m = (m_1, m_2, m_3) : \mathbf{R}^2 \times [0, \infty) \to S^2$, where S^2 is the unit sphere in $\mathbf{R}^3$. We assume $m(x, y, t) = m(x, t)$ for any $(x, y, t) \in \mathbf{R}^2 \times [0, \infty)$. Let $H = (0, h, 0) \in \mathbf{R}^3$ be a constant vector corresponding to the external magnetic field. The micromagnetic energy $E(m)$ is given by the following [7]:

$$E(m) = \frac{1}{2}\int_{\mathbf{R}} |\partial_x m|^2 dx + \frac{a}{2}\int_{\mathbf{R}} m_1^2\, dx + \frac{1}{2}\int_{\mathbf{R}} (1 - m_2^2) dx - \int_{\mathbf{R}} m \cdot H\, dx,$$

where $a > 0$ is a constant. The first term is called the exchange energy which prefers the constant vectors. The magnetization vector m causes a magnetic field called the stray field or demagnetizing field. The second term is the contribution from the stray field. The third term is caused by the crystalline anisotropies of the ferromagnetic material. The fourth term is due to the external magnetic field H. With $h = 0$ note that $m = (0, \pm 1, 0)$ achieve minimum energy and E has the bi-stable structure.

Recall the Landau-Lifshitz equation:

$$\partial_t m = m \times \nabla_{L^2} E(m) + \eta m \times \partial_t m, \tag{1}$$

where $\eta > 0$ is a constant called the Gilbert loss parameter. $\nabla_{L^2} E(m)$ is the L^2-gradient of $E(\cdot)$ at m. The standard functional variation yields

$$\nabla_{L^2} E(m) = -\partial_x^2 m + a m_1 e_1 - m_2 e_2 - H,$$

here $\{e_1, e_2, e_3\}$ are the standard basis in $\mathbf{R}^3$. Let m be smooth and $|m(x,0)| = 1$ for any $x \in \mathbf{R}$. Taking the scalar product between m and (1), we obtain $m \cdot \partial_t m = 0$. Hence

$$|m(x,t)| = 1$$

for any $(x,t) \in \mathbf{R} \times [0,\infty)$. From $|m| = 1$ and $m \cdot \partial_t m = 0$, we have $m \times (m \times \partial_t m) = -\partial_t m$. Thus by substituting $\partial_t m$ to its own right-hand side of (1) we obtain

$$(1 + \eta^2)\partial_t m = m \times \nabla_{L^2} E(m) + \eta m \times (m \times \nabla_{L^2} E(m)). \tag{2}$$

The two equations (2) and (1) are equivalent.

Suppose $h = 0$, that is, there is no external magnetic field. We first find a heteroclinic solution which connects the two energy minima $m = (0, \pm 1, 0)$. To do so, assume that $m_1 = 0$. Then from $m_3^2 = 1 - m_2^2$ we have $(\partial_x m_3)^2 = \dfrac{m_2^2 (\partial_x m_2)^2}{1 - m_2^2}$ and $|\partial_x m|^2 = \dfrac{(\partial_x m_2)^2}{1 - m_2^2}$. From Young's inequality we obtain

$$\begin{aligned} E(m) &= \frac{1}{2}\int_{\mathbf{R}} |\partial_x m|^2 + (1 - m_2^2)dx \geq \int_{\mathbf{R}} |\partial_x m| \sqrt{1 - m_2^2}\, dx \\ &= \int_{\mathbf{R}} \frac{|\partial_x m_2|}{\sqrt{1 - m_2^2}} \sqrt{1 - m_2^2}\, dx = \int_{\mathbf{R}} |\partial_x m_2| dx = 2. \end{aligned} \tag{3}$$

The equality holds if and only if $|\partial_x m| = \sqrt{1 - m_2^2}$ which is $|\partial_x m_2| = 1 - m_2^2$. The inequality (3) shows that 2 is the least energy with the boundary conditions $m_2(\pm\infty) = \mp 1$. Such minima is achieved by $m_2(x) = -\tanh(x - \alpha)$ for arbitrary $\alpha \in \mathbf{R}$, which is the solution of $\dfrac{|\partial_x m_2|}{\sqrt{1 - m_2^2}} = \sqrt{1 - m_2^2}$. The resulting function $m(x) = (0, -\tanh(x - \alpha), \operatorname{sech}(x - \alpha))$ is called the Bloch wall. It is a stationary solution for the Landau-Lifshitz equation.

When the external magnetic field is switched on ($h > 0$), then $m = (0, 1, 0)$ has the lower energy than that of $m = (0, -1, 0)$ even though they remain local energy minima. Due to the energy difference, one expects that the domain occupied by state close to $(0, 1, 0)$ should expand, resulting in the motion of wall towards the positive direction. One also expects that there should be a travelling wave solution for such phenomena. One of such travelling waves is the Walker wall. For the notational convenience denote

$$E_0(m) = \frac{1}{2}\int_{\mathbf{R}} |\partial_x m|^2 dx + \frac{a}{2}\int_{\mathbf{R}} m_1^2\, dx + \frac{1}{2}\int_{\mathbf{R}} (1 - m_2^2) dx.$$

With this notation we have

$$\nabla_{L^2} E(m) = \nabla_{L^2} E_0(m) - he_2.$$

Take the cross product between m and (1). Utilizing the formula $m \times (m \times b) = -(1 - m \otimes m)b$ and $\partial_t m \cdot m = 0$ (both due to $|m| = 1$), we obtain another equivalent form of the Landau-Lifshitz equation:

$$\eta \partial_t m + m \times \partial_t m + (1 - m \otimes m)\nabla_{L^2} E_0(m) = (1 - m \otimes m)he_2. \tag{4}$$

For some $M_0 = (m_{01}, m_{02}, m_{03})$ which is a function depending only on x, assume that the solution of (4) is expressed as $m(x, t) = M_0(x - vt)$, thus we have

$$\begin{aligned} &-v\eta \partial_x M_0 - v M_0 \times \partial_x M_0 + (1 - M_0 \otimes M_0)\nabla_{L^2} E_0(M_0) \\ &\quad = (1 - M_0 \otimes M_0)he_2. \end{aligned} \tag{5}$$

Furthermore assume that M_0 has the particular form of

$$M_0(x) = \left(\operatorname{sech}\left(\frac{x}{\delta}\right) \sin\theta, -\tanh\left(\frac{x}{\delta}\right), \operatorname{sech}\left(\frac{x}{\delta}\right)\cos\theta\right), \tag{6}$$

where $\delta > 0$ and $|\theta| \le \frac{\pi}{2}$. Note that the particular choice of $\theta = 0$ and $\delta = 1$ corresponds to the Bloch wall solution. As a vector, note that each term of (5) is orthogonal to M_0. Note also that M_0, $\partial_x M_0$ and $M_0 \times \partial_x M_0$ form a system of orthogonal basis of $\mathbf{R}^3$. We project the equation to the latter two linear spaces. Take the scalar product between $\partial_x M_0$ and (5). Then we obtain

$$-v\eta |\partial_x M_0|^2 + \nabla_{L^2} E_0(M_0) \cdot \partial_x M_0 = h\partial_x m_{02}. \tag{7}$$

The direct calculation shows

$$\nabla_{L^2} E_0(M_0) \cdot \partial_x M_0 = \delta^{-1} \sinh\left(\frac{x}{\delta}\right) \operatorname{sech}^3\left(\frac{x}{\delta}\right)(\delta^{-2} - 1 - a\sin^2\theta). \tag{8}$$

Since we are seeking a travelling wave solution, we may assume that $\nabla_{L^2} E_0(M_0) \cdot \partial_x M_0 = 0$. Hence this assumption with (8) leads us to

$$\delta^{-2} - 1 - a \sin^2 \theta = 0. \tag{9}$$

Again by direct calculation one can check that $|\partial_x M_0|^2 = -\delta^{-1} \partial_x m_{02}$. Substituting this into (7) we obtain

$$v\eta = h\delta. \tag{10}$$

Next by taking the scalar product between $M_0 \times \partial_x M_0$ and (5) we obtain

$$\nabla_{L^2} E_0(M_0) \cdot M_0 \times \partial_x M_0 = v|\partial_x M_0|^2. \tag{11}$$

Here $|M_0 \times \partial_x M_0| = |\partial_x M_0|$ and $e_2 \cdot M_0 \times \partial_x M_0 = 0$ are used. The second identity can be deduced intuitively since the image of M_0 lies in a tilted plane which includes e_2 axis. With the direct calculations

$$\begin{cases} \nabla_{L^2} E_0(M_0) \cdot M_0 \times \partial_x M_0 = a\delta^{-1} \sin\theta \cos\theta \operatorname{sech}^2\left(\dfrac{x}{\delta}\right), \\ |\partial_x M_0|^2 = \delta^{-2} \operatorname{sech}^2\left(\dfrac{x}{\delta}\right), \end{cases}$$

and (11) we derive

$$a\delta \sin 2\theta = 2v. \tag{12}$$

By re-arranging (9), (10) and (12), we obtain

$$\sin 2\theta = \frac{2h}{a\eta}, \qquad \delta = (1 + a \sin^2 \theta)^{-1/2}, \qquad v = \frac{a \sin 2\theta}{2\sqrt{1 + a \sin^2 \theta}}. \tag{13}$$

Note that θ, δ and v are determined when h, a and η are given. It is clear that M_0 with these choices is a solution of (4). The resulting travelling wave solution is called the Walker wall. One point to note is that we need to have $|h| \le \frac{a\eta}{2}$ for making a proper choice of θ in (13). The constant $h_w = \frac{a\eta}{2}$ is called the Walker limit (see [10]).

Throughout the rest of the paper we denote the Walker wall solution above by M_0. We denote

$$X = \{m \in H^2_{loc}(\mathbf{R}; S^2); (m - M_0) \in H^2(\mathbf{R}; \mathbf{R}^3), m(\pm\infty) = (0, \mp 1, 0)\}.$$

Definition 1. $m = m(x, t)$ is called a solution of (2) with initial data $m_0 \in X$ if $(m - M_0) \in C([0, \infty); H^2(\mathbf{R}; \mathbf{R}^3)) \cap C^1((0, \infty); L^2(\mathbf{R}; \mathbf{R}^3))$ and m satisfies (2) and $m(0) = m_0$.

Definition 2. M_0 is called asymptotically stable if there exist $\varepsilon = \varepsilon(a, \eta, h) > 0$ and $\alpha = \alpha(a, \eta, h, m_0) \in \mathbf{R}$ such that

$$\sup_{\|m_0 - M_0\|_{H^1} < \varepsilon} \|m(t) - M_0(\cdot - \alpha, t)\|_{H^1} \to 0 \qquad \text{as } t \to \infty, \tag{14}$$

where m is a solution of (2) with initial data $m_0 \in X$.

Our main result is the following:

Theorem 1. *For any $a > 0$ and $\eta > 0$, there exists $K = K(a, \eta) > 0$ with the following property: Assume $|h| < K$. Then there exists $\varepsilon = \varepsilon(a, \eta, h) > 0$ of Definition 2 and M_0 is asymptotically stable. Moreover there exist $C = C(a, \eta, h) > 0$ and $\gamma = \gamma(a, \eta, h) > 0$ such that*

$$\sup_{\|m_0 - M_0\|_{H^1} < \varepsilon} \|m(t) - M_0(\cdot - \alpha, t)\|_{H^1} \le Ce^{-\gamma t}, \qquad \textit{for } t > 0,$$

where m is a solution of (2) with initial data $m_0 \in X$.

The result shows that m converges exponentially to a shifted Walker wall (by α) as $t \to \infty$ if it is close to M_0 at $t = 0$. The result is in the same spirit as [1] where they studied $a = h = 0$, the case of the Bloch wall. We remark that K is in principle a computable number. It is expected that K is much smaller, on the other hand, than the Walker limit $h_w = \frac{a\eta}{2}$.

In the following we collect notations we use for the readers' convenience.

$a, \eta > 0$: material constants,
$H = (0, h, 0)$: external magnetic field,
$v = v(a, \eta, h)$: velocity of Walker wall,
$\theta = \theta(a, \eta, h)$: angle of Walker wall,
$\delta = \delta(a, \eta, h)$: $\approx$ thickness of Walker wall,
$M_0 = M_0(v, \theta, \delta)$: Walker wall,
$\{M_1, M_0, M_2\}$: orthonormal basis in $\mathbf{R}^3$ defined by (15),
$R(\alpha)$: function defined by (29),
$\lambda(r) = (1 - |r|^2)^{1/2}$,
$s_\delta(x) = \operatorname{sech}\left(\frac{x}{\delta}\right) = \cosh^{-1}\left(\frac{x}{\delta}\right)$,
$t_\delta(x) = \tanh\left(\frac{x}{\delta}\right)$,
L: operator defined by (20),
$p = 2 + a - 2\delta^{-2} = a \cos 2\theta > 0$,
$\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{M}$: operators defined by (21) and (22),
$\varphi = \frac{1}{\sqrt{2\delta}}\,{}^t(0, s_\delta)$: normalized base vector of $\ker \mathscr{L}$,
Q: orthogonal projection onto $(\ker \mathscr{L})^\perp$,
$\langle \cdot, \cdot \rangle$: L^2 inner product.

3. Linearized Landau-Lifshitz equation

In this section we linearize the Landau-Lifshitz equation (2) around the Walker wall solution. The similar computation has been carried out by

Carbou and Labbé [1]. First we set

$$\begin{cases} M_1 = (\cos\theta, 0, -\sin\theta), \\ M_2(x,t) = (-t_\delta(x-vt)\sin\theta, -s_\delta(x-vt), -t_\delta(x-vt)\cos\theta), \end{cases} \tag{15}$$

where we use the notations $s_\delta(x) = \mathrm{sech}\big(\frac{x}{\delta}\big)$ and $t_\delta(x) = \tanh\big(\frac{x}{\delta}\big)$. With M_0 defined as in (6), the set $\{M_1, M_0, M_2\}$ forms a positively oriented orthonormal basis in $\mathbf{R}^3$ for each $x \in \mathbf{R}$. The direct computations show that

$$\begin{cases} \partial_x M_0 = \delta^{-1} s_\delta M_2, \\ \partial_x^2 M_0 = -\delta^{-2} t_\delta s_\delta M_2 - \delta^{-2} s_\delta^2 M_0, \\ \partial_x M_2 = -\delta^{-1} s_\delta M_0, \\ \partial_x^2 M_2 = -\delta^{-2} s_\delta^2 M_2 + \delta^{-2} t_\delta s_\delta M_0. \end{cases}$$

In the following we write all the relevant quantities in terms of this frame. For a solution m of (2), define r_1, r_2 and λ as

$$m = r_1 M_1 + r_2 M_2 + \lambda M_0, \tag{16}$$

where $\lambda = (1 - r_1^2 - r_2^2)^{1/2}$. Furthermore define f_1, f_2 and f_3 as

$$\nabla_{L^2} E(m) = f_1 M_1 + f_2 M_2 + f_0 M_0.$$

The direct computations show that

$$\begin{cases} f_1 = -\partial_x^2 r_1 + a\cos\theta(r_1\cos\theta - r_2 t_\delta \sin\theta + \lambda s_\delta \sin\theta), \\ f_2 = -\partial_x^2 r_2 + \delta^{-2} r_2 s_\delta^2 - 2\delta^{-1}\partial_x \lambda s_\delta + \delta^{-2}\lambda t_\delta s_\delta + (-r_2 s_\delta - t_\delta \lambda + h) s_\delta \\ \qquad - a t_\delta \sin\theta(r_1\cos\theta - r_2 t_\delta \sin\theta + \lambda s_\delta \sin\theta), \\ f_0 = 2\delta^{-1}\partial_x r_2 s_\delta - \delta^{-2} r_2 t_\delta s_\delta - \partial_x^2 \lambda + \delta^{-2}\lambda s_\delta^2 + (-r_2 s_\delta - t_\delta\lambda + h) t_\delta \\ \qquad + a s_\delta \sin\theta(r_1\cos\theta - r_2 t_\delta\sin\theta + \lambda s_\delta \sin\theta). \end{cases} \tag{17}$$

We also compute that

$$\begin{aligned} \partial_t m &= \partial_t(r_1 M_1 + r_2 M_2 + \lambda M_0) \\ &= \partial_t r_1 M_1 + \partial_t r_2 M_2 + r_2(v\delta^{-1} s_\delta M_0) + \partial_t \lambda M_0 + \lambda(-v\delta^{-1} s_\delta M_2) \\ &= \partial_t r_1 M_1 + (\partial_t r_2 - v\delta^{-1}\lambda s_\delta) M_2 + (\partial_t \lambda + v\delta^{-1} r_2 s_\delta) M_0. \end{aligned}$$

Now, rewriting (2) in terms of r_1 and r_2, we obtain

$$\begin{cases} (1+\eta^2)\partial_t r_1 = \lambda f_2 - r_2 f_0 + \eta(\lambda r_1 f_0 - \lambda^2 f_1 - r_2^2 f_1 + r_1 r_2 f_2), \\ (1+\eta^2)\partial_t r_2 = r_1 f_0 - \lambda f_1 + \eta(r_1 r_2 f_1 - r_1^2 f_2 - \lambda^2 f_2 + \lambda r_2 f_0) + (1+\eta^2) v\delta^{-1}\lambda s_\delta. \end{cases}$$

Since it is natural to adopt the coordinate which moves with the travelling wave, we define $z = x - vt$ and replace the parameter (x,t) by (z,t). This results in the replacements

$$\partial_x \to \partial_z, \qquad \partial_t \to \partial_t - v\partial_z.$$

With this we obtain

$$\begin{cases} (1+\eta^2)\partial_t r_1 = \lambda f_2 - r_2 f_0 + \eta(\lambda r_1 f_0 - \lambda^2 f_1 - r_2^2 f_1 + r_1 r_2 f_2) \\ \qquad\qquad\qquad + (1+\eta^2) v \partial_z r_1, \\ (1+\eta^2)\partial_t r_2 = r_1 f_0 - \lambda f_1 + \eta(r_1 r_2 f_1 - r_1^2 f_2 - \lambda^2 f_2 + \lambda r_2 f_0) \\ \qquad\qquad\qquad + (1+\eta^2) v \partial_z r_2 + (1+\eta^2) v \delta^{-1} \lambda s_\delta. \end{cases} \tag{18}$$

We linearize (18) around $(r_1, r_2, \lambda) = (0, 0, 1)$. The direct computations show that we have the following system of equations,

$$(1+\eta^2)\partial_t r = \begin{pmatrix} \eta & -1 \\ 1 & \eta \end{pmatrix} \begin{pmatrix} L-p & 0 \\ 0 & L \end{pmatrix} r + \begin{pmatrix} \eta^2 M_+ - M_- & 0 \\ \eta(M_+ + M_-) & (1+\eta^2) M_+ \end{pmatrix} r, \tag{19}$$

where $p = 2 + a - 2\delta^{-2} = a\cos 2\theta$ and

$$L = \partial_z^2 - \delta^{-2}(1 - 2s_\delta^2), \qquad M_\pm = v(\delta^{-1} t_\delta \pm \partial_z). \tag{20}$$

We denote

$$\mathscr{L} = \begin{pmatrix} \eta & -1 \\ 1 & \eta \end{pmatrix} \begin{pmatrix} L-p & 0 \\ 0 & L \end{pmatrix}, \qquad \mathscr{M} = \begin{pmatrix} \eta^2 M_+ - M_- & 0 \\ \eta(M_+ + M_-) & (1+\eta^2) M_+ \end{pmatrix}, \tag{21}$$

and

$$\mathscr{L}_1 = \begin{pmatrix} \eta(L-p) & 0 \\ 0 & \eta L \end{pmatrix}, \qquad \mathscr{L}_2 = \begin{pmatrix} 0 & -L \\ L-p & 0 \end{pmatrix}. \tag{22}$$

Observe that $(1+\eta^2)\partial_t r = \mathscr{L} r + \mathscr{M} r$ and $\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2$ with the above notations.

The next Lemma follows from [3] and gives precise information on the eigenvalues of $-L$.

Lemma 1. *For $-L = -\partial_z^2 + \delta^{-2}(1 - 2s_\delta^2)$, we have the following properties.*

(i) *The first eigenvalue λ_1 of $-L$ equals 0 and λ_1 is simple. Furthermore* $\ker(-L) = \mathrm{span}\{s_\delta\}$.

(ii) $\sigma(-L) = \{0\} \cup [\delta^{-2}, \infty)$.

(iii) *For $u \in H^1(\mathbf{R})$ with $\langle u, s_\delta \rangle = 0$, $\langle -Lu, u \rangle \geq \delta^{-2} \|u\|_{L^2}^2$.*

Proof. We only need to see the case $\delta = 1$. Since $-L$ is self-adjoint, $\sigma(-L)$ is the union of the discrete and essential spectrum. We have $f(z) = \mathrm{sech}\, z$ in the kernel of $-L$. Since $f > 0$, (i) follows from the standard argument ([9]). It is known that the discrete spectrum of $-L$ is only $\{0\}$ (see [3]).

For (ii), since $\sigma_{ess}(-\partial_z^2) = [0, \infty)$, we have $\sigma_{ess}(-\partial_z^2 + 1) = [1, \infty)$. Define the operator $B : H^2(\mathbf{R}) \to L^2(\mathbf{R})$ by $Bu(z) = (\operatorname{sech} z)u(z)$ $(u \in H^2(\mathbf{R}))$. B is compact for the graph norm of $-\partial_z^2 + 1$, $\|u\|_G := \|u\|_{L^2} + \|(-\partial_z^2 + 1)u\|_{L^2}$. Hence, by Weyl's theorem we have $\sigma_{ess}(-L) = [1, \infty)$. From (ii), (iii) follows.

Next, since $p = a\cos 2\theta > 0$, we may conclude from Lemma 1 the following

Lemma 2.

$$\ker \mathscr{L} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ s_\delta \end{pmatrix} \right\}.$$

We denote $\varphi = \dfrac{1}{\sqrt{2\delta}} \begin{pmatrix} 0 \\ s_\delta \end{pmatrix}$, which is normalized as $\|\varphi\|_{L^2} = 1$.

Lemma 3. *On the closed subspace $H^2(\mathbf{R}; \mathbf{R}^2) \cap (\ker \mathscr{L})^\perp$ of $H^2(\mathbf{R}; \mathbf{R}^2)$, the norms $\|\cdot\|_{H^2}$ and $\|\mathscr{L}_1 \cdot\|_{L^2}$ are equivalent.*

Furthermore, there exists $C > 0$ such that

$$\|u\|_{H^2} \le C\eta^{-1}(1 + p^{-1})\|\mathscr{L}_1 u\|_{L^2}, \tag{23}$$

for any $u \in H^2(\mathbf{R}; \mathbf{R}^2) \cap (\ker \mathscr{L})^\perp$. Here C depends only on $a > 0$.

Proof. From the definition of $\mathscr{L}_1$ there exists a constant $C > 0$ such that

$$\|\mathscr{L}_1 u\|_{L^2} \le C\|u\|_{H^2},$$

for any $u \in H^2(\mathbf{R}; \mathbf{R}^2) \cap (\ker \mathscr{L})^\perp$. We only need to establish (23). By $u \in (\ker \mathscr{L})^\perp$, Lemma 1 and Hölder's inequality, we have

$$\|u_2\|_{L^2} \le \delta^2 \|-Lu_2\|_{L^2}.$$

Hence, by $\delta = (1 + a\sin\theta)^{-1/2} \le 1$ we get

$$\|u_2\|_{L^2} \le \|-Lu_2\|_{L^2}.$$

We denote $-L = -\partial_z^2 + g_\delta$. By using $\delta^{-2} = 1 + a\sin\theta \le 1 + a$, we have $\|g_\delta\|_\infty \le 1 + a$. Hence

$$\begin{aligned} \|\partial_z^2 u_2\|_{L^2} &= \|-\partial_z^2 u_2 + g_\delta u_2 - g_\delta u_2\|_{L^2} \\ &\le \|-Lu_2\|_{L^2} + \|g_\delta\|_\infty \|u_2\|_{L^2} \\ &\le (2 + a)\|-Lu_2\|_{L^2}. \end{aligned}$$

By

$$\|\partial_z u_2\|_{L^2}^2 = -\langle u_2, \partial_z^2 u_2 \rangle \le \frac{1}{2}(\|u_2\|_{L^2}^2 + \|\partial_z^2 u_2\|_{L^2}^2),$$

we have a constant $C_1 = C_1(a) > 0$ such that

$$\|u_2\|_{H^2} \leq \eta^{-1} C_1 \|-\eta L u_2\|_{L^2}. \tag{24}$$

Since $p = a\cos 2\theta > 0$ and $\langle -Lu_1, u_1\rangle \geq 0$, we get

$$\begin{aligned}\|(-L+p)u_1\|_{L^2}^2 &= \|-Lu_1\|_{L^2}^2 + 2p\langle -Lu_1, u_1\rangle + p^2\|u_1\|_{L^2}^2 \\ &\geq p^2\|u_1\|_{L^2}^2.\end{aligned}$$

Hence

$$\|u_1\|_{L^2} \leq p^{-1}\|(-L+p)u_1\|_{L^2}.$$

Furthermore

$$\begin{aligned}\|\partial_z^2 u_1\|_{L^2} &= \|-\partial_z^2 u_1 + (g_\delta + p)u_1 - (g_\delta + p)u_1\|_{L^2} \\ &\leq \|(-L+p)u_1\|_{L^2} + \|g_\delta + p\|_\infty \|u_1\|_{L^2} \\ &\leq \|(-L+p)u_1\|_{L^2} + (1+a+p)p^{-1}\|(-L+p)u_1\|_{L^2} \\ &\leq \{1 + (1+a+p)p^{-1}\}\|(-L+p)u_1\|_{L^2}.\end{aligned}$$

From $p = a\cos 2\theta \leq a$, we get the constant $C_2 = C_2(a) > 0$ such that

$$\|u_1\|_{H^2} \leq \eta^{-1} C_2(1 + p^{-1})\|\eta(-L+p)u_1\|_{L^2}. \tag{25}$$

By (24) and (25), we obtain (23).

We can also prove the following lemma.

Lemma 4. *On the closed subspace $H^1(\mathbf{R};\mathbf{R}^2) \cap (\ker \mathscr{L})^\perp$ of $H^1(\mathbf{R};\mathbf{R}^2)$, the norms $\|\cdot\|_{H^1}$ and $\langle -\mathscr{L}_1 \cdot, \cdot\rangle^{1/2}$ are equivalent.*

Furthermore, there exists $C > 0$ such that

$$\|u\|_{H^1} \leq C\eta^{-1/2}(1 + p^{-1})^{1/2}\langle -\mathscr{L}_1 u, u\rangle^{1/2}, \tag{26}$$

for any $u \in H^1(\mathbf{R};\mathbf{R}^2) \cap (\ker \mathscr{L})^\perp$. Here C depends only on $a > 0$.

Remark 1. *We fix $a > 0$ and $\eta > 0$. From $p = a\cos 2\theta$, we have $p \to 0$ as $\theta \to \frac{\pi}{4}$. By (13) we have $\theta \to \frac{\pi}{4}$ as $|h| \to h_w = \frac{a\eta}{2}$. Hence we obtain $p^{-1} \to \infty$ as $|h| \to h_w$. Remark that the equivalence of these norms deteriorates as $|h| \to h_w$.*

4. Stability analysis

In this section we estimate the perturbation of the Walker wall and work with the Landau-Lifshitz equation in the form of (18). We remark that m is a solution of (2) in the sense of Definition 1 if and only if there exists a solution

$r \in C([0,+\infty); H^2(\mathbf{R};\mathbf{R}^2)) \cap C^1((0,+\infty); L^2(\mathbf{R};\mathbf{R}^2))$ of (18). Here

$$m = r_1 M_1 + r_2 M_2 + \lambda(r) M_0.$$

We express (18) by

$$(1+\eta^2)\partial r = \mathscr{L}r + \mathscr{M}r + \tilde{\mathscr{N}}r, \tag{27}$$

where $\tilde{\mathscr{N}}r$ is the nonlinear term of (18). We denote

$$\tilde{\mathscr{N}}r = \begin{pmatrix} \tilde{N}_1 r \\ \tilde{N}_2 r \end{pmatrix}.$$

The direct computations show that

$$\begin{aligned}
\tilde{N}_1 r = {} & -2\delta^{-1}\partial_z\lambda s_\delta + \delta^{-2}(\lambda-1)t_\delta s_\delta - (\lambda-1)t_\delta s_\delta - a(\lambda-1)t_\delta s_\delta \sin^2\theta \\
& + (\lambda-1)f_2 + (-r_2+\eta r_1)\{2\delta^{-1}\partial_z r_2 s_\delta - \delta^{-2} r_2 t_\delta s_\delta - \partial_z^2\lambda + \delta^{-2}(\lambda-1)s_\delta^2 \\
& + (-r_2 s_\delta - (\lambda-1)t_\delta)t_\delta + a s_\delta \sin\theta(r_1\cos\theta - r_2 t_\delta \sin\theta + (\lambda-1)s_\delta\sin\theta)\} \\
& + \eta(\lambda-1)r_1 f_0 - \eta a(\lambda-1)s_\delta\cos\theta\sin\theta - \eta(\lambda^2-1)f_1 \\
& - \eta r_2^2 f_1 + \eta r_1 r_2 f_2,
\end{aligned} \tag{28}$$

$$\begin{aligned}
\tilde{N}_2 r = {} & r_1\{2\delta^{-1}\partial_z r_2 s_\delta - \delta^{-2} r_2 t_\delta s_\delta - \partial_z^2\lambda + \delta^{-2}(\lambda-1)s_\delta^2 + (-r_2 s_\delta - (\lambda-1)t_\delta)t_\delta \\
& + a s_\delta \sin\theta(r_1\cos\theta - r_2 t_\delta\sin\theta + (\lambda-1)s_\delta\sin\theta)\} \\
& - a(\lambda-1)s_\delta\cos\theta\sin\theta - (\lambda-1)f_1 + \eta r_1 r_2 f_1 - \eta r_1^2 f_2 \\
& - \eta\{-2\delta^{-1}\partial_z\lambda s_\delta + \delta^{-2}(\lambda-1)t_\delta s_\delta - (\lambda-1)t_\delta s_\delta - a(\lambda-1)t_\delta s_\delta\sin^2\theta\} \\
& - \eta(\lambda^2-1)f_2 + (\lambda-1)r_2 f_0 \\
& + \eta r_2\{2\delta^{-1}\partial_z r_2 s_\delta - \delta^{-2} r_2 t_\delta s_\delta - \partial_z^2\lambda + \delta^{-2}(\lambda-1)s_\delta^2 \\
& + (-r_2 s_\delta - (\lambda-1)t_\delta)t_\delta + a s_\delta\sin\theta(r_1\cos\theta - r_2 t_\delta\sin\theta + (\lambda-1)s_\delta\sin\theta)\},
\end{aligned}$$

where f_0, f_1 and f_2 are given by (17).

Definition 3. For any $\alpha \in \mathbf{R}$, we define

$$R(\alpha, z) = \begin{pmatrix} M_0(z-\alpha)\cdot M_1(z) \\ M_0(z-\alpha)\cdot M_2(z) \end{pmatrix} = \begin{pmatrix} 0 \\ t_\delta(z-\alpha)s_\delta(z) - s_\delta(z-\alpha)t_\delta(z) \end{pmatrix}. \tag{29}$$

Since s_δ and t_δ are smooth and bounded we can check that there exists $C > 0$ such that

$$\|R(\alpha_1) - R(\alpha_2)\|_{H^2} < C|\alpha_1 - \alpha_2|, \tag{30}$$

for any $\alpha_1, \alpha_2 \in \mathbf{R}$.

Lemma 5. *There exists* $\varepsilon_0 > 0$ *such that the following holds. For any* $r \in B(0, \varepsilon_0) \subset L^2(\mathbf{R}; \mathbf{R}^2)$ *there exists a unique* $(\alpha, W) \in \mathbf{R} \times (\ker \mathscr{L})^{\perp}$ *such that*

$$r(z) = R(\alpha, z) + W(z).$$

Here, $B(0, \varepsilon_0) = \{u \in L^2(\mathbf{R}; \mathbf{R}^2) \mid \|u\|_{L^2} < \varepsilon_0\}$. *Furthermore, there exist open sets* $U \subset H^k(\mathbf{R}; \mathbf{R}^2)$ *and* $V \subset \mathbf{R} \times ((\ker \mathscr{L})^{\perp} \cap H^k(\mathbf{R}; \mathbf{R}^2))$ *such that the map* $F; r \mapsto (\alpha, W)$ *is a homeomorphism from* U *to* V *for* $k = 1, 2$. *Moreover* $F(0) = (0, 0)$.

Proof. For any $\alpha \in \mathbf{R}$, we define

$$\Psi(\alpha) = \langle R(\alpha), \varphi \rangle.$$

We remark that $\Psi(0) = \langle 0, \varphi \rangle = 0$ and $\Psi'(0) < 0$. From the inverse function theorem, there exist neighborhoods $A, B \subset \mathbf{R}$ of $0 \in \mathbf{R}$ such that

$$\Psi : A \to B$$

is a homeomorphism. Let $\varepsilon_0 > 0$ satisfy $(-\varepsilon_0, \varepsilon_0) \subset B$ and fix $r \in B(0, \varepsilon_0) \subset L^2(\mathbf{R}; \mathbf{R}^2)$. From $\|r\|_{L^2} < \varepsilon_0$, we get

$$|\langle r, \varphi \rangle| \le \|r\|_{L^2} \|\varphi\|_{L^2} = \|r\|_{L^2} < \varepsilon_0.$$

Hence from $\langle r, \varphi \rangle \in B$ we have a unique $\alpha \in A$ such that $\langle r, \varphi \rangle = \Psi(\alpha)$. Denoting $W = r - R(\alpha)$, we have

$$\langle W, \varphi \rangle = \langle r, \varphi \rangle - \Psi(\alpha) = 0.$$

Thus for any $r \in B(0, \varepsilon_0)$ there exists a unique $(\alpha, W) \in \mathbf{R} \times (\ker \mathscr{L})^{\perp}$ such that $r = R(\alpha) + W$. Furthermore, we can check that the map $F : r \mapsto (\alpha, W)$ is a homeomorphism from $U \subset H^2(\mathbf{R}; \mathbf{R}^2)$ to $F(U) \subset \mathbf{R} \times ((\ker \mathscr{L})^{\perp} \cap H^2(\mathbf{R}; \mathbf{R}^2))$.

Lemma 6. *Let* $r \in C([0, \infty); H^2(\mathbf{R}; \mathbf{R}^2)) \cap C^1((0, \infty); L^2(\mathbf{R}; \mathbf{R}^2))$ *satisfy* $\|r(0)\|_{L^2} < \varepsilon_0$, *where* ε_0 *is given by Lemma* 5. *Then there exists* $T > 0$ *and the following holds. There exist unique* $\alpha \in C([0, T)) \cap C^1((0, T))$ *and* $W \in C([0, T); (\ker \mathscr{L})^{\perp} \cap H^2(\mathbf{R}; \mathbf{R}^2)) \cap C^1((0, T); (\ker \mathscr{L})^{\perp})$ *such that*

$$\begin{cases} r(t) = R(\alpha(t)) + W(t), & t \in [0, T), \\ \partial_t r(t) = \partial_\alpha R(\alpha(t)) \alpha'(t) + \partial_t W(t), & t \in (0, T). \end{cases}$$

Proof. Since $\|r(t)\|_{L^2}$ is continuous, there exists $T > 0$ such that $\|r(t)\|_{L^2} < \varepsilon_0$ for any $t \in [0, T)$. By Lemma 5, for any $t \in [0, T)$, there exists a unique $(\alpha(t), W(t)) \in \mathbf{R} \times ((\ker \mathscr{L})^{\perp} \cap H^2(\mathbf{R}; \mathbf{R}^2))$ such that $r(t) = R(\alpha(t)) + W(t)$, $\alpha \in C([0, T))$ and $W \in C([0, T); ((\ker \mathscr{L})^{\perp} \cap H^2(\mathbf{R}; \mathbf{R}^2))$.

On the other hand, from $\langle r(t), \varphi \rangle = \langle R(\alpha(t)), \varphi \rangle = \Psi(\alpha(t))$ for any $t \in [0, T))$, we get

$$\frac{d}{dt} \Psi(\alpha(t)) = \frac{d}{dt} \langle r(t), \varphi \rangle = \langle \partial_t r(t), \varphi \rangle.$$

Then we obtain

$$\begin{aligned}\alpha'(t) &= \frac{d}{dt}(\Psi^{-1}(\Psi(\alpha(t)))) \\ &= (\Psi^{-1})'(\Psi(\alpha(t)))\frac{d}{dt}(\Psi(\alpha(t))) \\ &= (\Psi^{-1})'(\langle r(t), \varphi\rangle)\langle \partial_t r(t), \varphi\rangle.\end{aligned}$$

Hence there exists $\alpha'(t)$ for any $t \in [0, T)$. Furthermore, since $(\Psi^{-1})'$, $\langle r(t), \varphi\rangle$ and $\langle \partial_t r(t), \varphi\rangle$ are continuous, we have $\alpha \in C^1((0, T))$. Hence there exists $\partial_t(R(\alpha(t))) \in L^2(\mathbf{R}; \mathbf{R}^2)$ with

$$\partial_t(R(\alpha(t))) = \partial_\alpha R(\alpha(t))\alpha'(t).$$

From $W(t) = r(t) - R(\alpha(t))$ and $r, R(\alpha(\cdot)) \in C^1((0, T); L^2(\mathbf{R}; \mathbf{R}^2))$ we obtain $W \in C^1((0, T); L^2(\mathbf{R}; \mathbf{R}^2))$. Since $\langle W(t), \varphi\rangle = 0$ for any $t \in [0, T)$, we get $\partial_t W \in (\ker \mathscr{L})^\perp$. Therefore $W \in C^1((0, T); (\ker \mathscr{L})^\perp)$.

Lemma 7. *The travelling wave M_0 is asymptotically stable in the sense of Definition 2 if and only if there exist $\varepsilon = \varepsilon(a, \eta, h) > 0$ and $\alpha = \alpha(a, \eta, h, r_0) \in \mathbf{R}$ such that*

$$\sup_{\|r_0\|_{H^1} < \varepsilon} \|r(t) - R(\alpha)\|_{H^1} \to 0 \qquad \text{as } t \to \infty, \tag{31}$$

where $r \in C([0, \infty); H^2(\mathbf{R}; \mathbf{R}^2)) \cap C^1((0, \infty); L^2(\mathbf{R}; \mathbf{R}^2))$ and r is a solution of (18) *with initial value r_0 and*

$$m = r_1 M_1 + r_2 M_2 + \lambda(r) M_0.$$

Proof. If α is a constant and $W \equiv 0$, then we have

$$m(z) = (M_0(z - \alpha) \cdot M_1(z))M_1(z) + (M_0(z - \alpha) \cdot M_2(z))M_2(z) + \lambda(R(\alpha, z))M_0(z).$$

Hence $m(z) = M_0(z - \alpha)$. Therefore (14) and (31) are equivalent.

Theorem 2. *For any $a > 0$ and $\eta > 0$, there exists $K = K(a, \eta) > 0$ such that M_0 is asymptotically stable if $|h| < K$. Furthermore there exist $C = C(a, \eta, h) > 0$ and $\gamma = \gamma(a, \eta, h) > 0$ such that*

$$\sup_{\|r_0\|_{H^1} < \varepsilon} \|r(t) - R(\alpha)\|_{H^1} \le Ce^{-\gamma t}, \qquad \text{for } t > 0,$$

where $r \in C([0, \infty); H^2(\mathbf{R}; \mathbf{R}^2)) \cap C^1((0, \infty); L^2(\mathbf{R}; \mathbf{R}^2))$ and r is a solution of (18) *with initial value r_0.*

Remark 2. *Theorem* 1 *and* 2 *are equivalent.*

To prove Theorem 2 we prepare some notations and lemmas. First let Q be the orthogonal projection onto $(\ker \mathscr{L})^{\perp} \subset L^2(\mathbf{R}^2; \mathbf{R})$ and we calculate (27) to estimate $|a(t)|$ and $\|W(t)\|_{H^1}$.

LEMMA 8. *Let* $r \in C([0,\infty); H^2(\mathbf{R}; \mathbf{R}^2)) \cap C^1((0,\infty); L^2(\mathbf{R}; \mathbf{R}^2))$ *be a solution of* (27) *and assume that there exists* $T > 0$ *such that* $\|r(t)\|_{L^2} < \varepsilon_0$ *for any* $t \in [0, T)$*, where* $\varepsilon_0 > 0$ *is given by Lemma* 5. *Then there exists* (α, W) *of Lemma* 6 *such that*

$$\begin{cases} \partial_t W(t) = (1+\eta^2)^{-1}\{Q\mathscr{L}W(t) + Q\mathscr{M}W(t) + Q\mathscr{N}(\alpha(t), W(t))\} \\ \qquad\qquad - \alpha'(t) Q\partial_\alpha R(\alpha(t)), \\ \alpha'(t) = \dfrac{1}{\sqrt{2\delta}}(1+\eta^2)^{-1}\langle \partial_\alpha R(\alpha(t)), \varphi\rangle^{-1}\{-p\langle w_1(t), s_\delta\rangle + 2h\langle w_1(t), s_\delta t_\delta\rangle \\ \qquad\qquad + 2h\eta^{-1}(1+\eta^2)\langle w_2(t), s_\delta t_\delta\rangle + \sqrt{2\delta}\langle \mathscr{N}(\alpha(t), W(t)), \varphi\rangle\}, \end{cases} \tag{32}$$

for any $t \in (0, T)$*. Here we denote* $W = {}^t(w_1, w_2)$ *and* $\mathscr{N}(\alpha, W) = \tilde{\mathscr{N}}(R(\alpha) + W) - \tilde{\mathscr{N}}(R(\alpha))$*.*

PROOF. By the assumption and Lemma 5, there exists a unique (α, W) with

$$r(z,t) = R(\alpha(t), z) + W(z,t),$$

for any $t \in [0, T)$. Fix any $t \in [0, T)$. Since $R(\alpha(t), z)$ is a solution of (27), we get

$$\mathscr{L}R(\alpha(t), z) + \mathscr{M}R(\alpha(t), z) + \tilde{\mathscr{N}}R(\alpha(t), z) = 0.$$

Hence (27) is expressed as follows:

$$\begin{aligned} (1+\eta^2)&\partial_t(R(\alpha(t), z) + W(z,t)) \\ &= \mathscr{L}W(z,t) + \mathscr{M}W(z,t) \\ &\quad + (\tilde{\mathscr{N}}(R(\alpha(t), z) + W(z,t)) - \tilde{\mathscr{N}}R(\alpha(t), z)) \\ &= \mathscr{L}W(z,t) + \mathscr{M}W(z,t) + \mathscr{N}(\alpha(t), W(z,t)). \end{aligned} \tag{33}$$

We operate Q on (33), then we have

$$(1+\eta^2)\partial_t W = Q\mathscr{L}W + Q\mathscr{M}W + Q\mathscr{N}(\alpha, W) - (1+\eta^2)\alpha' Q\partial_\alpha R(\alpha).$$

Next, the direct computations show that

$$\langle \mathscr{L}W, \varphi\rangle = \langle W, \mathscr{L}_1\varphi\rangle + \frac{1}{\sqrt{2\delta}}\langle (L-p)w_1, s_\delta\rangle = \frac{-p}{\sqrt{2\delta}}\langle w_1, s_\delta\rangle, \tag{34}$$

here we remark that $\mathscr{L}_1\varphi = 0$ and $Ls_\delta = 0$. From (10), $M_+s_\delta = 0$ and $M_-s_\delta = 2v\delta^{-1}t_\delta s_\delta$ we obtain

$$\begin{aligned}\langle \mathscr{M}W, \varphi\rangle &= \frac{1}{\sqrt{2\delta}}\langle \eta(M_+ + M_-)w_1 + (1+\eta^2)M_+w_2, s_\delta\rangle \\ &= \frac{1}{\sqrt{2\delta}}\{\eta\langle w_1, (M_- + M_+)s_\delta\rangle + (1+\eta^2)\langle w_2, M_-s_\delta\rangle\} \\ &= \frac{1}{\sqrt{2\delta}}\{2h\langle w_1, t_\delta s_\delta\rangle + 2h\eta^{-1}(1+\eta^2)\langle w_2, t_\delta s_\delta\rangle\}. \end{aligned} \tag{35}$$

Take the L^2 inner product between φ and (33). From $\langle \partial_t W, \varphi\rangle = 0$, (34) and (35) we obtain

$$\begin{aligned}\sqrt{2\delta}(1+\eta^2)\alpha'\langle \partial_\alpha R(\alpha), \varphi\rangle &= -p\langle w_1, s_\delta\rangle + 2h\langle w_1, s_\delta t_\delta\rangle \\ &\quad + 2h\eta^{-1}(1+\eta^2)\langle w_2, s_\delta t_\delta\rangle + \sqrt{2\delta}\langle \mathscr{N}(\alpha, W), \varphi\rangle.\end{aligned}$$

Lemma 9. *There exist $C > 0$ and $\alpha_0 > 0$ such that*

$$\|\mathscr{N}(\alpha, W)\|_{L^2} \le C(1 + a + \eta + a\eta)(|h| + |\alpha| + \|W\|_{H^1})\|W\|_{H^2},$$

for any $(\alpha, W) \in \mathbf{R} \times ((\ker \mathscr{L})^\perp \cap H^2(\mathbf{R}; \mathbf{R}^2))$ *with* $|\alpha| < \alpha_0$, $\|W\|_\infty \le \frac{1}{4}$ *and* $\|W\|_{H^1} \le 1$.

Proof. From the mean-value theorem, there exists a constant $C > 0$ such that

$$\|\partial_z^k R(\alpha)\|_\infty \le C|\alpha|, \tag{36}$$

for $k = 0, 1, 2$. Let α satisfy $C|\alpha| \le \frac{1}{4}$. By (36), we have

$$\|\partial_z^k R(\alpha)\|_\infty^l \le C^l|\alpha|^l \le C|\alpha| \le \frac{1}{4}, \tag{37}$$

for $l = 1, 2, 3\ldots$, and $k = 0, 1, 2$. From (37) and a, η which are coefficients of $\mathscr{N}$ we obtain

$$\begin{aligned}|\mathscr{N}(\alpha, W)| &\le C(1 + a + \eta + a\eta) \\ &\quad \cdot \{|h|\,|W| + |\alpha|(|W| + |\partial_z W| + |\partial_z^2 W|) + |W\partial_z^2 W| + |\partial_z W|^2\}. \end{aligned} \tag{38}$$

For example, we will estimate the following term

$$-2\delta^{-1}\partial_z\{\lambda(R(\alpha) + W) - \lambda(R(\alpha))\}s_\delta.$$

Here, this is one of the terms of $\mathscr{N}(\alpha, W)$. We remark that $\mathscr{N}(\alpha, W) = \tilde{\mathscr{N}}(R(\alpha) + W) - \tilde{\mathscr{N}}(R(\alpha))$.

First, we note that $\delta^{-1} \le (1+a)^{1/2}$ and $|s_\delta| \le 1$. From (36) and (37), we get

$$\begin{cases} (1 - |R(\alpha) + W|^2) > \dfrac{1}{2}, \\ (1 - |R(\alpha)|^2) > \dfrac{1}{2}, \\ \partial_z \lambda(r) = (1 - |r|^2)^{-1/2} r \cdot \partial_z r. \end{cases}$$

Therefore

$$\begin{aligned} &|\partial_z(\lambda(R(\alpha) + W) - \lambda(R(\alpha)))| \\ &\quad = |(1 - |R(\alpha) + W|^2)^{-1/2}(R(\alpha) + W) \cdot \partial_z(R(\alpha) + W) \\ &\qquad - (1 - |R(\alpha)|^2)^{-1/2}(R(\alpha)) \cdot \partial_z(R(\alpha))| \\ &\quad \le (1 - |R(\alpha) + W|^2)^{-1/2}|(R(\alpha) + W) \cdot \partial_z(R(\alpha) + W) - (R(\alpha)) \cdot \partial_z(R(\alpha))| \\ &\qquad + |(1 - |R(\alpha) + W|^2)^{-1/2} - (1 - |R(\alpha)|^2)^{-1/2}| \\ &\qquad \cdot |(R(\alpha) + W) \cdot \partial_z(R(\alpha) + W)| \\ &\quad \le C(|\alpha|\,|W| + |\alpha|\,|\partial_z W| + |W|\,|\partial_z W|) + C|(R(\alpha) + W)^2 - R(\alpha)^2| \\ &\quad \le C(|\alpha|\,|W| + |W|^2 + |\alpha|\,|\partial_z W| + |W|\,|\partial_z W|). \end{aligned}$$

Hence

$$\begin{aligned} &|-2\delta^{-1}\partial_z\{\lambda(R(\alpha) + W) - \lambda(R(\alpha))\}s_\delta| \\ &\quad \le C(1 + a)(|\alpha|\,|W| + |W|^2 + |\alpha|\,|\partial_z W| + |W|\,|\partial_z W|). \end{aligned}$$

As above we can obtain (38) by similar estimates. On the other hand, we have

$$\begin{cases} \|W\partial_z^2 W\|_{L^2}^2 = \displaystyle\int_{\mathbf{R}} |W|^2|\partial_z^2 W|^2 dz \le \|W\|_\infty^2 \int_{\mathbf{R}} \|\partial_z^2 W\|^2 dz \\ \qquad\qquad\quad \le C\|W\|_{H^1}^2 \|W\|_{H^2}^2, \\ \|(\partial_z W)^2\|_{L^2}^2 = \displaystyle\int_{\mathbf{R}} |\partial_z W|^2|\partial_z W|^2 dz \le \|\partial_z W\|_\infty^2 \int_{\mathbf{R}} |\partial_z W|^2 dz \\ \qquad\qquad\quad \le C\|\partial_z W\|_{H^1}^2 \|W\|_{H^1}^2 \le C\|W\|_{H^2}^2 \|W\|_{H^1}^2. \end{cases}$$

Therefore we obtain

$$\|\mathcal{N}(\alpha, W)\|_{L^2} \le C(1 + a + \eta + a\eta)(|h| + |\alpha| + \|W\|_{H^1})\|W\|_{H^2}.$$

Lemma 10. *Fix $a > 0$ and $\eta > 0$. Let $|h|$ be sufficiently small. Then there exists $\varepsilon > 0$ such that for the solution (α, W) of (32) with $\|R(\alpha(0)) + W(0)\|_{H^1} < \varepsilon$ the following hold:*

(i) *There exists $\alpha \in \mathbf{R}$ such that*

$$\alpha(t) \to \alpha \qquad \text{exponentially as } t \to \infty.$$

(ii) *There exist $C, \gamma > 0$ depending only on a and η such that*

$$\|W(t)\|_{H^1} \le Ce^{-\gamma t}\|W(0)\|_{H^1}, \qquad \text{for } t > 0.$$

(iii) *For any $t \in [0, \infty)$, we have*

$$\|R(\alpha(t)) + W(t)\|_{H^1} < \varepsilon_0,$$

where $\varepsilon_0 > 0$ is given by Lemma 5.

Proof. First, we fix $a > 0$ and $\eta > 0$. From (23) and (26), there exists $d = d(a, \eta, p) > 0$ such that

$$\|u\|_{H^1} \le d\langle -\mathscr{L}_1 u, u\rangle^{1/2}, \qquad \|u\|_{H^2} \le d\|\mathscr{L}_1 u\|_{L^2}, \tag{39}$$

for any $u \in (\ker \mathscr{L})^{\perp} \cap H^2(\mathbf{R}; \mathbf{R}^2)$. We remark that p depends only on a, η and h. From (13), (23), (26) and Remark 1 there exists a constant $C = C(a, \eta) > 0$ such that $d(h) < C$ for any $h \in \left(-\frac{h_w}{2}, \frac{h_w}{2}\right)$. Let $|h| < \frac{h_w}{2}$. We assume that there exist $T > 0$, $\alpha \in C([0, \infty)) \cap C^1((0, \infty))$ and $W \in C([0, \infty); (\ker \mathscr{L})^{\perp} \cap H^2(\mathbf{R}; \mathbf{R}^2)) \cap C^1((0, \infty); (\ker \mathscr{L})^{\perp})$ such that α and W satisfy (32) for any $t \in (0, T)$. Take the L^2 inner product between $-\mathscr{L}_1 W$ and the first equation of (32). Since $\mathscr{L}_1$ is a self adjoint operator, we have

$$\frac{1}{2}\frac{d}{dt}\langle W, -\mathscr{L}_1 W\rangle = \langle \partial_t W, -\mathscr{L}_1 W\rangle. \tag{40}$$

By $\mathscr{L}_1 \varphi = 0$ and (22), we obtain $Q\mathscr{L}_1 = \mathscr{L}_1$ and $\langle \mathscr{L}_2 W, \mathscr{L}_1 W\rangle = 0$. Then we get

$$\begin{aligned}\langle Q\mathscr{L}W, -\mathscr{L}_1 W\rangle &= \langle (Q\mathscr{L}_1 + Q\mathscr{L}_2)W, -\mathscr{L}_1 W\rangle \\ &= \langle \mathscr{L}_1 W, -\mathscr{L}_1 W\rangle + \langle \mathscr{L}_2 W, -Q\mathscr{L}_1 W\rangle = -\|\mathscr{L}_1 W\|_{L^2}^2. \end{aligned}\tag{41}$$

From (10) we have $|v| \le |h|\eta^{-1}$. By (21) there exists $C = C(\eta) >$ such that

$$|\langle Q\mathscr{M}W, -\mathscr{L}_1 W\rangle| \le \|\mathscr{M}W\|_{L^2}\|\mathscr{L}_1 W\|_{L^2} \le C|h|\,\|W\|_{H^1}\|\mathscr{L}_1 W\|_{L^2}. \tag{42}$$

If $\|W\|_{H^1}$ is sufficiently small, then from Lemma 9 there exists $C = C(a, \eta) > 0$ such that

$$
\begin{aligned}
|\langle Q\mathcal{N}(\alpha, W), -\mathcal{L}_1 W\rangle| &\le \|\mathcal{N}(\alpha, W)\|_{L^2}\|\mathcal{L}_1 W\|_{L^2} \\
&\le C(1 + a + \eta + a\eta)(|h| + |\alpha| + \|W\|_{H^1})\|W\|_{H^2}\|\mathcal{L}_1 W\|_{L^2} \\
&\le C(|h| + |\alpha| + \|W\|_{H^1})\|W\|_{H^2}\|\mathcal{L}_1 W\|_{L^2}. \qquad (43)
\end{aligned}
$$

The direct calculation shows that

$$
\partial_\alpha R(\alpha(t), z) = -\delta^{-1}\begin{pmatrix} 0 \\ s_\delta(z - \alpha(t))\{t_\delta(z)t_\delta(z - \alpha(t)) + s_\delta(z)s_\delta(z - \alpha(t))\} \end{pmatrix}. \qquad (44)
$$

From (44) we obtain

$$
\sup_{\alpha\in\mathbf{R}} |\langle \partial_\alpha R(\alpha), \varphi\rangle| = |\langle \partial_\alpha R(0), \varphi\rangle| = \sqrt{2}\delta^{-1/2}.
$$

Hence if $|\alpha|$ is sufficiently small, we have

$$
1 < |\langle \partial_\alpha R(\alpha), \varphi\rangle|. \qquad (45)
$$

Here we remark that $\delta \le 1$. We denote $\hat{\varphi} = -\sqrt{2}\delta^{-1/2}\varphi$. From (44) we obtain

$$
\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2} \to 0 \qquad \text{as } \alpha \to 0. \qquad (46)
$$

From (32) and (45), there exists $C = C(a, \eta) > 0$ such that

$$
\begin{aligned}
|\alpha'| &\le (1 + \eta^2)^{-1}|\langle \partial_\alpha R(\alpha), \varphi\rangle|^{-1}(2\delta)^{-1/2} \\
&\quad \cdot \{a\|s_\delta\|_{L^2}\|W\|_{L^2} + 2|h|\, \|s_\delta\|_{L^2}\|W\|_{L^2} \\
&\quad + 2|h|\eta^{-1}(1 + \eta^2)\|s_\delta\|_{L^2}\|W\|_{L^2} + \sqrt{2\delta}\|\mathcal{N}(\alpha, W)\|_{L^2}\} \\
&\le C[\{a + 2|h|(1 + \eta^{-1}(1 + \eta^2))\}\|W\|_{L^2} + \|\mathcal{N}(\alpha, W)\|_{L^2}] \\
&\le C(\|W\|_{L^2} + \|\mathcal{N}(\alpha, W)\|_{L^2}), \qquad (47)
\end{aligned}
$$

here $p \le a$, $(1 + a)^{-1/2} \le \delta \le 1$ and $|h| < h_w = \frac{a\eta}{2}$ are used. Hence, by (32), (43), (47), $Q\mathcal{L}_1 = \mathcal{L}_1$ and $\mathcal{L}_1\hat{\varphi} = 0$ we get

$$
\begin{aligned}
&(1 + \eta^2)|\alpha'\langle Q\partial_\alpha R(\alpha), -\mathcal{L}_1 W\rangle| \\
&\quad \le (1 + \eta^2)|\alpha'|\,|\langle \partial_\alpha R(\alpha), \mathcal{L}_1 W\rangle| = (1 + \eta^2)|\alpha'|\,|\langle \mathcal{L}_1\partial_\alpha R(\alpha), W\rangle| \\
&\quad = (1 + \eta^2)|\alpha'|\,|\langle \mathcal{L}_1(\partial_\alpha R(\alpha) - \hat{\varphi}), W\rangle| = (1 + \eta^2)|\alpha'|\,|\langle \partial_\alpha R(\alpha) - \hat{\varphi}, \mathcal{L}_1 W\rangle| \\
&\quad \le (1 + \eta^2)|\alpha'|\,\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|\mathcal{L}_1 W\|_{L^2} \\
&\quad \le C(\|W\|_{L^2} + \|\mathcal{N}(\alpha, W)\|_{L^2})\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|\mathcal{L}_1 W\|_{L^2} \\
&\quad = C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|W\|_{L^2}\|\mathcal{L}_1 W\|_{L^2} \\
&\qquad + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|\mathcal{N}(\alpha, W)\|_{L^2}\|\mathcal{L}_1 W\|_{L^2}, \qquad (48)
\end{aligned}
$$

here $C > 0$ depends only on a and η. Therefore, from (26), (39), (40), (41), (42), (43) and (48), there exist $C_1, C_2, C_3 > 0$ depending only on $a, \eta > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}(1+\eta^2)\frac{d}{dt}\langle W, -\mathscr{L}_1 W\rangle \\
&\quad \le -\|\mathscr{L}_1 W\|_{L^2}^2 + C|h|\,\|W\|_{H^1}\|\mathscr{L}_1 W\|_{L^2} \\
&\qquad + (1 + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2})\|\mathscr{N}(\alpha, W)\|_{L^2}\|\mathscr{L}_1 W\|_{L^2} \\
&\qquad + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|W\|_{L^2}\|\mathscr{L}_1 W\|_{L^2} \\
&\quad \le -\|\mathscr{L}_1 W\|_{L^2}^2 + C_2|h|\,\|W\|_{H^1}\|\mathscr{L}_1 W\|_{L^2} \\
&\qquad + C(1 + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2})(|h| + |\alpha| + \|W\|_{H^1})\|W\|_{H^2}\|\mathscr{L}_1 W\|_{L^2} \\
&\qquad + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|W\|_{L^2}\|\mathscr{L}_1 W\|_{L^2} \\
&\quad \le -\|\mathscr{L}_1 W\|_{L^2}^2 + C|h|\,\|\mathscr{L}_1 W\|_{L^2}^2 \\
&\qquad + C(1 + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2})(|h| + |\alpha| + \|W\|_{H^1})\|\mathscr{L}_1 W\|_{L^2}^2 \\
&\qquad + C\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|\mathscr{L}_1 W\|_{L^2}^2 \\
&\quad \le -\|\mathscr{L}_1 W\|_{L^2}^2 + C_1|h|\,\|\mathscr{L}_1 W\|_{L^2}^2 \\
&\qquad + C_2(1 + \|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2})(|h| + |\alpha| + \langle -\mathscr{L}_1 W, W\rangle^{1/2})\|\mathscr{L}_1 W\|_{L^2}^2 \\
&\qquad + C_3\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2}\|\mathscr{L}_1 W\|_{L^2}^2.
\end{aligned}
$$

From (46), if $|h|$ and $\Lambda > 0$ are sufficiently small, we obtain

$$
\begin{cases}
C_1|h| < \dfrac{1}{4}, \\[2mm]
C_2(1 + \|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2})(|h| + |\alpha|) < \dfrac{1}{4}, \\[2mm]
C_3\|\partial_\alpha R(\alpha) - \hat{\varphi}\|_{L^2} < \dfrac{1}{4},
\end{cases}
\tag{49}
$$

for any α with $|\alpha| < \Lambda$. By (49) there exists $C_4 = C_4(a, \eta) > 0$ such that

$$
\frac{1}{2}(1+\eta^2)\frac{d}{dt}\langle W, -\mathscr{L}_1 W\rangle \le \left(-\frac{1}{4} + C_4\langle W, -\mathscr{L}_1 W\rangle^{1/2}\right)\|\mathscr{L}_1 W\|_{L^2}^2. \tag{50}
$$

From Lemma 4, if $\|W\|_{H^1}$ is sufficiently small, then

$$
\left(-\frac{1}{4} + C_4\langle W, -\mathscr{L}_1 W\rangle^{1/2}\right) < -\frac{1}{8}. \tag{51}
$$

Hence from Lemma 3, Lemma 4, (50) and (51), there exists $C_5 = C_5(a,\eta) > 0$ such that

$$\frac{d}{dt}\langle W, -\mathscr{L}_1 W\rangle \le C_5\left(-\frac{1}{4} + C_4\langle W, -\mathscr{L}_1 W\rangle^{1/2}\right)\langle W, -\mathscr{L}_1 W\rangle.$$

On the other hand, by (32) we have

$$(1+\eta^2)\alpha'\langle \partial_\alpha R(\alpha), \varphi\rangle = -p\langle w_1, s_\delta\rangle + 2\eta v\delta^{-1}\langle w_1, s_\delta t_\delta\rangle + 2v\delta^{-1}\langle w_2, s_\delta t_\delta\rangle + \langle \mathscr{N}(\alpha, W), \varphi\rangle. \tag{52}$$

From (28), (36) and integration by parts, there exists $C_6 = C_6(a,\eta) > 0$ such that

$$|\langle \mathscr{N}(\alpha, W), \varphi\rangle| \le C_6(1+\Lambda)(\|W\|_{H^1} + \|W\|_{H^1}^2 + \|W\|_{H^1}^3), \tag{53}$$

here we remark that

$$\partial_z^2 \lambda(r) = -\frac{|\partial r|^2 + r\cdot\partial_z^2 r}{\lambda} - \frac{(r\cdot\partial_z r)^2}{\lambda^3}.$$

Let $\|W\|_{H^1} \le 1$ then from (45), (52) and (53) we have

$$|\alpha'| \le C_6'(1+\Lambda)\|W\|_{H^1},$$

here $C_6' > 0$ depends only on a and η. Therefore, if $|h|$ and $\|W(t)\|_{H^1}$ are sufficiently small and $|\alpha(t)| < \Lambda$ for any $t \in (0,T)$, then we have

$$\begin{cases} \dfrac{d}{dt}\langle W(t), -\mathscr{L}_1 W(t)\rangle \le C_5\left(-\dfrac{1}{4} + C_4\langle W(t), -\mathscr{L}_1 W(t)\rangle^{1/2}\right) \\ \qquad\qquad\qquad\qquad \times \langle W(t), -\mathscr{L}_1 W(t)\rangle, \\ |\alpha'(t)| \le C_6'(1+\Lambda)\|W(t)\|_{H^1}, \end{cases} \tag{54}$$

for any $t \in [0,T)$. From (51) there exist $C_7 > 0$ and $C_8 > 0$ depending only on a and η such that

$$\|W(t)\|_{H^1} \le C_7 e^{-C_8 t}\|W(0)\|_{H^1}, \tag{55}$$

for any $t \in [0,T)$. Then by (54) and (55), there exists $C_9 > 0$ such that

$$|\alpha(t)| \le |\alpha_0| + C_9(1+\Lambda)(1-e^{-C_8 t})\|W(0)\|_{H^1}, \tag{56}$$

for any $t \in [0,T)$. Let $W(0)$ and $\alpha(0)$ satisfy

$$\|W(0)\|_{H^1} < \frac{\Lambda}{2C_9(1+\Lambda)}, \qquad |\alpha(0)| < \frac{\Lambda}{2}.$$

Then we have

$$|\alpha(t)| < \Lambda, \tag{57}$$

for any $t \in [0, T)$. Therefore from (55), (56) and (57) we get $T = +\infty$. Hence $\|W(t)\|_{H^1}$ has an exponential decay, and there exists $\alpha \in \mathbf{R}$ such that

$$\alpha(t) \to \alpha \qquad \text{exponentially as } t \to \infty.$$

Furthermore we replace $\Lambda > 0$ and $W(0)$ to satisfy

$$\sup_{|\beta|<\Lambda} \|R(\beta)\|_{H^1} < \frac{\varepsilon_0}{2}, \qquad \|W(0)\|_{H^1} < \frac{\varepsilon_0}{2C_7}. \tag{58}$$

Hence we obtain

$$\|R(\alpha(t)) + W(t)\|_{H^1} \leq \|R(\alpha(t))\|_{H^1} + \|W(t)\|_{H^1} < \varepsilon_0,$$

for any $t \in [0, \infty)$. From Lemma 5 there exists $\varepsilon > 0$ such that if $\|R(\alpha(0)) + W(0)\|_{H^1} < \varepsilon$ then (58) holds. Therefore (i), (ii) and (iii) are satisfied.

Proof of Theorem 2. Let $|h|$ and $\|r_0\|_{H^1}$ be sufficiently small. Then from (iii) of Lemma 10 we have

$$\|r(t)\|_{L^2} \leq \|r(t)\|_{H^1} < \varepsilon_0,$$

for any $t \in [0, \infty)$. Here $\varepsilon_0 > 0$ is given by Lemma 5. Hence Lemma 6 holds for any $t \in [0, \infty)$. Furthermore by (i) and (ii) of Lemma 10 and (30), there exists $\alpha \in \mathbf{R}$ such that

$$r(t) \to R(\alpha) \qquad \text{exponentially as } t \to \infty \text{ in } H^1.$$

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