Hiroshima Math. J.
41 (2011), 235–273

A quantitative result on Sendov's conjecture for a zero near the unit circle

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(Received December 14, 2010)
(Revised January 21, 2011)

Abstract. On Sendov's conjecture, V. Vâjâitu and A. Zaharescu (and M. J. Miller, independently) state the following in their paper: if one zero a of a polynomial which has all the zeros in the closed unit disk is sufficiently close to the unit circle, then the distance from a to the closest critical point is less than 1. It is desirable to quantify this assertion. In the author's previous paper, we obtained an upper bound on the radius of the disk centered at the origin which contains all the critical points. In this paper, we improve it, and then, estimate the range of the zero a satisfying the above. This result, moreover, implies that if a zero of a polynomial is close to the unit circle and all the critical points are far from the zero, then the polynomial must be close to $P(z) = z^n - c$ with $|c| = 1$.

1. Introduction

Let $\mathscr{P}_n$ be the set of monic complex polynomials of degree n with all the zeros in the closed unit disk. For a polynomial $P \in \mathscr{P}_n$, let $z_1, \dots, z_n$ and $w_1, \dots, w_{n-1}$ be the zeros of P and P', respectively: w_j's are also called critical. Define $I_P(z_i) = \min_{1 \le j \le n-1} |z_i - w_j|$, $I(P) = \max_{1 \le i \le n} I_P(z_i)$ and $I(\mathscr{P}_n) = \sup_{P \in \mathscr{P}_n} I(P)$. Under the notation, Sendov's conjecture (see [7, p. 25 Problem 4.5]) is stated as follows:

Conjecture (Sendov). *For all positive integers $n \ge 2$, $I(\mathscr{P}_n) = 1$.*

A polynomial $P^* \in \mathscr{P}_n$ is said to be extremal if $I(P^*) = I(\mathscr{P}_n)$. Since $\mathscr{P}_n$ is a compact family, we can see the existence of extremal polynomials (see [13]). In addition to Sendov's conjecture, it is conjectured that if P^* is extremal, then $P^*(z) = z^n - c$ with $|c| = 1$. The example $P^*(z) = z^n - c$ shows that $I(\mathscr{P}_n) \ge 1$.

Sendov's conjecture is true in the case that $2 \le n \le 8$ (see [1]–[4], [10], [15] and [14], for example). It is also true in the case when one zero a is close to 0 even if $n > 8$ (see [3], for example). We should, therefore, show the case when $|a|$ is close to 1. In this situation, V. Vâjâitu and A. Zaharescu [16], and M. J. Miller [11] independently proved the following.

2010 *Mathematics Subject Classification.* Primary 12D10; Secondary 26C10, 30C15.
Key words and phrases. Sendov's conjecture; polynomial; zero; critical point.

THEOREM A. *For any $n \geq 2$, there exist $c_n > 0$ and $\varepsilon_n > 0$ such that, for every polynomial $P \in \mathscr{P}_n$, and for any zero a of P such that $1 - \varepsilon_n \leq |a|$, $I_P(a) \leq 1 - c_n(1 - |a|)$.*

Vâjâitu and Zaharescu pointed out that $\lim_{n\to\infty} \varepsilon_n = 0$, and that c_n can be chosen to be close to $1/4$ for large n and to be $1/4$ for n divisible by 4.

Their results imply that Sendov's conjecture is true if a zero of a polynomial is sufficiently close to 1. We, however, know neither how close the zero is to the unit circle nor how small the constant c_n is, in order that the conclusion of Theorem A may hold. In particular, not knowing the size of ε_n, we cannot apply Theorem A to the proof of Sendov's conjecture. Furthermore, in their proofs, they use the following fact under the assumption that one zero a of a polynomial is close to the unit circle: if all the critical points are far from a, then the polynomial must be close to the polynomial $P^*(z) = z^n - c$ with $|c| = 1$. It is, however, not stated as any proposition. Nevertheless, it is a key to Theorem A.

In the author's previous paper [5], we quantified the radius of a disk centered at the origin which contains all the critical points of a polynomial. Our aim in this paper is to improve the result of the previous paper and moreover, to quantify Theorem A.

Our result is stated as follows:

MAIN THEOREM. *Let P be a polynomial of degree $n \geq 4$ with all the zeros in the closed unit disk. If one zero a of P satisfies $|a| \geq 1 - \varepsilon_n$, where*

$$\varepsilon_n = \frac{1}{2n^9 4^n},$$

then there exists a critical point w such that

$$|w - a| \leq 1 - c_n(1 - |a|),$$

where

$$c_n = \begin{cases} \dfrac{1}{4}, & \text{if } n = 4s, \\ \dfrac{n-3}{4(n-1)}, & \text{if } n = 4s+1, \\ \dfrac{n-6}{4(n-1)}, & \text{if } n = 4s+2, \\ \dfrac{n-9}{4(n-1)}, & \text{if } n = 4s+3 \text{ and } n \neq 7, \\ \dfrac{1}{1159}, & \text{if } n = 7 \text{ (exceptionally)} \end{cases}$$

for every positive integer s.

Since this result is quantitative, we can partly solve Sendov's conjecture. Define $S(n,a)$ to be the set of polynomials in $\mathscr{P}_n$ with at least one zero at a. There exists $A_n \in (0,1]$ such that for a with $|a| < A_n$, $I_P(a) \le 1$ whenever $P \in S(n,a)$ (see [3] and Section 4, for example). On the other hand, our result shows that for a with $1 - \varepsilon_n \le |a| < 1$, $I_P(a) \le 1$ whenever $P \in S(n,a)$. This means that Sendov's conjecture is still open for polynomials which have one zero $a \in [A_n, 1 - \varepsilon_n)$.

In this paper, we shall prepare a series of lemmas in Section 2, and Main Theorem will be proven in Section 3. In order to visualize the open gap interval of $|a|$ not covered by our results, we exhibit several numerical values of A_n and $1 - \varepsilon_n$ for some n in Section 4.

2. Preliminary results

We follow the arguments developed in V. Vâjâitu and A. Zaharescu [16]. Let P be a polynomial with all the zeros in the closed unit disk, and of the form

$$P(z) = (z - a)\prod_{i=2}^{n}(z - z_i)$$

and let $w_1, \dots, w_{n-1}$ be the critical points of P. Hereafter, we may assume that $n \ge 4$, $z_1 = a = 1 - \varepsilon$ with $0 < \varepsilon \le \varepsilon_n$ and $0 \le \arg z_2 \le \cdots \le \arg z_n < 2\pi$. To prove Main Theorem by contradiction, we also suppose that for $j = 1, \dots, n-1$,

$$|w_j - a| > 1 - c_n(1 - a) = 1 - c_n\varepsilon. \tag{1}$$

Note that $0 \le c_n \le 1/4$. First, we shall use the following lemma to prove Lemma 1.

Lemma A (Kumar and Shenoy [8]). *Let $p(z) = (z - a)\prod_{i=1}^{n-1}(z - z_i)$ be any polynomial with $0 \le a \le 1$ and $|z_i| \le 1$, $1 \le i \le n-1$. If the disk $|z - a| \le r$ contains no zero of p', then $|z_i - a| > 2r\sin(\pi/n)$ for any i.*

S. Kumar and B. G. Shenoy proved the original lemma in the case that $r = 1$. It is not difficult to modify their proof for our case. Lemma 1 is used to prove Lemma 2.

Lemma 1. *For $j = 1, \dots, n-1$ and $i = 2, \dots, n$,*

$$1 - k_3\varepsilon < \operatorname{Re}\frac{1}{a - w_j} < 1 + k_1\varepsilon,$$

$$\frac{1}{2} - k_2\varepsilon < \operatorname{Re}\frac{1}{a - z_i} < \frac{1}{2} + k_4\varepsilon,$$

where

$$k_1 := \frac{c_n}{1 - c_n\varepsilon_n} \in \left[0, \frac{1}{4} + \frac{\varepsilon_n}{12}\right),$$

$$k_2 := \frac{2 - \varepsilon_n}{8(1 - \varepsilon_n)(1 - c_n\varepsilon_n)^2 \sin^2(\pi/n)} \in \left(\frac{n^2}{4\pi^2}, 0.0313n^2\right),$$

$$k_3 := (n-2)k_1 + 2(n-1)k_2 \in \left(\frac{3n^3}{8\pi^2}, 0.0626n^3\right),$$

$$k_4 := \frac{(n-1)k_1 + 2(n-2)k_2}{2} < 0.0313n^3.$$

PROOF. For $j = 1, \dots, n-1$, we have

$$\operatorname{Re}\frac{1}{a - w_j} \le \frac{1}{|a - w_j|} < \frac{1}{1 - c_n\varepsilon} = 1 + \frac{c_n}{1 - c_n\varepsilon}\varepsilon \le 1 + \frac{c_n}{1 - c_n\varepsilon_n}\varepsilon$$
$$= 1 + k_1\varepsilon. \tag{2}$$

Since $|a - z_i| > 2(1 - c_n\varepsilon)\sin(\pi/n) \ge 2(1 - c_n\varepsilon_n)\sin(\pi/n)$ by Lemma A,

$$\operatorname{Re}\frac{1}{a - z_i} = \frac{1}{2a} - \frac{|z_i|^2 - a^2}{2a|a - z_i|^2}$$
$$> \frac{1}{2} - \frac{1 - a^2}{2a \cdot 4(1 - c_n\varepsilon_n)^2 \sin^2(\pi/n)}$$
$$= \frac{1}{2} - \frac{1 + a}{a} \cdot \frac{\varepsilon}{8(1 - c_n\varepsilon_n)^2 \sin^2(\pi/n)}$$
$$\ge \frac{1}{2} - \frac{2 - \varepsilon_n}{8(1 - \varepsilon_n)(1 - c_n\varepsilon_n)^2 \sin^2(\pi/n)}\varepsilon = \frac{1}{2} - k_2\varepsilon. \tag{3}$$

Writing $P(z) = (z - a)Q(z)$, we obtain $P''(a)/P'(a) = 2Q'(a)/Q(a)$, and we know

$$\frac{P''(a)}{P'(a)} = \sum_{j=1}^{n-1} \frac{1}{a - w_j} \qquad \text{and} \qquad \frac{Q'(a)}{Q(a)} = \sum_{i=2}^{n} \frac{1}{a - z_i}.$$

By (2) and (3), we also obtain

$$(n-2)(1 + k_1\varepsilon) + \operatorname{Re}\frac{1}{a - w_{j_0}} > \sum_{j=1}^{n-1} \operatorname{Re}\frac{1}{a - w_j}$$
$$= 2\sum_{i=2}^{n} \operatorname{Re}\frac{1}{a - z_i} > (n-1)(1 - 2k_2\varepsilon), \tag{4}$$

$$(n-1)(1+k_1\varepsilon) > \sum_{j=1}^{n-1} \operatorname{Re} \frac{1}{a-w_j}$$

$$= 2\sum_{i=2}^{n} \operatorname{Re} \frac{1}{a-z_i} > (n-2)(1-2k_2\varepsilon) + 2 \operatorname{Re} \frac{1}{a-z_{i_0}} \qquad (5)$$

for $j_0 = 1, \ldots, n-1$ and $i_0 = 2, \ldots, n$. By (4),

$$\operatorname{Re} \frac{1}{a-w_{j_0}} > (n-1)(1-2k_2\varepsilon) - (n-2)(1+k_1\varepsilon)$$

$$= 1 - ((n-2)k_1 + 2(n-1)k_2)\varepsilon = 1 - k_3\varepsilon.$$

By (5),

$$\operatorname{Re} \frac{1}{a-z_{i_0}} < \frac{1}{2}((n-1)(1+k_1\varepsilon) - (n-2)(1-2k_2\varepsilon))$$

$$= \frac{1}{2} + \frac{(n-1)k_1 + 2(n-2)k_2}{2}\varepsilon = \frac{1}{2} + k_4\varepsilon.$$

Combining these results with (2) and (3), we complete the proof.

Finally, we quantify the constants k_1, k_2, k_3 and k_4. Since $c_n(c_n + \varepsilon_n/12) < 1/12$, namely, $c_n^2/(1-c_n\varepsilon_n) < 1/12$,

$$k_1 = c_n + \frac{c_n^2}{1-c_n\varepsilon_n}\varepsilon_n < \frac{1}{4} + \frac{\varepsilon_n}{12}.$$

It is trivial that $k_1 \geq 0$. Since $\sin(\pi/n) \geq 2\sqrt{2}/n$ for $n \geq 4$ and $c_n \leq 1/4$,

$$k_2 \leq \frac{2-\varepsilon_n}{8(1-\varepsilon_n)(1-\varepsilon_n/4)^2(2\sqrt{2}/n)^2} = \frac{n^2}{4} \cdot \frac{2-\varepsilon_n}{(1-\varepsilon_n)(4-\varepsilon_n)^2}$$

$$= \frac{n^2}{4} \cdot \left(1 + \frac{1}{1-\varepsilon_n}\right) \frac{1}{(4-\varepsilon_n)^2} < 0.0313n^2.$$

Note that $\sin(\pi/n) < \pi/n$ and $c_n\varepsilon_n > 0$ imply

$$k_2 > \frac{2-\varepsilon_n}{8(1-\varepsilon_n)(\pi/n)^2} = \frac{n^2}{8\pi^2}\left(1 + \frac{1}{1-\varepsilon_n}\right) > \frac{n^2}{4\pi^2}.$$

Since $k_1 < 1/4 + \varepsilon_n/12 < 1/2$, we have

$$k_3 < (n-2)/2 + 2(n-1) \cdot 0.0313n^2$$

$$= n^3\left(0.0626 - \frac{0.0626}{n} + \frac{1}{2n^2} - \frac{1}{n^3}\right)$$

$$= n^3\left[0.0626 - \frac{1}{n}\left(\left(\frac{1}{n} - \frac{1}{4}\right)^2 + 0.0001\right)\right] < 0.0626n^3.$$

On the other hand,

$$k_3 > (n-2)\cdot 0 + 2(n-1)\cdot \frac{n^2}{4\pi^2} = \frac{n^3}{2\pi^2}\left(1-\frac{1}{n}\right) \geq \frac{3n^3}{8\pi^2}.$$

Since $k_1 < 1/4 + \varepsilon_n/12 < 1/2$,

$$\begin{aligned} k_4 &< \frac{1}{2}((n-1)/2 + 2(n-2)\cdot 0.032n^2) \\ &= n^3\left(0.0313 - \frac{0.0626}{n} + \frac{1}{4n^2} - \frac{1}{4n^3}\right) \\ &= n^3\left[0.0313 - \frac{1}{4n}\left(\left(\frac{1}{n}-\frac{1}{2}\right)^2 + 0.0004\right)\right] < 0.0313n^3. \qquad \square \end{aligned}$$

The next lemma is used frequently in this paper, especially in the proof of Main Theorem. It is also an improvement of the result in the author's previous paper [5].

LEMMA 2. *For* $j = 1, \dots, n-1$,

$$|w_j| < k_7\sqrt{\varepsilon},$$

where

$$k_7 := k_6 + k_5\sqrt{\varepsilon_n} \in \left(\frac{\sqrt{3}}{2\pi}n\sqrt{n}, 0.3649n\sqrt{n}\right)$$

with

$$k_5 := 1 + \frac{k_3}{1-k_3\varepsilon_n} \in \left(\frac{3n^3}{8\pi^2}, 0.0783n^3\right),$$

$$k_6 := \sqrt{\left(\frac{k_3}{1-k_3\varepsilon_n} + c_n\right)\left(\frac{1}{1-k_3\varepsilon_n} + 1\right)} \in \left(\frac{\sqrt{3}}{2\pi}n\sqrt{n}, 0.3648n\sqrt{n}\right).$$

PROOF. We define $h_j \in \mathbf{R}$ by

$$\frac{1}{a-h_j} := \operatorname{Re}\frac{1}{a-w_j}.$$

Since $1/(a-h_j) > 1 - k_3\varepsilon$ by Lemma 1 and $1 - k_3\varepsilon > 0$,

$$0 < a - h_j < \frac{1}{1-k_3\varepsilon}. \tag{6}$$

Thus

$$h_j > -\left(1 + \frac{k_3}{1-k_3\varepsilon}\right)\varepsilon \geq -\left(1 + \frac{k_3}{1-k_3\varepsilon_n}\right)\varepsilon = -k_5\varepsilon.$$

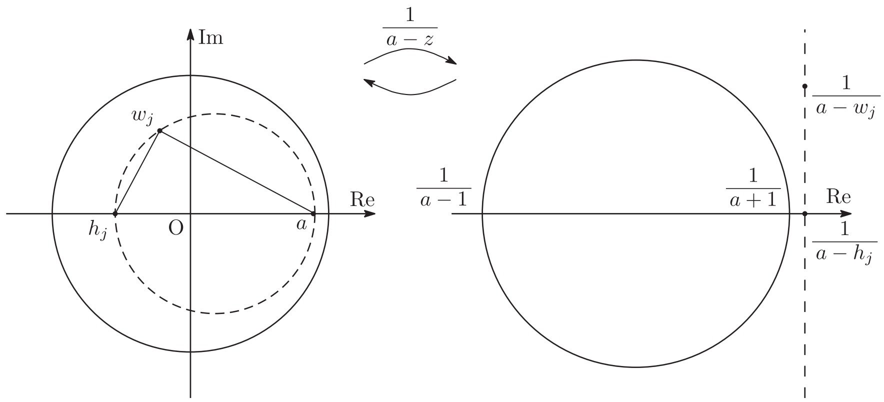

Fig. 1. w_j, h_j and their images by $1/(a-z)$.

On the other hand, since $h_j \leq a - (1 - c_n\varepsilon) = -(1 - c_n)\varepsilon$,

$$-k_5\varepsilon < h_j \leq -(1 - c_n)\varepsilon.$$

If $w_j \neq h_j$, then by Pythagoras' theorem and (6) (see Figure 1),

$$|w_j - h_j|^2 + |a - w_j|^2 = |a - h_j|^2 < \left(\frac{1}{1 - k_3\varepsilon}\right)^2.$$

This implies that

$$\begin{aligned} |w_j - h_j|^2 &< \left(\frac{1}{1 - k_3\varepsilon}\right)^2 - (1 - c_n\varepsilon)^2 \\ &= \left(\frac{k_3}{1 - k_3\varepsilon} + c_n\right)\left(\frac{1}{1 - k_3\varepsilon} + 1 - c_n\varepsilon\right)\varepsilon \\ &< \left(\frac{k_3}{1 - k_3\varepsilon_n} + c_n\right)\left(\frac{1}{1 - k_3\varepsilon_n} + 1\right)\varepsilon = k_6^2\varepsilon. \end{aligned}$$

Therefore, $|w_j| - |h_j| \leq |w_j - h_j| < k_6\sqrt{\varepsilon}$, that is,

$$|w_j| < k_6\sqrt{\varepsilon} + |h_j| < k_6\sqrt{\varepsilon} + k_5\varepsilon \leq (k_6 + k_5\sqrt{\varepsilon_n})\sqrt{\varepsilon} = k_7\sqrt{\varepsilon}.$$

If $w_j = h_j$, then $|w_j| = |h_j| < k_5\varepsilon < k_7\sqrt{\varepsilon}$ also holds.

Finally, we quantify the constants k_5, k_6 and k_7. We have

$$\begin{aligned} k_5 &< 1 + \frac{0.0626n^3}{1 - 0.0626n^3/(2n^9 4^n)} = n^3\left(\frac{1}{n^3} + \frac{0.0626}{1 - 0.0626/(2n^6 4^n)}\right) \\ &\leq n^3\left(\frac{1}{4^3} + \frac{0.0626}{1 - 0.0626/(2 \cdot 4^6 \cdot 4^4)}\right) < 0.0783n^3. \end{aligned}$$

On the other hand, since $\varepsilon_n > 0$, $k_5 > 1 + k_3 > k_3 > 3n^3/(8\pi^2)$. We have

$$k_6^2 < \left(\frac{0.0626n^3}{1 - 0.0626n^3/(2n^9 4^n)} + \frac{1}{4}\right)\left(\frac{1}{1 - 0.0626n^3/(2n^9 4^n)} + 1\right)$$
$$= n^3\left(\frac{0.0626}{1 - 0.0626/(2n^6 4^n)} + \frac{1}{4n^3}\right)\left(\frac{1}{1 - 0.0626/(2n^6 4^n)} + 1\right)$$
$$\leq n^3\left(\frac{0.0626}{1 - 0.0626/(2 \cdot 4^6 \cdot 4^4)} + \frac{1}{4 \cdot 4^3}\right)\left(\frac{1}{1 - 0.0626/(2 \cdot 4^6 \cdot 4^4)} + 1\right)$$
$$< 0.3648^2 n^3,$$

that is, $k_6 < 0.3648n\sqrt{n}$. On the other hand, since $k_3\varepsilon_n > 0$,

$$k_6 > \sqrt{(k_3 + 0)(1 + 1)} = \sqrt{2k_3} > \sqrt{\frac{3n^3}{4\pi^2}} = \frac{\sqrt{3}}{2\pi} n\sqrt{n}.$$

We also have

$$k_7 < 0.3648n\sqrt{n} + \frac{0.0783n^3}{\sqrt{2}n^4\sqrt{n}2^n} = n\sqrt{n}\left(0.3648 + \frac{0.0783}{\sqrt{2}n^3 2^n}\right)$$
$$\leq n\sqrt{n}\left(0.3648 + \frac{0.0783}{\sqrt{2} \cdot 4^3 \cdot 2^4}\right) < 0.3649n\sqrt{n}.$$

On the other hand,

$$k_7 > \frac{\sqrt{3}}{2\pi} n\sqrt{n} + \frac{3n^3}{8\pi^2} \cdot \frac{1}{\sqrt{2}n^4\sqrt{n}2^n} > \frac{\sqrt{3}}{2\pi} n\sqrt{n}. \qquad \square$$

For the real parts of the critical points, we can improve the order of the estimate in Lemma 2. The next Lemma 3 is used in the proof of Lemma 7 and Main Theorem.

LEMMA 3. *For* $j = 1, \ldots, n-1$,

$$-k_5\varepsilon < \mathrm{Re}\, w_j < k_8\varepsilon,$$

where

$$k_8 := \frac{k_7^2}{1 - \varepsilon_n - k_7\sqrt{\varepsilon_n}} - (1 - c_n) \in \left(\left(\frac{3}{4\pi^2} - \frac{1}{64}\right)n^3, 0.1332n^3\right).$$

PROOF. Since $\mathrm{Re}\, w_j \geq h_j > -k_5\varepsilon$, the first inequality holds. If $w_j \neq h_j$, then by the power-of-a-point theorem and Lemma 2,

$$(\mathrm{Re}\, w_j - h_j)(a - \mathrm{Re}\, w_j) = (\mathrm{Im}\, w_j)^2 < k_7^2\varepsilon.$$

Furthermore, since

$$\varepsilon_n + k_7\sqrt{\varepsilon_n} < \frac{1}{2n^9 4^n} + \frac{0.3649 n\sqrt{n}}{\sqrt{2}n^4\sqrt{n}2^n}$$
$$= \frac{1}{n^3 2^n}\left(\frac{1}{2n^6 2^n} + \frac{0.3649}{\sqrt{2}}\right) < \frac{0.2581}{n^3 2^n} < 1,$$

it follows that

$$a - \operatorname{Re} w_j = 1 - \varepsilon - \operatorname{Re} w_j > 1 - \varepsilon - k_7\sqrt{\varepsilon} \geq 1 - (\varepsilon_n + k_7\sqrt{\varepsilon_n}) > 0.$$

Therefore,

$$\operatorname{Re} w_j < \frac{k_7^2\varepsilon}{1 - \varepsilon - k_7\sqrt{\varepsilon}} - (1 - c_n)\varepsilon \leq \left(\frac{k_7^2}{1 - \varepsilon_n - k_7\sqrt{\varepsilon_n}} - (1 - c_n)\right)\varepsilon = k_8\varepsilon.$$

If $w_j = h_j$, then $\operatorname{Re} w_j = h_j \leq -(1 - c_n\varepsilon) < 0 < k_8\varepsilon$ also holds.

Finally, we quantify the constant k_8. We have

$$k_8 = \frac{k_7^2}{1 - \varepsilon_n - k_7\sqrt{\varepsilon_n}} - (1 - c_n)$$
$$< \frac{k_7^2}{1 - 0.2581/(n^3 2^n)} - \frac{3}{4} < n^3\left(\frac{0.3649^2}{1 - 0.2581/(n^3 2^n)} - \frac{3}{4n^3}\right).$$

For $n \geq 4$, $2^n \geq 2^4$ and hence

$$\frac{0.3649^2}{1 - 0.2581/(n^3 2^n)} \leq \frac{0.3649^2}{1 - 0.2581/(n^3 2^4)}.$$

Since

$$f_8(x) := \frac{0.3649^2}{1 - (0.2581/2^4)x} - \frac{3}{4}x$$

is monotone decreasing on $(0, 1/64]$, we have $f_8(x) < f_8(0)$. Thus

$$k_8 < n^3 \cdot f_8(1/n^3) < n^3 \cdot f_8(0) < 0.3649^2 n^3 < 0.1332 n^3.$$

On the other hand,

$$k_8 > k_7^2 - 1 > \frac{3n^3}{4\pi^2} - 1 = n^3\left(\frac{3}{4\pi^2} - \frac{1}{n^3}\right) \geq n^3\left(\frac{3}{4\pi^2} - \frac{1}{4^3}\right)$$
$$= n^3\left(\frac{3}{4\pi^2} - \frac{1}{64}\right). \qquad \square$$

REMARK 1. *As a matter of fact,* $k_5 < k_8$. *By the proof of Lemma* 2, *if* $n \geq 6$, *then*

$$k_5 < n^3\left(\frac{1}{6^3} + \frac{0.0626}{1 - 0.0626/(2 \cdot 6^6 \cdot 4^6)}\right) < 0.0673n^3.$$

By the proof of Lemma 3, *if* $n \geq 6$, *then*

$$k_8 > n^3\left(\frac{3}{4\pi^2} - \frac{1}{6^3}\right) > 0.0713n^3.$$

This means that $k_5 < k_8$ *if* $n \geq 6$. *Since* $c_4 = 1/4$ *and* $c_5 = 1/8$, *a numerical computation gives the approximate values of each constant for* $n = 4,\ 5$;

n	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
4	0.25	0.50	3.50	0.88	4.50	2.74	2.74	6.75
5	0.25	0.72	6.54	1.25	7.54	3.68	3.68	12.8

This shows that $k_5 < k_8$ *also holds for* $n = 4$ *and* $n = 5$.

To prove Lemma 4, we shall use the next lemma.

LEMMA B (Miller [11]). *Let P be a polynomial of degree n. If* $P'(w_0) \neq 0$, *then there exists a zero* z_0 *of P such that* $|z_0 - w_0| \leq n|P(w_0)/P'(w_0)|$.

Define $\xi_\nu = \exp(2(\nu - 1)\pi\sqrt{-1}/n)$ for $\nu = 1, \ldots, n$. We shall use Lemma 4 for Lemma 6. Lemma 4 shall be improved by making use of other lemmas.

LEMMA 4. *For* $i = 1, \ldots, n$,

$$|z_i - \xi_i| < k_{10}\sqrt{\varepsilon},$$

where

$$k_{10} = \frac{k_9}{k_9'} < 0.7315n^2\sqrt{n}$$

with

$$k_9 = n\sqrt{\varepsilon_n} + 2nk_7 + 2k_7^2\sqrt{\varepsilon_n}(2^n - 2 - n) < 0.7306n^2\sqrt{n},$$

$$k_9' = (1 - k_7\sqrt{\varepsilon_n})^{n-1} > 0.9989.$$

PROOF. Writing $P(z) = \sum_{\nu=0}^{n} C_\nu z^\nu$, we have for fixed i,

$$P(\xi_i) = P(\xi_i) - P(a) = \sum_{\nu=1}^{n} C_\nu(\xi_i^\nu - (1-\varepsilon)^\nu)$$

$$= (1 - (1-\varepsilon)^n) + \sum_{\nu=1}^{n-1} C_\nu(\xi_i^\nu - (1-\varepsilon)^\nu).$$

Now since

$$P'(z) = \sum_{v=0}^{n} vC_v z^{v-1} = n\prod_{j=1}^{n-1}(z - w_j),$$

we have

$$vC_v = n \sum_{1 \le j_1 < \cdots < j_{n-v} \le n-1} (-1)^{n-v} w_{j_1} \dots w_{j_{n-v}},$$

and by Lemma 2, $|vC_v| < n\binom{n-1}{v-1}(k_7\sqrt{\varepsilon})^{n-v}$, namely, $|C_v| < \binom{n}{v}(k_7\sqrt{\varepsilon})^{n-v}$. Hence,

$$\begin{aligned}
|P(\xi_i)| &< (1-(1-\varepsilon)^n) + \sum_{v=1}^{n-1}\binom{n}{v}(k_7\sqrt{\varepsilon})^{n-v}|\xi_i^v - (1-\varepsilon)^v| \\
&< n\varepsilon + 2nk_7\sqrt{\varepsilon} + 2(k_7\sqrt{\varepsilon})^2\sum_{v=1}^{n-2}\binom{n}{v} \\
&= n\varepsilon + 2nk_7\sqrt{\varepsilon} + 2(k_7\sqrt{\varepsilon})^2(2^n - 2 - n) \\
&\le (n\sqrt{\varepsilon_n} + 2nk_7 + 2k_7^2\sqrt{\varepsilon_n}(2^n - 2 - n))\sqrt{\varepsilon} = k_9\sqrt{\varepsilon}.
\end{aligned}$$

In addition, for $j = 1, \dots, n-1$, $|\xi_i - w_j| > 1 - k_7\sqrt{\varepsilon}$ by Lemma 2. We have

$$|P'(\xi_i)| = \left|n\prod_{j=1}^{n-1}(\xi_i - w_j)\right| > n(1 - k_7\sqrt{\varepsilon})^{n-1} \ge n(1 - k_7\sqrt{\varepsilon_n})^{n-1} = nk_9'.$$

Hence, by Lemma B, there exists a zero z_{i_0} of P such that

$$|z_{i_0} - \xi_i| \le n\left|\frac{P(\xi_i)}{P'(\xi_i)}\right| < n \cdot \frac{k_9\sqrt{\varepsilon}}{nk_9'} = k_{10}\sqrt{\varepsilon}.$$

Now we quantify the constants k_9 and k_9' to estimate k_{10}. We have

$$\begin{aligned}
k_9 &< \frac{n}{\sqrt{2}n^4\sqrt{n}2^n} + 2n(0.3649n\sqrt{n}) + \frac{2(0.3649n\sqrt{n})^2(2^n - 2 - n)}{\sqrt{2}n^4\sqrt{n}2^n} \\
&< n^2\sqrt{n}\left(\frac{1}{\sqrt{2}n^6 2^n} + 0.7298 + \frac{0.2664(2^n - 2 - n)}{\sqrt{2}n^4 2^n}\right) \\
&< n^2\sqrt{n}\left(\frac{1}{\sqrt{2}n^6 2^n} + 0.7298 + \frac{0.2664 \cdot 2^n}{\sqrt{2}n^4 2^n}\right) \\
&\le n^2\sqrt{n}\left(\frac{1}{\sqrt{2}\cdot 4^6 \cdot 2^4} + 0.7298 + \frac{0.2664}{\sqrt{2}\cdot 4^4}\right) < 0.7306n^2\sqrt{n}.
\end{aligned}$$

On the other hand, since $1 - 0.3649/(\sqrt{2}n^3 2^n) < 1$,

$$k_9' = \left(1 - \frac{0.3649n\sqrt{n}}{\sqrt{2}n^4\sqrt{n}2^n}\right)^{n-1} > \left(1 - \frac{0.2581}{n^3 2^n}\right)^n.$$

Put $f_9(x) := x \log(1 - 0.2581/(x^3 2^x))$ for $x \geq 4$. Then

$$f_9'(x) = \log(1 - 0.2581/(x^3 2^x)) + \frac{0.2581(3 + x \log 2)}{x^3 2^x - 0.2581}.$$

Since

$$f_9''(x) = -\frac{0.2581[0.2581(3 + 2x \log 2) + x^3 2^x(6 + 4x \log 2 + (x \log 2)^2)]}{x(x^3 2^x - 0.2581)^2} < 0,$$

$f_9'(x)$ is monotone decreasing. We have

$$f_9'(x) > \lim_{x \to \infty} f_9'(x) = 0,$$

which implies that $f_9(x)$ is monotone increasing, so is $e^{f_9(x)}$. Thus it follows that

$$k_9' > \left(1 - \frac{0.2581}{n^3 2^n}\right)^n = e^{f_9(n)} \geq \left(1 - \frac{0.2581}{4^3 \cdot 2^4}\right)^4 > 0.9989.$$

Hence

$$k_{10} = \frac{k_9}{k_9'} < \frac{0.7306n^2\sqrt{n}}{0.9989} < 0.7315n^2\sqrt{n}.$$

Now,

$$k_{10}\sqrt{\varepsilon} \leq k_{10}\sqrt{\varepsilon_n} < \frac{0.7315n^2\sqrt{n}}{\sqrt{2}n^4\sqrt{n}2^n} < \frac{2\sqrt{2}}{n} \leq \sin(\pi/n)$$

for $n \geq 4$, and therefore, the n disks $\{|z - \xi_i| < k_{10}\sqrt{\varepsilon}\}_{i=1}^n$ are mutually disjoint. This implies $i_0 = i$. □

The next lemma shall be used for Lemma 6.

LEMMA 5. *For* $n \geq 4$,

$$\sum_{v=2}^{n} \frac{1}{\sin((v-1)\pi/n)} < \frac{n(n+11)}{8}.$$

PROOF. Let $N = \lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer not greater than x. Since $\sin((v-1)\pi/n) \geq 2(v-1)/n$ for $v \in [2, N+1]$,

$$\sum_{v=2}^{n} \frac{1}{\sin((v-1)\pi/n)}$$

$$= \begin{cases} 1 + 2\sum_{v=2}^{N} \frac{1}{\sin((v-1)\pi/n)} \le 1 + 2\sum_{v=2}^{N} \frac{n}{2(v-1)} = 1 + n\sum_{v=1}^{N-1} \frac{1}{v}, & \text{if } n \text{ is even,} \\ 2\sum_{v=2}^{N+1} \frac{1}{\sin((v-1)\pi/n)} \le 2\sum_{v=2}^{N+1} \frac{n}{2(v-1)} = n\sum_{v=1}^{N} \frac{1}{v}, & \text{if } n \text{ is odd.} \end{cases}$$

For even integer $n > 4$,

$$\sum_{v=1}^{N-1} \frac{1}{v} < 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{N-1}\right)(N-3)$$

$$= \frac{N^2 + 4N - 9}{4(N-1)}$$

$$< \frac{N^2 + 4N - 5}{4(N-1)} - \frac{1}{2N} = \frac{N+5}{4} - \frac{1}{2N} = \frac{n+10}{8} - \frac{1}{n},$$

which implies that

$$1 + n\sum_{v=1}^{N-1} \frac{1}{v} < 1 + \frac{n(n+10)}{8} - 1 = \frac{n(n+10)}{8}.$$

Since

$$\sum_{v=2}^{n} \frac{1}{\sin((v-1)\pi/n)} = 1 + 2\sqrt{2} < \frac{n(n+10)}{8}$$

for $n = 4$,

$$\sum_{v=2}^{n} \frac{1}{\sin((v-1)\pi/n)} < \frac{n(n+10)}{8}$$

holds also for even integer $n \ge 4$. For odd integer $n \ge 5$,

$$\sum_{v=1}^{N} \frac{1}{v} < 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{N}\right)(N-2)$$

$$= \frac{N^2 + 6N - 4}{4N} < \frac{N+6}{4} = \frac{(n-1)/2 + 6}{4} = \frac{n+11}{8}.$$

As a result, for $n \geq 4$

$$\sum_{v=2}^{n} \frac{1}{\sin((v-1)\pi/n)} < \max\left\{\frac{n(n+10)}{8}, \frac{n(n+11)}{8}\right\} = \frac{n(n+11)}{8}. \qquad \square$$

Lemma 6 and Remark 2 shall be used for Lemma 7 and Lemma 8. It shall be also improved for the proof of Main Theorem.

LEMMA 6. *For* $i = 1, \dots, n$,

$$\left| \xi_i \prod_{v \neq i} (\xi_i - z_v) - n \right| < k_{11}\sqrt{\varepsilon},$$

where

$$k_{11} := 0.0460 n^4 (n+11)\sqrt{n}.$$

In particular,

$$n - k_{11}\sqrt{\varepsilon} \leq \mathrm{Re}\left(\xi_i \prod_{v \neq i} (\xi_i - z_v) \right) \leq n + k_{11}\sqrt{\varepsilon},$$

$$-k_{11}\sqrt{\varepsilon} \leq \mathrm{Im}\left(\xi_i \prod_{v \neq i} (\xi_i - z_v) \right) \leq k_{11}\sqrt{\varepsilon},$$

$$n - k_{11}\sqrt{\varepsilon} \leq \left| \xi_i \prod_{v \neq i} (\xi_i - z_v) \right| \leq n + k_{11}\sqrt{\varepsilon}.$$

PROOF. Writing $P_0(z) = z^n - 1 = (z - \xi_i)Q_0(z)$, we obtain $P_0'(z) = Q_0(z) + (z - \xi_i)Q_0'(z)$, and therefore $P_0'(\xi_i) = n\xi_i^{n-1} = Q_0(\xi_i)$. Let $P(z) = (z - z_i)Q_i(z)$. If we put

$$K := \left| \prod_{v \neq i} (\xi_i - z_v) - n\xi_i^{n-1} \right| = |Q_i(\xi_i) - Q_0(\xi_i)|,$$

then it is sufficient to show that $K < k_{11}\sqrt{\varepsilon}$. We have

$$Q_i(\xi_i) = \prod_{v \neq i} (\xi_i - z_v) = \prod_{v \neq i} \left[(\xi_i - \xi_v) \left(\frac{\xi_i - z_v}{\xi_i - \xi_v} \right) \right] = Q_0(\xi_i) \cdot \prod_{v \neq i} \left(1 + \frac{\xi_v - z_v}{\xi_i - \xi_v} \right). \quad (7)$$

Put $\Delta_v = (\xi_v - z_v)/(\xi_i - z_v)$ for $v = 2, \dots, n$ $(v \neq i)$. By Lemma 4,

$$|\Delta_v| = \frac{|\xi_v - z_v|}{|\xi_i - \xi_v|} < \frac{k_{10}\sqrt{\varepsilon}}{2\sin(\pi/n)} < \frac{0.7315 n^2 \sqrt{n}\sqrt{\varepsilon}}{4\sqrt{2}/n} = \frac{0.7315 n^3 \sqrt{n}}{4\sqrt{2}} \sqrt{\varepsilon},$$

and $\varepsilon \leq \varepsilon_n$ implies that

$$|\Delta_v| < \frac{0.7315}{4\sqrt{2}} \cdot \frac{n^3\sqrt{n}}{\sqrt{2}n^4\sqrt{n}2^n} = \frac{0.7315}{8} \cdot \frac{1}{n2^n} < 0.0015.$$

Since

$$
\begin{aligned}
|\log(1+\Delta_\nu)| &\le \sum_{r=1}^{\infty} \frac{|\Delta_\nu|^r}{r} \\
&< |\Delta_\nu|\left(\frac{1}{2} + \sum_{r=1}^{\infty} \frac{|\Delta_\nu|^{r-1}}{2}\right) = \frac{|\Delta_\nu|}{2}\left(1 + \frac{1}{1-|\Delta_\nu|}\right),
\end{aligned}
$$

we have

$$
|\log(1+\Delta_\nu)| < \frac{|\Delta_\nu|}{2}\left(1 + \frac{1}{1-0.0015}\right) < 1.0008|\Delta_\nu|.
$$

By making use of Lemma 5,

$$
\begin{aligned}
\left|\sum_{\nu \ne i} \log(1+\Delta_\nu)\right| &< 1.0008 \sum_{\nu \ne i} |\Delta_\nu| \\
&< \frac{1.0008 k_{10}\sqrt{\varepsilon}}{2} \sum_{\nu \ne i} \frac{1}{|\sin(\nu - i)\pi/n)|} \\
&< \frac{1.0008 k_{10}\sqrt{\varepsilon}}{2} \cdot \frac{n(n+11)}{8} \\
&< 0.0458 n^3 (n+11)\sqrt{n}\sqrt{\varepsilon}.
\end{aligned}
$$

Since $\varepsilon \le \varepsilon_n$,

$$
\left|\sum_{\nu \ne i} \log(1+\Delta_\nu)\right| < \frac{0.0458 n^3 (n+11)\sqrt{n}}{\sqrt{2} n^4 \sqrt{n} 2^n} = \frac{0.0458(n+11)}{\sqrt{2} n 2^n} < 0.0076.
$$

Furthermore, since $|e^x - 1| \le e^{|x|} - 1$ for $x \in \mathbf{C}$,

$$
\begin{aligned}
K &= |Q_i(\xi_i) - Q_0(\xi_i)| \\
&= |Q_0(\xi_i)|\,|e^{\sum_{\nu \ne i} \log(1+\Delta_\nu)} - 1| \le n(e^{|\sum_{\nu \ne i} \log(1+\Delta_\nu)|} - 1),
\end{aligned}
\tag{8}
$$

and since $e^{|x|} - 1 \le 1.0039|x|$ for $|x| < 0.0076$,

$$
\begin{aligned}
K &< n \cdot 1.0039 \left|\sum_{\nu \ne i} \log(1+\Delta_\nu)\right| \\
&< 1.0039 n \cdot 0.0458 n^3 (n+11)\sqrt{n}\sqrt{\varepsilon} < k_{11}\sqrt{\varepsilon}.
\end{aligned}
$$

□

Remark 2. *We have*

$$n - k_{11}\sqrt{\varepsilon} \geq n - k_{11}\sqrt{\varepsilon_n} > n - \frac{0.0460n^4(n+11)\sqrt{n}}{\sqrt{2}n^4\sqrt{n}2^n}$$

$$= n\left(1 - \frac{0.0460(n+11)}{\sqrt{2}n2^n}\right) > 0.9923n.$$

We also have

$$n + k_{11}\sqrt{\varepsilon} \leq n + k_{11}\sqrt{\varepsilon_n} < n + \frac{0.0460n^4(n+11)\sqrt{n}}{\sqrt{2}n^4\sqrt{n}2^n}$$

$$= n\left(1 + \frac{0.0460(n+11)}{\sqrt{2}n2^n}\right) < 1.0077n.$$

The next lemma shall be used in the proof of Lemma 8 and Main Theorem.

Lemma 7. *Put* $S_1 = \sum_{j=1}^{n-1} w_j$. *Then*

$$|S_1| < k_{14}\varepsilon,$$

where

$$k_{14} := (n-1)\sqrt{\left(\frac{2nk_8 + k_{13}}{n\sin(2\pi/n)}\right)^2 + k_8^2} < 0.6130n^3(n-1)2^n$$

with

$$k_{13} := \frac{k_{12}(n + k_{11}\sqrt{\varepsilon_n}) + 2nk_7k_{11}}{n - k_{11}\sqrt{\varepsilon_n}} < 2.3850n^3 2^n,$$

$$k_{12} := n + 2(2^n - n - 2)k_7^2 < 0.2664n^3 2^n.$$

Proof. For $v = 1, \ldots, n-1$, denote by S_v the v-th elementary symmetric polynomials in $w_1, \ldots, w_{n-1}$. Since

$$P'(z) = n\prod_{j=1}^{n-1}(z - w_j) = n\sum_{v=0}^{n-1}(-1)^v S_v z^{n-1-v},$$

with $S_0 = 1$,

$$P(\xi_i) = \int_{1-\varepsilon}^{\xi_i} P'(\zeta)\, d\zeta = \sum_{v=0}^{n-1}(-1)^v \frac{nS_v}{n-v}(\xi_i^{n-v} - (1-\varepsilon)^{n-v}). \tag{9}$$

Since

$$\frac{\xi_i - z_i}{\xi_i} = \frac{P(\xi_i)}{\xi_i \prod_{v\neq i}(\xi_i - z_v)}$$
$$= \frac{1}{\xi_i \prod_{v\neq i}(\xi_i - z_v)} \cdot \left[(1-(1-\varepsilon)^n) - \frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1}) + \sum_{v=2}^{n-1}(-1)^v \frac{nS_v}{n-v}(\xi_i^{n-v} - (1-\varepsilon)^{n-v}) \right],$$

we have

$$\operatorname{Re}\frac{\xi_i - z_i}{\xi_i} = \operatorname{Re}\frac{T_1}{\xi_i \prod_{v\neq i}(\xi_i - z_v)} + \operatorname{Re}\frac{T_2}{\xi_i \prod_{v\neq i}(\xi_i - z_v)}$$
$$\leq \operatorname{Re}\frac{T_1}{\xi_i \prod_{v\neq i}(\xi_i - z_v)} + \left|\frac{T_2}{\xi_i \prod_{v\neq i}(\xi_i - z_v)}\right|,$$

where

$$T_1 := -\frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1}),$$
$$T_2 := 1-(1-\varepsilon)^n + \sum_{v=2}^{n-1}(-1)^v \frac{n}{n-v} S_v(\xi_i^{n-v} - (1-\varepsilon)^{n-v}).$$

On the other hand, since

$$\operatorname{Re}\frac{\xi_i - z_i}{\xi_i} = 1 - \operatorname{Re}(z_i\bar{\xi}_i) \geq 1 - |z_i\bar{\xi}_i| \geq 0,$$

it successively follows that

$$\operatorname{Re}\frac{T_1}{\xi_i \prod_{v\neq i}(\xi_i - z_v)} + \left|\frac{T_2}{\xi_i \prod_{v\neq i}(\xi_i - z_v)}\right| \geq 0,$$

$$\frac{\operatorname{Re} T_1 \cdot \operatorname{Re}(\xi_i \prod_{v\neq i}(\xi_i - z_v)) + \operatorname{Im} T_1 \cdot \operatorname{Im}(\xi_i \prod_{v\neq i}(\xi_i - z_v))}{|\xi_i \prod_{v\neq i}(\xi_i - z_v)|^2}$$
$$\geq -\left|\frac{T_2}{\xi_i \prod_{v\neq i}(\xi_i - z_v)}\right|,$$

$$\operatorname{Re} T_1 \geq -\frac{|T_2|\,|\xi_i \prod_{v\neq i}(\xi_i - z_v)| + \operatorname{Im} T_1 \cdot \operatorname{Im}(\xi_i \prod_{v\neq i}(\xi_i - z_v))}{\operatorname{Re}(\xi_i \prod_{v\neq i}(\xi_i - z_v))}. \tag{10}$$

Here, note that $\mathrm{Re}(\xi_i \prod_{v \neq i}(\xi_i - z_v)) > 0$ by Lemma 6 together Remark 2. Lemma 2 implies the following two inequalities. First one is that

$$|\mathrm{Im}\ T_1| < \frac{2n}{n-1} \cdot (n-1)k_7\sqrt{\varepsilon} = 2nk_7\sqrt{\varepsilon}. \tag{11}$$

The other is that since

$$\begin{aligned}
|T_2 - (1-(1-\varepsilon)^n)| &= \left|\sum_{v=2}^{n-1}(-1)^v \frac{n}{n-v} S_v(\xi_i^{n-v} - (1-\varepsilon)^{n-v})\right| \\
&\leq \sum_{v=2}^{n-1} \frac{n}{n-v} |S_v|\, |\xi_i^{n-v} - (1-\varepsilon)^{n-v}| \\
&< \sum_{v=2}^{n-1} \frac{n}{n-v} \binom{n-1}{v} (k_7\sqrt{\varepsilon})^v \cdot 2 \\
&< 2(k_7\sqrt{\varepsilon})^2 \sum_{v=2}^{n-1} \binom{n}{v} = 2(2^n - n - 2)(k_7\sqrt{\varepsilon})^2,
\end{aligned}$$

$$\begin{aligned}
|T_2| &< 1 - (1-\varepsilon)^n + 2(2^n - n - 2)k_7^2\varepsilon < n\varepsilon + 2(2^n - n - 2)k_7^2\varepsilon \\
&= (n + 2(2^n - n - 2)k_7^2)\varepsilon = k_{12}\varepsilon.
\end{aligned} \tag{12}$$

Substituting (11), (12) and Lemma 6 for (10), we obtain

$$\begin{aligned}
\mathrm{Re}\ T_1 &> -\frac{k_{12}\varepsilon \cdot (n + k_{11}\sqrt{\varepsilon}) + 2nk_7\sqrt{\varepsilon} \cdot k_{11}\sqrt{\varepsilon}}{n - k_{11}\sqrt{\varepsilon}} \\
&\geq -\frac{k_{12}(n + k_{11}\sqrt{\varepsilon_n}) + 2nk_7k_{11}}{n - k_{11}\sqrt{\varepsilon_n}}\varepsilon = -k_{13}\varepsilon.
\end{aligned}$$

By making use of Lemma 3 and Remark 1, we obtain

$$\begin{aligned}
0 &< \mathrm{Re}\ T_1 + k_{13}\varepsilon \\
&= \mathrm{Re}\left[-\frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1})\right] + k_{13}\varepsilon \\
&= -\frac{n}{n-1}\ \mathrm{Re}(S_1\xi_i^{n-1}) + \frac{n(1-\varepsilon)^{n-1}}{n-1}\ \mathrm{Re}\ S_1 + k_{13}\varepsilon \\
&< (n(1-\varepsilon)^{n-1}k_8 + k_{13})\varepsilon - \frac{n}{n-1}(\mathrm{Re}\ S_1 \cdot \mathrm{Re}\ \xi_i^{n-1} - \mathrm{Im}\ S_1 \cdot \mathrm{Im}\ \xi_i^{n-1}) \\
&< (n(1-\varepsilon)^{n-1}k_8 + k_{13} + nk_8)\varepsilon + \frac{n}{n-1}\ \mathrm{Im}\ S_1 \cdot \mathrm{Im}\ \xi_i^{n-1}
\end{aligned}$$

$$< (2nk_8 + k_{13})\varepsilon + \frac{n}{n-1} \text{ Im } S_1 \cdot \text{Im } \xi_i^{n-1}$$
$$= (2nk_8 + k_{13})\varepsilon - \frac{n}{n-1} \text{ Im } S_1 \cdot \text{Im } \xi_i.$$

Thus

$$\text{Im } S_1 \cdot \text{Im } \xi_i < \frac{n-1}{n}(2nk_8 + k_{13})\varepsilon.$$

If we choose i such that Im ξ_i and Im S_1 have the same sign, we deduce that

$$|\text{Im } S_1 \cdot \text{Im } \xi_i| < \frac{n-1}{n}(2nk_8 + k_{13})\varepsilon,$$

namely,

$$|\text{Im } S_1| < \frac{(n-1)(2nk_8 + k_{13})}{n \sin(2\pi/n)}\varepsilon. \tag{13}$$

Since $|\text{Re } S_1| < (n-1)k_8\varepsilon$ by Lemma 3 and Remark 1,

$$|S_1| < (n-1)\varepsilon\sqrt{\left(\frac{2nk_8 + k_{13}}{n \sin(2\pi/n)}\right)^2 + k_8^2} = k_{14}\varepsilon.$$

Finally, we quantify the constants k_{12}, k_{13} and k_{14}. We have

$$k_{12} < n + 2(2^n - n - 2) \cdot 0.3649^2 n^3$$
$$< n^3 2^n \left(\frac{1}{n^2 2^n} + 0.2664\left(1 - \frac{n}{2^n} - \frac{2}{2^n}\right)\right)$$
$$= n^3 2^n \left(0.2664 - \frac{1}{2^n}\left(-\frac{1}{n^2} + 0.2664n + 0.2664 \cdot 2\right)\right) < 0.2664n^3 2^n.$$

By Remark 2, we have

$$k_{13} < \frac{(0.2664n^3 2^n)1.0077n + 2n(0.3649n\sqrt{n})(0.0460n^4(n+11)\sqrt{n})}{0.9923n}$$
$$= n^3 2^n \cdot \frac{0.2664 \cdot 1.0077 + 2 \cdot 0.3649 \cdot 0.0460n^3(n+11)/2^n}{0.9923}.$$

Putting $f_{13}(x) := x^3(x+11)/2^x$, we have

$$f_{13}'(x) = \frac{x^2}{2^x}(-(\log 2)x^2 - (11 \log 2 - 4)x + 33).$$

If $x \geq 4$, $\tilde{f}_{13}(x) := -(\log 2)x^2 - (11 \log 2 - 4)x + 33$ is monotone decreasing. Since $\tilde{f}_{13}(4) > 0$ and $\tilde{f}_{13}(5) < 0$, there exists $x_0 \in (4, 5)$ such that $\tilde{f}_{13}(x_0) = 0$, namely, $f_{13}'(x_0) = 0$. Therefore, $f_{13}(x)$ attains the maximum value at $x = x_0$. Since $f_{13}(4) < f_{13}(5)$, $f_{13}(n) \le f_{13}(5)$ holds for any positive integer n. Hence,

$$k_{13} < n^3 2^n \cdot \frac{0.2664 \cdot 1.0077 + 2 \cdot 0.3649 \cdot 0.0460 f_{13}(5)}{0.9923} < 2.3850 n^3 2^n.$$

Since $\sin(2\pi/n) \ge 4/n$ for $n \ge 4$,

$$\begin{aligned}
k_{14} &< (n-1)\sqrt{\left(\frac{2n(0.1332n^3) + (2.3850n^3 2^n)}{4}\right)^2 + (0.1332n^3)^2} \\
&= (n-1)\frac{0.2664n^4 + 2.3850n^3 2^n}{4}\sqrt{1 + \left(\frac{4 \cdot 0.1332n^3}{0.2664n^4 + 2.3850n^3 2^n}\right)^2} \\
&= n^3(n-1)2^n \cdot \frac{0.2664n/2^n + 2.3850}{4}\sqrt{1 + \left(\frac{0.5328}{0.2664n + 2.3850 \cdot 2^n}\right)^2} \\
&\le n^3(n-1)2^n \cdot \frac{0.2664 \cdot 4/2^4 + 2.3850}{4}\sqrt{1 + \left(\frac{0.5328}{0.2664 \cdot 4 + 2.3850 \cdot 2^4}\right)^2} \\
&< 0.6130 n^3(n-1)2^n.
\end{aligned}$$

□

We can improve Lemma 4. Lemma 8 shall be used for Lemma 9.

Lemma 8. *For* $i = 1, \dots, n$,

$$|z_i - \xi_i| < k_{15}\varepsilon,$$

where

$$k_{15} := \frac{1}{n - k_{11}\sqrt{\varepsilon_n}}\left(\frac{2nk_{14}}{n-1} + k_{12}\right) < 1.3027 n^3 2^n.$$

Proof. By Lemma 6 and the proof of Lemma 7,

$$|z_i - \xi_i| = \left|\frac{z_i - \xi_i}{\xi_i}\right| = \left|\frac{T_1 + T_2}{\xi_i \prod_{\nu \ne i}(\xi_i - z_\nu)}\right| \le \frac{|T_1| + |T_2|}{|\xi_i \prod_{\nu \ne i}(\xi_i - z_\nu)|} < \frac{|T_1| + |T_2|}{n - k_{11}\sqrt{\varepsilon}}.$$

By Lemma 7 and (12),

$$\begin{aligned}
|z_i - \xi_i| &< \frac{1}{n - k_{11}\sqrt{\varepsilon}}\left(\frac{2n}{n-1}|S_1| + k_{12}\varepsilon\right) \\
&< \frac{1}{n - k_{11}\sqrt{\varepsilon_n}}\left(\frac{2nk_{14}}{n-1} + k_{12}\right)\varepsilon = k_{15}\varepsilon.
\end{aligned}$$

Now, we quantify the constant k_{15}. By Remark 2, we have

$$k_{15} < \frac{1}{0.9923n}\left(\frac{2n \cdot 0.6130n^3(n-1)2^n}{n-1} + 0.2664n^3 2^n\right)$$
$$= \frac{n^3 2^n}{0.9923}\left(1.2260 + \frac{0.2664}{n}\right) < 1.3027n^3 2^n.$$

Note that

$$k_{15}\varepsilon \le k_{15}\varepsilon_n < \frac{1.3027n^3 2^n}{2n^9 4^n} < \frac{2\sqrt{2}}{n} \le \sin(\pi/n)$$

for $n \ge 4$. This implies that the $(n-1)$ disks $\{|z - \xi_i| < k_{15}\varepsilon\}_{i=2}^n$ are also disjoint. □

We can also improve Lemma 6. The next Lemma 9 shall be used in the proof of Main Theorem.

LEMMA 9. *For* $i = 1, \dots, n$,

$$\left|\xi_i \prod_{v \ne i}(\xi_i - z_v) - n\right| < k'_{11}\varepsilon,$$

where

$$k'_{11} := 0.0816n^5(n+11)2^n.$$

In particular,

$$n - k'_{11}\varepsilon \le \mathrm{Re}\left(\xi_i \prod_{v \ne i}(\xi_i - z_v)\right) \le n + k'_{11}\varepsilon,$$
$$-k'_{11}\varepsilon \le \mathrm{Im}\left(\xi_i \prod_{v \ne i}(\xi_i - z_v)\right) \le k'_{11}\varepsilon,$$
$$n - k'_{11}\varepsilon \le \left|\xi_i \prod_{v \ne i}(\xi_i - z_v)\right| \le n + k'_{11}\varepsilon.$$

PROOF. We may replace $k_{10}\sqrt{\varepsilon}$ with $k_{15}\varepsilon$ in the proof of Lemma 6 since $|z_i - \xi_i| < k_{15}\varepsilon$ for $i = 2, \dots, n$ by Lemma 8. Since

$$|\Delta_v| < \frac{k_{15}\varepsilon}{2\sin(\pi/n)} < \frac{1.3027n^4 2^n}{4\sqrt{2}}\varepsilon \le \frac{1.3027n^4 2^n}{4\sqrt{2} \cdot 2n^9 4^n} = \frac{1.3027}{8\sqrt{2} \cdot n^5 2^n} < 0.00001,$$

we have

$$|\log(1 + \Delta_v)| < \frac{|\Delta_v|}{2}\left(1 + \frac{1}{1 - 0.00001}\right) < 1.00001|\Delta_v|,$$

and then,

$$\left|\sum_{\nu \neq i} \log(1+\Delta_\nu)\right| < \frac{1.00001 k_{15}\varepsilon}{2} \cdot \frac{n(n+11)}{8} < 0.0815 n^4 (n+11) 2^n \varepsilon$$

$$< 0.00004.$$

Since $e^{|x|} - 1 < 1.00003|x|$ for $|x| < 0.00004$, by (8),

$$\begin{aligned} K = |Q_i(\xi_i) - Q_0(\xi_i)| &\leq n(e^{|\sum_{\nu \neq i} \log(1+\Delta_\nu)|} - 1) \\ &< 1.00003 n \cdot \left|\sum_{\nu \neq i} \log(1+\Delta_\nu)\right| \\ &< 1.00003 n \cdot 0.0815 n^4 (n+11) 2^n \varepsilon \\ &< 0.0816 n^5 (n+11) 2^n \varepsilon = k'_{11}\varepsilon. \end{aligned}$$

□

REMARK 3. *We have*

$$\begin{aligned} n - k'_{11}\varepsilon \geq n - k'_{11}\varepsilon_n &> n - 0.0816 n^5 (n+11) 2^n \cdot \frac{1}{2n^9 4^n} \\ &= n\left(1 - \frac{0.0816(n+11)}{2n^5 2^n}\right) > 0.9999n. \end{aligned}$$

We also have

$$\begin{aligned} n + k'_{11}\varepsilon \leq n + k'_{11}\varepsilon_n &< n + 0.0816 n^5 (n+11) 2^n \cdot \frac{1}{2n^9 4^n} \\ &= n\left(1 + \frac{0.0816(n+11)}{2n^5 2^n}\right) < 1.0001n. \end{aligned}$$

Among the lemmas which have been shown, Lemmas 2, 3, 7 and 9 shall be also used to prove Main Theorem.

3. Proof of Main Theorem

We now prove Main Theorem. By (9) in the proof of Lemma 7,

$$P(\xi_i) = \left[(1-(1-\varepsilon)^n) - \frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1}) + \frac{nS_2}{n-2}(\xi_i^{n-2} - (1-\varepsilon)^{n-2})\right]$$

$$+ \sum_{\nu=3}^{n-1} (-1)^\nu \frac{nS_\nu}{n-\nu}(\xi_i^{n-\nu} - (1-\varepsilon)^{n-\nu}).$$

Putting

$$T_1' := (1-(1-\varepsilon)^n) - \frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1}) + \frac{nS_2}{n-2}(\xi_i^{n-2} - (1-\varepsilon)^{n-2}),$$

$$T_2' := \sum_{v=3}^{n-1}(-1)^v \frac{nS_v}{n-v}(\xi_i^{n-v} - (1-\varepsilon)^{n-v}),$$

we have

$$\operatorname{Re} T_1' \geq -\frac{|T_2'|\,|\xi_i \prod_{v\neq i}(\xi_i - z_v)| + \operatorname{Im} T_1' \cdot \operatorname{Im}(\xi_i \prod_{v\neq i}(\xi_i - z_v))}{\operatorname{Re}(\xi_i \prod_{v\neq i}(\xi_i - z_v))} \tag{14}$$

in a similar way as in the proof of Lemma 7. By (13) and Lemma 2, we have

$$\begin{aligned}
|\operatorname{Im} T_1'| &= \left|\frac{n}{n-1}\operatorname{Im} S_1(\xi_i^{n-1} - (1-\varepsilon)^{n-1}) - \frac{n}{n-2}\operatorname{Im} S_2(\xi_i^{n-2} - (1-\varepsilon)^{n-2})\right| \\
&< \frac{2n}{n-1}\cdot\frac{(n-1)(2nk_8 + k_{13})}{4}\varepsilon + \frac{2n}{n-2}\binom{n-1}{2}(k_7\sqrt{\varepsilon})^2 \\
&= \frac{n}{2}((2nk_8 + k_{13}) + 2(n-1)k_7^2)\varepsilon \\
&< \frac{n}{2}(2n(0.1332n^3) + (2.3850n^3 2^n) + 2(n-1)(0.3649n\sqrt{n})^2)\varepsilon \\
&= \frac{n^4 2^n}{2}\left(\frac{0.2664n}{2^n} + 2.3850 + \frac{0.2664(n-1)}{2^n}\right)\varepsilon \\
&< 1.2508n^4 2^n \varepsilon.
\end{aligned}$$

By Lemma 2, we also have

$$\begin{aligned}
|T_2'| &< \sum_{v=3}^{n-1}\frac{n}{n-v}\binom{n-1}{v}(k_7\sqrt{\varepsilon})^v(2-\varepsilon) \\
&< 2k_7^3(\varepsilon\sqrt{\varepsilon})\sum_{v=3}^{n-1}\binom{n}{v} = 2k_7^3(\varepsilon\sqrt{\varepsilon})\left(2^n - \frac{n^2}{2} - \frac{n}{2} - 2\right) \\
&= (2^{n+1} - n^2 - n - 4)k_7^3(\varepsilon\sqrt{\varepsilon}) \\
&< (2^{n+1} - n^2 - n - 4)(0.3649n\sqrt{n})^3(\varepsilon\sqrt{\varepsilon}) \\
&= n^4\sqrt{n}2^n\left(2 - \frac{n^2+n+4}{2^n}\right)0.3649^3(\varepsilon\sqrt{\varepsilon}) \\
&< 0.0972n^4\sqrt{n}2^n(\varepsilon\sqrt{\varepsilon}).
\end{aligned}$$

Substituting these results and Lemma 9 for (14), we obtain

$$
\begin{aligned}
\mathrm{Re}\ T_1' &> -\frac{|T_2'| \cdot (n + k_{11}'\varepsilon) + |\mathrm{Im}\ T_1'| \cdot k_{11}'\varepsilon}{n - k_{11}'\varepsilon} \\
&> -\frac{1}{0.9999n}[0.0972n^4\sqrt{n}2^n(\varepsilon\sqrt{\varepsilon}) \cdot 1.0001n \\
&\qquad + 1.2508n^4 2^n\varepsilon \cdot 0.0816n^5(n+11)2^n\varepsilon] \\
&> -\frac{0.0973n^4\sqrt{n}2^n + 0.1021n^4 2^n \cdot n^4(n+11)2^n\sqrt{\varepsilon_n}}{0.9999}(\varepsilon\sqrt{\varepsilon}) \\
&= -nk_{16}(\varepsilon\sqrt{\varepsilon}),
\end{aligned}
$$

where

$$
\begin{aligned}
k_{16} &:= \frac{0.0973n^3\sqrt{n}2^n + 0.1021n^3 2^n \cdot n^4(n+11)2^n/(\sqrt{2}n^4\sqrt{n}2^n)}{0.9999} \\
&< n^3\sqrt{n}2^n \cdot \frac{0.0973 + 0.0722(n+11)/n}{0.9999} \\
&\le n^3\sqrt{n}2^n \cdot \frac{0.0973 + 0.0722(4+11)/4}{0.9999} \\
&< 0.36809n^3\sqrt{n}2^n.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathrm{Re}\ T_1' &= \mathrm{Re}\left[(1-(1-\varepsilon)^n) - \frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1}) \right. \\
&\qquad \left. + \frac{nS_2}{n-2}(\xi_i^{n-2} - (1-\varepsilon)^{n-2})\right] \\
&< \mathrm{Re}\left[n\varepsilon - \frac{nS_1}{n-1}(\xi_i^{n-1} - (1-\varepsilon)^{n-1}) + \frac{nS_2}{n-2}(\xi_i^{n-2} - (1-\varepsilon)^{n-2})\right] \\
&= n\varepsilon + \frac{n(1-\varepsilon)^{n-1}}{n-1}\ \mathrm{Re}\ S_1 - \frac{n}{n-1}(\mathrm{Re}\ S_1 \cdot \mathrm{Re}\ \overline{\xi_i} - \mathrm{Im}\ S_1 \cdot \mathrm{Im}\ \overline{\xi_i}) \\
&\quad - \frac{n(1-\varepsilon)^{n-2}}{n-2}\ \mathrm{Re}\ S_2 + \frac{n}{n-2}(\mathrm{Re}\ S_2 \cdot \mathrm{Re}\ \overline{\xi_i^2} - \mathrm{Im}\ S_2 \cdot \mathrm{Im}\ \overline{\xi_i^2}) \\
&= n\varepsilon + na_1(1-\varepsilon)^{n-1} - n(a_1 \cos\theta_i + b_1 \sin\theta_i) \\
&\quad - na_2(1-\varepsilon)^{n-2} + n(a_2 \cos 2\theta_i + b_2 \sin 2\theta_i),
\end{aligned}
$$

where

$$a_1 := \frac{\mathrm{Re}\ S_1}{n-1}, \qquad b_1 := \frac{\mathrm{Im}\ S_1}{n-1}, \qquad a_2 := \frac{\mathrm{Re}\ S_2}{n-2}, \qquad b_2 := \frac{\mathrm{Im}\ S_2}{n-2}$$

and $\theta_v := 2(v-1)\pi/n$. We, therefore, obtain

$$\begin{aligned} 0 &< \mathrm{Re}\ T_1' + nk_{16}(\varepsilon\sqrt{\varepsilon}) \\ &< n[\varepsilon + a_1(1-\varepsilon)^{n-1} - (a_1 \cos\theta_i + b_1 \sin\theta_i) - a_2(1-\varepsilon)^{n-2} \\ &\quad + (a_2 \cos 2\theta_i + b_2 \sin 2\theta_i)] + nk_{16}(\varepsilon\sqrt{\varepsilon}), \end{aligned}$$

that is,

$$\begin{aligned} 0 &< \varepsilon + a_1(1-\varepsilon)^{n-1} - (a_1 \cos\theta_i + b_1 \sin\theta_i) - a_2(1-\varepsilon)^{n-2} \\ &\quad + (a_2 \cos 2\theta_i + b_2 \sin 2\theta_i) + k_{16}(\varepsilon\sqrt{\varepsilon}). \end{aligned}$$

Replacing θ_i by $2\pi - \theta_i$, we also obtain

$$\begin{aligned} 0 &< \varepsilon + a_1(1-\varepsilon)^{n-1} - (a_1 \cos\theta_i - b_1 \sin\theta_i) - a_2(1-\varepsilon)^{n-2} \\ &\quad + (a_2 \cos 2\theta_i - b_2 \sin 2\theta_i) + k_{16}(\varepsilon\sqrt{\varepsilon}). \end{aligned}$$

Adding the resulting two inequalities, we obtain

$$\begin{aligned} 0 &< \varepsilon + a_1(1-\varepsilon)^{n-1} - a_1 \cos\theta_i - a_2(1-\varepsilon)^{n-2} + a_2 \cos 2\theta_i + k_{16}(\varepsilon\sqrt{\varepsilon}) \\ &= \varepsilon + a_1[(1-\varepsilon)^{n-1} - \cos\theta_i] - a_2[(1-\varepsilon)^{n-2} - \cos 2\theta_i] + k_{16}(\varepsilon\sqrt{\varepsilon}) \\ &= \varepsilon + a_1(1-\cos\theta_i) - a_2(1-\cos 2\theta_i) - a_1(1-(1-\varepsilon)^{n-1}) \\ &\quad + a_2(1-(1-\varepsilon)^{n-2}) + k_{16}(\varepsilon\sqrt{\varepsilon}) \\ &\le \varepsilon + a_1(1-\cos\theta_i) - a_2(1-\cos 2\theta_i) \\ &\quad + |a_1(1-(1-\varepsilon)^{n-1}) - a_2(1-(1-\varepsilon)^{n-2})| + k_{16}(\varepsilon\sqrt{\varepsilon}). \end{aligned} \tag{15}$$

Since $|\mathrm{Re}\ w_j| < k_8\varepsilon$ and $|\mathrm{Re}\ w_i w_j| \le |w_i w_j| < (k_7\sqrt{\varepsilon})^2$ by Lemma 3 and Lemma 2, respectively,

$$\begin{aligned} &|a_1(1-(1-\varepsilon)^{n-1}) - a_2(1-(1-\varepsilon)^{n-2})| \\ &\quad \le |a_1|(1-(1-\varepsilon)^{n-1}) + |a_2|(1-(1-\varepsilon)^{n-2}) \\ &\quad < \frac{(n-1)k_8\varepsilon}{n-1} \cdot (n-1)\varepsilon + \frac{\binom{n-1}{2}k_7^2\varepsilon}{n-2} \cdot (n-2)\varepsilon \end{aligned}$$

$$= (n-1)k_8\varepsilon^2 + \frac{(n-1)(n-2)}{2}k_7^2\varepsilon^2$$
$$\le \left(k_8 + \frac{n-2}{2}k_7^2\right)(n-1)\sqrt{\varepsilon_n}(\varepsilon\sqrt{\varepsilon}).$$

Therefore, if we put

$$k_{16}' := \left(k_8 + \frac{n-2}{2}k_7^2\right)(n-1)\sqrt{\varepsilon_n} + k_{16},$$

then

$$k_{16}' < \left((0.1332n^3) + \frac{n-2}{2}(0.3649n\sqrt{n})^2\right)(n-1)\frac{1}{\sqrt{2}n^4\sqrt{n}2^n} + 0.36809n^3\sqrt{n}2^n$$
$$= n^3\sqrt{n}2^n\left[\left(\frac{0.1332}{n} + \frac{n-2}{2n}\cdot 0.3649^2\right)\frac{n-1}{\sqrt{2}n^4 4^n} + 0.36809\right]$$
$$< n^3\sqrt{n}2^n\left[\left(0.0666 + \frac{0.00005}{n}\right)\frac{n-1}{\sqrt{2}n^4 4^n} + 0.36809\right]$$
$$\le n^3\sqrt{n}2^n\left[\left(0.0666 + \frac{0.00005}{4}\right)\frac{4-1}{\sqrt{2}\cdot 4^4\cdot 4^4} + 0.36809\right]$$
$$< 0.3681n^3\sqrt{n}2^n,$$

and we can rewrite (15) with

$$0 < \varepsilon + a_1(1-\cos\theta_i) - a_2(1-\cos 2\theta_i) + k_{16}'(\varepsilon\sqrt{\varepsilon}).$$

Taking θ_i to be $2\lfloor n/2\rfloor\pi/n$ and $2\lfloor n/4\rfloor\pi/n$, we obtain

$$\varepsilon + a_1[1+\cos(l\pi/n)] - a_2[1-\cos(2l\pi/n)] + k_{16}'(\varepsilon\sqrt{\varepsilon}) > 0, \tag{16}$$
$$\varepsilon + a_1[1-\sin(r\pi/(2n))] - a_2[1+\cos(r\pi/n)] + k_{16}'(\varepsilon\sqrt{\varepsilon}) > 0, \tag{17}$$

respectively, where $l := n - 2\lfloor n/2\rfloor$ and $r := n - 4\lfloor n/4\rfloor$. Note that for positive integers s, we have Table 1.

n	$4s$	$4s+1$	$4s+2$	$4s+3$
l	0	1	0	1
r	0	1	2	3

Table 1

Now by (1), we successively obtain

$$|w_j - a| > 1 - c_n(1-a),$$

$$|w_j - (1-\varepsilon)| > 1 - c_n\varepsilon,$$

$$|w_j|^2 - 2(1-\varepsilon)\ \mathrm{Re}\ w_j + (1-\varepsilon)^2 > 1 - 2c_n\varepsilon + (c_n\varepsilon)^2,$$

$$(\mathrm{Re}\ w_j)^2 + (\mathrm{Im}\ w_j)^2 - 2(1-\varepsilon)\ \mathrm{Re}\ w_j + (1-\varepsilon)^2 > 1 - 2c_n\varepsilon + (c_n\varepsilon)^2.$$

Thus,

$$\begin{aligned} 0 &< (\mathrm{Re}\ w_j)^2 + (\mathrm{Im}\ w_j)^2 - 2(1-\varepsilon)\ \mathrm{Re}\ w_j + (1-\varepsilon)^2 - 1 + 2c_n\varepsilon - (c_n\varepsilon)^2 \\ &= (\mathrm{Re}\ w_j)^2 + (\mathrm{Im}\ w_j)^2 - 2\ \mathrm{Re}\ w_j + 2\varepsilon\ \mathrm{Re}\ w_j - 2(1-c_n)\varepsilon + (1-c_n^2)\varepsilon^2. \end{aligned}$$

By Lemma 3, we have

$$\begin{aligned} 0 &< k_8^2\varepsilon^2 + (\mathrm{Im}\ w_j)^2 - 2\ \mathrm{Re}\ w_j + 2k_8\varepsilon^2 - 2(1-c_n)\varepsilon + (1-c_n^2)\varepsilon^2 \\ &= (\mathrm{Im}\ w_j)^2 - 2\ \mathrm{Re}\ w_j - 2(1-c_n)\varepsilon + (k_8^2 + 2k_8 + 1 - c_n^2)\varepsilon^2 \\ &< (\mathrm{Im}\ w_j)^2 - 2\ \mathrm{Re}\ w_j - 2(1-c_n)\varepsilon + (k_8+1)^2\varepsilon^2. \end{aligned}$$

Taking the sum of these inequalities for $j = 1, \dots, n-1$, we obtain

$$\begin{aligned} 0 &< \sum_{j=1}^{n-1} (\mathrm{Im}\ w_j)^2 - 2\ \mathrm{Re}\ S_1 - 2(n-1)(1-c_n)\varepsilon + (n-1)(k_8+1)^2\varepsilon^2 \\ &= \sum_{j=1}^{n-1} (\mathrm{Im}\ w_j)^2 - 2(n-1)a_1 - 2(n-1)(1-c_n)\varepsilon + (n-1)(k_8+1)^2\varepsilon^2. \end{aligned} \tag{18}$$

On the other hand, By Lemma 3 and (13),

$$\begin{aligned} (n-2)a_2 = \mathrm{Re}\ S_2 &= \mathrm{Re}\left(\sum_{i<j} w_i w_j\right) \\ &= \sum_{i<j} (\mathrm{Re}\ w_i \cdot \mathrm{Re}\ w_j - \mathrm{Im}\ w_i \cdot \mathrm{Im}\ w_j) \\ &> \sum_{i<j} (-k_5 k_8 \varepsilon^2 - \mathrm{Im}\ w_i \cdot \mathrm{Im}\ w_j) \\ &= -\frac{1}{2}\left(\left(\sum_{j=1}^{n-1} \mathrm{Im}\ w_j\right)^2 - \sum_{j=1}^{n-1} (\mathrm{Im}\ w_j)^2\right) - \binom{n-1}{2} k_5 k_8 \varepsilon^2 \\ &= -\frac{1}{2}\left((\mathrm{Im}\ S_1)^2 - \sum_{j=1}^{n-1} (\mathrm{Im}\ w_j)^2\right) - \binom{n-1}{2} k_5 k_8 \varepsilon^2 \end{aligned}$$

$$> \frac{1}{2}\sum_{j=1}^{n-1}(\mathrm{Im}\ w_j)^2 - \frac{1}{2}\left(\frac{(n-1)(2nk_8+k_{13})}{n\sin(2\pi/n)}\right)^2\varepsilon^2$$

$$- \binom{n-1}{2}k_5k_8\varepsilon^2$$

$$= \frac{1}{2}\sum_{j=1}^{n-1}(\mathrm{Im}\ w_j)^2 - \frac{1}{2}k_{17}\varepsilon^2,$$

where

$$k_{17} := \left(\frac{(n-1)(2nk_8+k_{13})}{n\sin(2\pi/n)}\right)^2 + (n-1)(n-2)k_5k_8.$$

As a result, we obtain

$$\sum_{j=1}^{n-1}(\mathrm{Im}\ w_j)^2 < 2(n-2)a_2 + k_{17}\varepsilon^2. \tag{19}$$

Note that

$$k_{17} \le \left(\frac{(n-1)(2nk_8+k_{13})}{4}\right)^2 + (n-1)(n-2)k_5k_8$$

$$< \frac{1}{16}(n-1)^2(2n(0.1332n^3) + (2.3850n^3 2^n))^2$$

$$+ (n-1)(n-2)(0.0783n^3)(0.1332n^3)$$

$$< \frac{n^6 4^n}{16}(n-1)^2\left(\left(\frac{0.2664n}{2^n} + 2.3850\right)^2 + \frac{0.1669(n-2)}{(n-1)4^n}\right)$$

$$\le \frac{n^6 4^n}{16}(n-1)^2\left(\left(\frac{0.2664\cdot 4}{2^4} + 2.3850\right)^2 + \frac{0.1669(4-2)}{(4-1)4^4}\right)$$

$$< 0.3757n^6(n-1)^2 4^n.$$

Substituting (19) for (18), we obtain

$$(2(n-2)a_2 + k_{17}\varepsilon^2) - 2(n-1)a_1 - 2(n-1)(1-c_n)\varepsilon + (n-1)(k_8+1)^2\varepsilon^2 > 0,$$

that is,

$$-(n-1)(1-c_n)\varepsilon - (n-1)a_1 + (n-2)a_2 + \frac{k_{17}+(n-1)(k_8+1)^2}{2}\varepsilon^2 > 0.$$

Putting

$$k_{18} := \frac{k_{17} + (n-1)(k_8+1)^2}{2} \sqrt{\varepsilon_n},$$

we have

$$-(n-1)(1-c_n)\varepsilon - (n-1)a_1 + (n-2)a_2 + k_{18}(\varepsilon\sqrt{\varepsilon}) > 0, \tag{20}$$

since $0 < \varepsilon \leq \varepsilon_n$. Note that

$$\begin{aligned}
k_{18} &< \frac{0.3757n^6(n-1)^2 4^n + (n-1)(0.1332n^3+1)^2}{2} \cdot \frac{1}{\sqrt{2}n^4\sqrt{n}2^n} \\
&= \frac{n(n-1)^2\sqrt{n}2^n}{2\sqrt{2}} \left(0.3757 + \frac{(0.1332 + 1/n^3)^2}{(n-1)4^n}\right) \\
&\leq \frac{n(n-1)^2\sqrt{n}2^n}{2\sqrt{2}} \left(0.3757 + \frac{(0.1332 + 1/4^3)^2}{(4-1)4^4}\right) \\
&< 0.1329n(n-1)^2\sqrt{n}2^n.
\end{aligned}$$

To eliminate the a_2 terms, multiplying (20) by $[1-\cos(2l\pi/n)]/(n-2) \geq 0$, and adding (16) to it, we obtain

$$\begin{aligned}
&\varepsilon\left[1 - \frac{n-1}{n-2}(1-c_n)(1-\cos(2l\pi/n))\right] \\
&\quad + a_1\left[1 + \cos(l\pi/n) - \frac{n-1}{n-2}(1-\cos(2l\pi/n))\right] \\
&\quad + \left[k_{16}' + \frac{k_{18}}{n-2}(1-\cos(2l\pi/n))\right](\varepsilon\sqrt{\varepsilon}) > 0.
\end{aligned}$$

Similarly, multiplying (20) by $[1+\cos(r\pi/n)]/(n-2) > 0$, and adding (17) to it, we obtain

$$\begin{aligned}
&\varepsilon\left[1 - \frac{n-1}{n-2}(1-c_n)(1+\cos(r\pi/n))\right] \\
&\quad - a_1\left[-1 + \sin(r\pi/(2n)) + \frac{n-1}{n-2}(1+\cos(r\pi/n))\right] \\
&\quad + \left[k_{16}' + \frac{k_{18}}{n-2}(1+\cos(r\pi/n))\right](\varepsilon\sqrt{\varepsilon}) > 0.
\end{aligned}$$

Putting

$$\alpha := 1 + \cos(l\pi/n) - \frac{n-1}{n-2}(1 - \cos(2l\pi/n)),$$

$$\beta := -1 + \sin(r\pi/(2n)) + \frac{n-1}{n-2}(1 + \cos(r\pi/n)),$$

$$k_{19} := k'_{16} + \frac{k_{18}}{n-2}(1 - \cos(2l\pi/n)),$$

$$k'_{19} := k'_{16} + \frac{k_{18}}{n-2}(1 + \cos(r\pi/n)),$$

we have

$$\varepsilon\left[1 - \frac{n-1}{n-2}(1 - c_n)(1 - \cos(2l\pi/n))\right] + \alpha a_1 + k_{19}(\varepsilon\sqrt{\varepsilon}) > 0, \tag{16'}$$

$$\varepsilon\left[1 - \frac{n-1}{n-2}(1 - c_n)(1 + \cos(r\pi/n))\right] - \beta a_1 + k'_{19}(\varepsilon\sqrt{\varepsilon}) > 0. \tag{17'}$$

Now, we quantify the constants α, β, k_{19} and k'_{19}. We have

$$\alpha = -\frac{1}{n-2} + \cos(l\pi/n) + \frac{n-1}{n-2}\cos(2l\pi/n) \le 2,$$

and equality holds if and only if $l = 0$, namely n is even. In addition, if n is odd, then since $n \ge 5$ and $l = 1$, we obtain

$$\begin{aligned}\alpha &= -\frac{1}{n-2} + \cos(\pi/n) + \frac{n-1}{n-2}\cos(2\pi/n)\\ &\ge -\frac{1}{n-2} + \cos(\pi/5) + \frac{n-1}{n-2}\cos(2\pi/5)\\ &= -\frac{1}{n-2} + \frac{\sqrt{5}+1}{4} + \frac{n-1}{n-2}\cdot\frac{\sqrt{5}-1}{4} > -\frac{1}{n-2} + \frac{\sqrt{5}}{2} > 0.\end{aligned}$$

Hence, $0 < \alpha \le 2$. We also have

$$\beta = \frac{1}{n-2} + \sin(r\pi/(2n)) + \frac{n-1}{n-2}\cos(r\pi/n) \le \frac{n}{n-2} + \frac{r\pi}{2n},$$

and equality holds if and only if $r = 0$. It is clear that $\beta > 0$. On the other hand,

$$\begin{aligned}k_{19} &< 0.3681n^3\sqrt{n}2^n + \frac{0.1329n(n-1)^2\sqrt{n}2^n}{n-2}\cdot(1 - \cos(2l\pi/n))\\ &= n^3\sqrt{n}2^n\left(0.3681 + \frac{0.1329(n-1)^2}{n^2(n-2)}\cdot(1 - \cos(2l\pi/n))\right).\end{aligned}$$

Put $f_{19}(x) := (x-1)^2/(x^2(x-2))$ for $x \geq 4$. Since $f_{19}'(x) = -(x-1)(x^2 - 3x+4)/(x^3(x-2)^2) < 0$, $f_{19}(x)$ is monotone decreasing, namely, $f_{19}(x) \leq f_{19}(4)$. Thus we obtain

$$\begin{aligned} k_{19} &< n^3\sqrt{n}2^n(0.3681 + 0.1329 f_{19}(4) \cdot (1 - \cos(2l\pi/n))) \\ &< n^3\sqrt{n}2^n(0.3681 + 0.0024(1 - \cos(2l\pi/n))). \end{aligned}$$

Similarly, we have

$$\begin{aligned} k_{19}' &< n^3\sqrt{n}2^n(0.3681 + 0.1329 f_{19}(4) \cdot (1 + \cos(r\pi/n))) \\ &\leq n^3\sqrt{n}2^n(0.3681 + 0.1329 f_{19}(4) \cdot (1+1)) \\ &< 0.3728 n^3\sqrt{n}2^n. \end{aligned}$$

Now, if we multiply (16′) and (17′) by $\beta/\varepsilon > 0$ and $\alpha/\varepsilon > 0$, respectively, and add the resulting two inequalities to eliminate the a_1 terms, then we obtain

$$Ac_n - B + (k_{19}\beta + k_{19}'\alpha)\sqrt{\varepsilon} > 0, \tag{21}$$

where

$$\begin{aligned} A &:= \frac{n-1}{n-2}[(1+\cos(r\pi/n))(1+\cos(l\pi/n)) \\ &\qquad - (1-\cos(2l\pi/n))(1-\sin(r\pi/(2n)))], \\ B &:= \frac{1}{n-2}(\cos(l\pi/n) + \sin(r\pi/(2n))) \\ &\quad + \frac{n-1}{n-2}(\cos(l\pi/n)\cos(r\pi/n) - \cos(2l\pi/n)\sin(r\pi/(2n))). \end{aligned}$$

Note that

$$0 < A \leq \frac{4(n-1)}{n-2},$$

and equality holds if and only if $l = 0$ and $r = 0$. We also quantify the coefficient of $\sqrt{\varepsilon}$ in (21). We have

$$\begin{aligned} &k_{19}\beta + k_{19}'\alpha \\ &\quad < n^3\sqrt{n}2^n(0.3681 + 0.0024(1-\cos(2l\pi/n))) \cdot \left(\frac{n}{n-2} + \frac{r\pi}{2n}\right) \\ &\qquad + 0.3728 n^3\sqrt{n}2^n \cdot 2 \end{aligned}$$

$$= \frac{n^4\sqrt{n}2^n}{n-2}\left[(0.3681 + 0.0024(1 - \cos(2l\pi/n)))\left(1 + \frac{(n-2)r\pi}{2n^2}\right)\right.$$
$$\left. + 0.3728 \cdot \frac{2(n-2)}{n}\right]$$
$$\leq \frac{n^4\sqrt{n}2^n}{n-2}\left[\left(0.3681 + 0.0024 \cdot \frac{2l^2\pi^2}{n^2}\right)\left(1 + \frac{(n-2)r\pi}{2n^2}\right) + 0.3728 \cdot \left(2 - \frac{4}{n}\right)\right]$$
$$= \frac{n^4\sqrt{n}2^n}{n-2}\left[1.1137 + \left(0.3681 + 0.0024 \cdot \frac{2l^2\pi^2}{n^2}\right)\frac{(n-2)r\pi}{2n^2}\right.$$
$$\left. + 0.0024 \cdot \frac{2l^2\pi^2}{n^2} - \frac{1.4912}{n}\right].$$

Since

$$\left(0.3681 + 0.0024 \cdot \frac{2l^2\pi^2}{n^2}\right)\frac{(n-2)r\pi}{2n^2} + 0.0024 \cdot \frac{2l^2\pi^2}{n^2} - \frac{1.4912}{n}$$
$$< \left(0.185 + \frac{0.003l^2\pi^2}{n^2}\right)\frac{(n-2)r\pi}{n^2} + \frac{0.005l^2\pi^2}{n^2} - \frac{1.491}{n},$$

by putting

$$F_{l,r}(n) := \left(0.185 + \frac{0.003l^2\pi^2}{n^2}\right)\frac{(n-2)r\pi}{n^2} + \frac{0.005l^2\pi^2}{n^2} - \frac{1.491}{n},$$

we obtain

$$k_{19}\beta + k'_{19}\alpha < \frac{n^4\sqrt{n}2^n}{n-2}(1.1137 + F_{l,r}(n)).$$

Now recall Table 1. If $n = 4s$ (note that $n \geq 4$), then

$$F_{0,0}(n) = -\frac{1.491}{n} < 0.$$

If $n = 4s + 1$ (note that $n \geq 5$), then

$$F_{1,1}(n) = \left(0.185 + \frac{0.003\pi^2}{n^2}\right)\frac{(n-2)\pi}{n^2} + \frac{0.005\pi^2}{n^2} - \frac{1.491}{n}$$
$$= \frac{0.185\pi}{n} - \frac{0.370\pi}{n^2} + \frac{0.003\pi^3}{n^3} - \frac{0.006\pi^3}{n^4} + \frac{0.005\pi^2}{n^2} - \frac{1.491}{n}$$
$$< -\frac{0.909}{n} - \frac{1.113}{n^2} + \frac{0.094}{n^3} - \frac{0.006\pi^3}{n^4} < 0.$$

If $n = 4s + 2$ (note that $n \geq 6$), then

$$F_{0,2}(n) = 0.185 \cdot \frac{(n-2)2\pi}{n^2} - \frac{1.491}{n} = \frac{0.370\pi}{n} - \frac{0.740\pi}{n^2} - \frac{1.491}{n}$$
$$< -\frac{0.328}{n} - \frac{0.74\pi}{n^2} < 0.$$

If $n = 4s + 3$ (suppose that $s > 1$, namely, $n \geq 11$), then

$$\begin{aligned} F_{1,3}(n) &= \left(0.185 + \frac{0.003\pi^2}{n^2}\right)\frac{(n-2)3\pi}{n^2} + \frac{0.005\pi^2}{n^2} - \frac{1.491}{n} \\ &= \frac{0.555\pi}{n} - \frac{1.110\pi}{n^2} + \frac{0.009\pi^3}{n^3} - \frac{0.018\pi^3}{n^4} + \frac{0.005\pi^2}{n^2} - \frac{1.491}{n} \\ &< \frac{0.009\pi^3}{n} - \frac{0.108\pi^3}{n^2} + \frac{0.009\pi^3}{n^3} - \frac{0.018\pi^3}{n^4} \\ &= \frac{0.009\pi^3}{n}\left(1 - \frac{12}{n} + \frac{1}{n^2} - \frac{2}{n^3}\right) < \frac{0.009\pi^3}{n} \leq \frac{0.009\pi^3}{11} < 0.026. \end{aligned}$$

Hence, we obtain

$$F_{l,r}(n) < \begin{cases} 0.026, & \text{if } n = 4s+3 \text{ and } n \neq 7, \\ 0, & \text{otherwise, if } n \neq 7, \end{cases}$$

that is,

$$k_{19}\beta + k'_{19}\alpha < \frac{n^4\sqrt{n}2^n}{n-2} \cdot \begin{cases} 1.140, & \text{if } n = 4s+3 \text{ and } n \neq 7, \\ 1.114, & \text{otherwise, if } n \neq 7. \end{cases}$$

Furthermore, since $\varepsilon \leq \varepsilon_n$ and

$$\frac{n^4\sqrt{n}2^n}{n-2} \cdot \frac{1}{\sqrt{2}n^4\sqrt{n}2^n} = \frac{1}{\sqrt{2}(n-2)},$$

$$(k_{19}\beta + k'_{19}\alpha)\sqrt{\varepsilon} < \frac{1}{\sqrt{2}(n-2)} \cdot \begin{cases} 1.140, & \text{if } n = 4s+3 \text{ and } n \neq 7, \\ 1.114, & \text{otherwise, if } n \neq 7, \end{cases}$$
$$< \frac{1}{n-2}.$$

Now, to prove Main Theorem, recall Table 1 and that

$$0 < A \leq \frac{4(n-1)}{n-2},$$

$$\begin{aligned} B &= \frac{1}{n-2}(\cos(l\pi/n) + \sin(r\pi/(2n))) \\ &\quad + \frac{n-1}{n-2}(\cos(l\pi/n)\cos(r\pi/n) - \cos(2l\pi/n)\sin(r\pi/(2n))), \end{aligned}$$

and moreover, by (21)

$$c_n > \frac{B}{A} - \frac{1}{A}(k_{19}\beta + k'_{19}\alpha)\sqrt{\varepsilon} > \frac{1}{A}\left(B - \frac{1}{n-2}\right), \qquad \text{if } n \neq 7,$$

since

$$(k_{19}\beta + k'_{19}\alpha)\sqrt{\varepsilon} < \frac{1}{n-2}, \qquad \text{if } n \neq 7.$$

If $n = 4s$ (note that $n \geq 4$), then

$$A = \frac{4(n-1)}{n-2} \qquad \text{and} \qquad B = \frac{n}{n-2}.$$

Thus, we have

$$c_n > \frac{n-2}{4(n-1)} \cdot \left(\frac{n}{n-2} - \frac{1}{n-2}\right) = \frac{1}{4},$$

which contradicts that $c_n = 1/4$.

If $n = 4s + 1$ (note that $n \geq 5$), then

$$\begin{aligned} B &= \frac{1}{n-2}(\cos(\pi/n) + \sin(\pi/(2n))) \\ &\quad + \frac{n-1}{n-2}(\cos(\pi/n)\cos(\pi/n) - \cos(2\pi/n)\sin(\pi/(2n))) \\ &= \frac{1}{n-2}(\cos(\pi/n) + \sin(\pi/(2n))) \\ &\quad + \frac{n-1}{n-2}\left(\frac{1+\cos(2\pi/n)}{2} - \cos(2\pi/n)\sin(\pi/(2n))\right) \\ &> \frac{1}{n-2}\left(1 - \frac{\pi^2}{2n^2} + \frac{3}{2n}\right) + \frac{n-1}{n-2}\left(\frac{1}{2} + \left(1 - \frac{2\pi^2}{n^2}\right)\left(\frac{1}{2} - \frac{\pi}{2n}\right)\right) \\ &= \frac{1}{n-2}\left(n - \left(\frac{\pi^3}{n^3} - \frac{\pi^3 + \pi^2/2}{n^2} + \frac{\pi^2 - \pi/2 - 3/2}{n} + \frac{\pi}{2}\right)\right). \end{aligned}$$

If we put $F_1(x) := \pi^3x^3 - (\pi^3 + \pi^2/2)x^2 + (\pi^2 - \pi/2 - 3/2)x + \pi/2$, then $F_1'(x) = 3\pi^3x^2 - (2\pi^3 + \pi^2)x + (\pi^2 - \pi/2 - 3/2)$. The derivative $F_1'(x)$ is monotone decreasing if $0 < x \leq 1/5$, and since $F_1'(1/10) > 0$ and $F_1'(1/9) < 0$, there exists $x_0 \in (1/10, 1/9)$ such that $F_1'(x_0) = 0$. Thus $F_1(x)$ attains the maximum value at $x = x_0$. If x is the reciprocal of some integer, then $F_1(1/10) < F_1(1/9)$ implies that $F_1(x) \leq F_1(1/9) < 2$. We, therefore, obtain

$$B > \frac{n-2}{n-2} = 1.$$

Hence, we have

$$c_n > \frac{n-2}{4(n-1)} \cdot \left(1 - \frac{1}{n-2}\right) = \frac{n-3}{4(n-1)},$$

which contradicts that $c_n = (n-3)/(4(n-1))$.

If $n = 4s+2$ (note that $n \geq 6$), then

$$\begin{aligned}
B &= \frac{1}{n-2}(1+\sin(\pi/n)) + \frac{n-1}{n-2}(\cos(2\pi/n) - \sin(\pi/n)) \\
&> \frac{1}{n-2}\left(1+\frac{3}{n}\right) + \frac{n-1}{n-2}\left(\left(1-\frac{2\pi^2}{n^2}\right) - \frac{\pi}{n}\right) \\
&= \frac{1}{n-2}\left(n - \left(\pi + \frac{2\pi^2-\pi-3}{n} - \frac{2\pi^2}{n^2}\right)\right) \\
&\geq \frac{1}{n-2}\left(n - \left(\pi + \frac{2\pi^2-\pi-3}{6} - \frac{2\pi^2}{6^2}\right)\right) \\
&> \frac{n-5}{n-2}.
\end{aligned}$$

Hence, we have

$$c_n > \frac{n-2}{4(n-1)} \cdot \left(\frac{n-5}{n-2} - \frac{1}{n-2}\right) = \frac{n-6}{4(n-1)},$$

which contradicts that $c_n = (n-6)/(4(n-1))$.

If $n = 4s+3$ and $n \neq 7$ (note that $n \geq 11$), then

$$\begin{aligned}
B &= \frac{1}{n-2}(\cos(\pi/n) + \sin(3\pi/(2n))) \\
&\quad + \frac{n-1}{n-2}(\cos(\pi/n)\cos(3\pi/n) - \cos(2\pi/n)\sin(3\pi/(2n))) \\
&> \frac{1}{n-2} \cdot \left(1 - \frac{\pi^2}{2n^2} + \frac{9}{2n}\right) + \frac{n-1}{n-2} \cdot \left(\left(1-\frac{\pi^2}{2n^2}\right)\left(1-\frac{9\pi^2}{2n^2}\right) - \frac{3\pi}{2n}\right) \\
&= \frac{1}{n-2}\left(n - \frac{1}{4}\left(\frac{9\pi^4}{n^4} - \frac{9\pi^4}{n^3} - \frac{18\pi^2}{n^2} + \frac{10\pi^2-6\pi-18}{n} + 6\pi\right)\right)
\end{aligned}$$

Since $F_3(x) := 9\pi^4x^4 - 9\pi^4x^3 - 18\pi^2x^2 + (10\pi^2-6\pi-18)x + 6\pi$ is monotone increasing if $0 < x \leq 1/11$, $F_3(x) \leq F_3(1/11) < 32$. Therefore,

$$B > \frac{n-8}{n-2}.$$

Hence, we have

$$c_n > \frac{n-2}{4(n-2)} \cdot \left(\frac{n-8}{n-2} - \frac{1}{n-2}\right) = \frac{n-9}{4(n-1)},$$

which contradicts that $c_n = (n-9)/(4(n-1))$.

If $n = 7$, then we directly estimate the constants. We have

$$\alpha = -\frac{1}{7-2} + \cos(\pi/7) + \frac{7-1}{7-2}\cos(2\pi/7) < 1.449157,$$

$$\beta = \frac{1}{7-2} + \sin(3\pi/14) + \frac{7-1}{7-2}\cos(3\pi/7) < 1.090515.$$

On the other hand, since $k_{16}' < 32872$ by $k_{16} < 0.2830n^3\sqrt{n}2^n$ and $k_{18}/(7-2) < 2172$ by $k_{17} < 0.3599n^6(n-1)^2 4^n$ if $n = 7$, we also have

$$k_{19}\sqrt{\varepsilon_n} < \frac{32872 + 2172(1 - \cos(2\pi/7))}{\sqrt{2} \cdot 7^4 \cdot \sqrt{7} \cdot 2^7} < 0.029298,$$

$$k_{19}'\sqrt{\varepsilon_n} < \frac{32872 + 2172(1 + \cos(3\pi/7))}{\sqrt{2} \cdot 7^4 \cdot \sqrt{7} \cdot 2^7} < 0.030896.$$

Thus,

$$\begin{aligned}(k_{19}\beta + k_{19}'\alpha)\sqrt{\varepsilon} &\leq (k_{19}\beta + k_{19}'\alpha)\sqrt{\varepsilon_n} \\ &< 0.029298 \cdot 1.090515 + 0.030896 \cdot 1.449157 \\ &< 0.076724.\end{aligned}$$

On the other hand, if $n = 7$, then

$$\begin{aligned}A &= \frac{n-1}{n-2}[(1 + \cos(3\pi/n))(1 + \cos(\pi/n)) \\ &\qquad - (1 - \cos(2\pi/n))(1 - \sin(3\pi/(2n)))] \\ &< 2.618658,\end{aligned}$$

$$\begin{aligned}B &= \frac{1}{n-2}(\cos(\pi/n) + \sin(3\pi/(2n))) \\ &\quad + \frac{n-1}{n-2}(\cos(\pi/n)\cos(3\pi/n) - \cos(2\pi/n)\sin(3\pi/(2n))) \\ &> 0.078985.\end{aligned}$$

Hence, we obtain

$$B - (k_{19}\beta + k_{19}'\alpha)\sqrt{\varepsilon} > 0.078985 - 0.076724 = 0.002261,$$

and then,

$$c_n > \frac{0.002261}{2.618658} > \frac{1}{1159},$$

which contradicts that $c_7 = 1/1159$.

We complete the proof of Main Theorem.

4. Some computations

In his paper [3], J. E. Brown proved the following. The definitions of $S(n,a)$ and $I_P(a)$ are given in Section 1.

Theorem B. *Let $P \in S(n,a)$ be an extremal polynomial. If $0 < a < A_n$, where A_n is the smallest positive root of*

$$n - (n-3)x - \frac{4x^2}{1+x^2} - (1 + x - x^2)^{n-1} = 0,$$

then $I_P(a) \le 1$.

We may assume that $I(\mathscr{P}_n) = I(P) = I_P(a)$. Then P is extremal and Main Theorem still holds. Under the assumption, combining our result with Theorem B, we can affirmatively solve Sendov's conjecture for polynomials which have one zero a with $0 < a < A_n$ or $1 - \varepsilon_n \le a < 1$. Numerical computations give the approximate values of A_n and ε_n (see Table 2).

If we assume $c_n = 0$, that is, $|w_j - a| \le 1$ for $j = 1, \dots, n-1$, then the statement of Main Theorem is equivalent to Sendov's conjecture for one zero a with $|a| \ge 1 - \varepsilon_n$. By making use of $c_n = 0$ and improving estimates for $n = 8, \dots, 17$, we get δ_n instead of ε_n (see Table 3).

Corollary 1. *Let P be a polynomial of degree $n = 8, \dots, 17$ with all the zeros in the closed unit disk. If one zero a of P satisfies $|a| \ge 1 - \delta_n$, where δ_n is given in Table 3, then there exists a critical point w such that $|w - a| \le 1$.*

n	A_n	ε_n	n	A_n	ε_n
8	0.491223	2.273737×10^{-14}	12	0.320704	2.310356×10^{-18}
9	0.431451	1.969280×10^{-15}	13	0.296483	2.810347×10^{-19}
10	0.385882	1.907349×10^{-16}	14	0.276030	3.606100×10^{-20}
11	0.349903	2.022255×10^{-17}	15	0.258502	4.845167×10^{-21}

Table 2

n	δ_n	n	δ_n
8	2.639966×10^{-11}	13	2.102111×10^{-15}
9	2.711327×10^{-12}	14	2.568821×10^{-16}
10	2.737107×10^{-13}	15	2.765708×10^{-17}
11	2.274622×10^{-14}	16	1.253953×10^{-17}
12	1.644964×10^{-14}	17	1.696073×10^{-18}

Table 3

Acknowledgement

The author would like to express his deepest gratitude to Professor Yoshihiro Mizuta for his helpful comments and enormous support. The author also appreciates Professor Hiroshi Yanagihara for his constructive suggestions. Special thanks are also due to Professor Toshiyuki Sugawa for his insightful ideas and invaluable suggestions.

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